5.3.6

(a) Let $g : [0, a] \to \mathbb{R}$ be differentiable, $g(0) = 0$, and $|g'(x)| \le M$ for all $x \in [0, a]$. Show $|g(x)| \le Mx$ for all $x \in [0, a]$.

Consider an arbitrary $x \in [0, a]$. We know $g \in C^0[0, x] \cap D^1(0, x)$. Apply the Mean Value Theorem and we have for some $c \in (0, x)$ that

$$\frac{g(x) - g(0)}{x - 0} = g'(c)$$

Taking absolute values on both sides of the equation yields

$$|g(x)| = |g(x) - g(0)| = |g'(c)||x| \le Mx$$

□

(b) Let $h : [0, a] \to \mathbb{R}$ be twice differentiable, $h'(0) = h(0) = 0$ and $|h''(x)| \le M$ for all $x \in [0, a]$. Show $h(x) \le Mx^2/2$ for all $x \in [0, a]$.

First observe that h' coincides with the function g from part (a). This means that $|h'(x)| \le Mx$ for all $x \in [0, a]$. Define $h_1(x) = x^2/2$ for $x \in \mathbb{R}$. Because $h', h_1 \in C^0[0, a] \cap D^2(0, a)$, we fix an arbitrary $x \in [0, a]$ and apply the Generalized Mean Value Theorem for $h', h_1 \in C^0[0, x] \cap C^2(0, x)$. It follows that for some $c \in [0, x]$,

$$\frac{h(x)}{x^2/2} = \frac{h(x) - h(0)}{h_1(x) - h_1(0)} = \frac{h'(c)}{h_1'(c)} = \frac{h'(c)}{c}$$

Taking absolute values on both sides of the equation then yields

$$|h(x)| = \underbrace{\frac{|h'(c)|}{c}}_{\le M} \frac{x^2}{2} \le \frac{Mx^2}{2}$$

□

(c) Conjecture and prove an analogous result for a function that is differentiable three times on $[0, a]$.

Conjecture: Let $f : [0, a] \to \mathbb{R}$ be three times differentiable, $f''(0) = f'(0) = f(0) = 0$ and $|f'''(x)| \le M$ for all $x \in [0, a]$. Show $|f(x)| \le Mx^3/3!$ for all $x \in [0, a]$.

Proof: The same argument as in part (b) shows that $|f'(x)| \le Mx^2/2$ for all $x \in [0, a]$. Define $f_1(x) = x^3/6 \in C^0[0, a] \cap D^1(0, a)$. Consider any $x \in [0, a]$. We know that $f_1, f' \in C^0[0, x] \cap D^1(0, x)$, so apply the Generalized Mean Value Theorem and we get for some $c \in [0, x]$ that

$$\frac{f(x)}{x^3/6} = \frac{f(x) - f(0)}{f_1(x) - f_1(0)} = \frac{f'(c)}{f_1'(c)} = \frac{f'(c)}{c^2/2}$$

Taking absolute values on both sides of the equation then yields

$$|f(x)| = \underbrace{\frac{|f'(c)|}{c^2/2}}_{\le M} \frac{x^3}{6} \le \frac{Mx^3}{3!}$$

□