MAT 127B
HW 8 Solutions(5.3.3/5.3.4/5.3.7)

Exercise 1 (5.3.3)

Let h be a differentiable function defined on the interval $[0, 3]$, and assume that $h(0) = 1, h(1) = 2$, and $h(3) = 2$.

a) Argue that there exists a point $d \in [0, 3]$ where $h(d) = d$.

b) Argue that at some point c we have $h'(c) = 1/3$.

c) Argue that $h'(x) = 1/4$ at some point in the domain.

Proof.

a) Consider the function

$$g(x) = h(x) - x.$$

Then, g is a differentiable function on $[0, 3]$ and $g(0) = 1, g(1) = 1$ and $g(3) = -1$.

Since g is differentiable, and hence continuous so there exists a $''c'' \in [1, 3] \subset [0, 3]$ such that $g(c) = 0$.

This implies

$$g(c) = 0 \implies h(c) = c.$$

with $c \in [0, 3]$.

b)

$$h(0) = 1; h(3) = 2.$$

By Mean Value Theorem, there exists $c \in [0, 3]$,

$$h'(c) = \frac{h(3) - h(1)}{3 - 1} = \frac{1}{3}.$$

c) Consider the function

$$g(x) = h(x) - \frac{x}{4} - \frac{5}{4}.$$

As before g is a differentiable function on $[0, 3]$, and

$$g(0) = -\frac{1}{4}, g(1) = \frac{1}{2} \text{ and } g(3) = 0.$$

Now, since g is differentiable and hence continuous, there exists $c \in [0, 1]$ such that $g(c) = 0$ (from a)).

Now $g(3) = 0$ and $g(c) = 0$.

By Rolle's Theorem there exists a $\tilde{c} \in [c, 3] \in [0, 3]$ such that $g'(\tilde{c}) = 0$.

Since $g'(\tilde{c}) = 0$, hence for $\tilde{c} \in [0, 3]$ we have

$$0 = g'(\tilde{c}) = h'(\tilde{c}) - \frac{1}{4} \implies h'(\tilde{c}) = \frac{1}{4}.$$

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Exercise 2 (5.3.4)

Let f be differentiable on an interval A containing zero, and assume (x_n) is a sequence in A with $(x_n) \longrightarrow 0$ and $x_n \neq 0$.

a) If $f(x_n) = 0$ for all $n \in \mathbb{N}$, show $f(0) = 0$ and $f'(0) = 0$.

b) Add the assumption that f is twice-differentiable at zero and show that $f''(0) = 0$ as well.

Proof.

a) Since f is differentiable on A, hence is continuous on A, hence

$$x_n \longrightarrow 0 \implies f(x_n) \longrightarrow 0.$$

Since by assumption, $f(x_n) = 0$ for all $n \in \mathbb{N}$ hence $f(0) = 0$.

Now since f is differentiable on A hence we have $f'(0)$ exists.

$$f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h}.$$

Since the limit exists, hence for any sequence $h_n \longrightarrow 0$, $(h_n \neq 0)$ the following sequence must converge to $f'(0)$. [f(0) = 0]

$$\frac{f(h_n)}{h_n} \longrightarrow f'(0).$$

Let $h_n = x_n$, then the sequence

$$\frac{f(x_n)}{x_n} = 0 \longrightarrow 0 = f'(0).$$

b) We have $f(x_n) = 0$. for all n and $f(0) = 0$.

For each n,

$f(x_n) = 0 = f(0)$.

Then by Rolle's Theorem, there exists $y_n \in (x_n, 0)$ or $y_n \in (0, x_n)$, depending on $x_n < 0$ or $x_n > 0$ such that $f'(y_n) = 0$. Note that $y_n \neq 0$.

Also since $0 < |y_n| < |x_n|$ and $x_n \longrightarrow 0$, implies $y_n \longrightarrow 0$.

Hence we have a sequence $y_n \longrightarrow 0$ and $y_n \neq 0$ with $f'(y_n) = 0$.

Since f' is differentiable at 0 hence we have $f''(0)$ exists.

$$f''(0) = \lim_{h \to 0} \frac{f'(h) - f'(0)}{h}.$$

Since the limit exists, hence for any sequence $h_n \longrightarrow 0$, $(h_n \neq 0)$ the following sequence must converge to $f''(0)$. [f'(0) = 0]

$$\frac{f'(h_n)}{h_n} \longrightarrow f''(0).$$

Let $h_n = y_n$, then the sequence

$$\frac{f'(y_n)}{y_n} = 0 \longrightarrow 0 = f''(0).$$

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Exercise 3 (5.3.7)

A fixed point of a function f is a value x where $f(x) = x$. Show that if f is differentiable on an interval with $f'(x) \neq 1$, then f can have at most one fixed point.

Proof.
We need to show that $f(x)$ cannot have more than 1 fixed point. Let $x_1 \neq x_2$ be two fixed points of $f(x)$. Say $f(x)$ is differentiable on the interval A and $x_1, x_2 \in A$.

Consider the function

$$g(x) = f(x) - x.$$

The function $g(x)$ is a differentiable function on an interval and we have

$$g'(x) = f'(x) - 1 \neq 0.$$

We have

$$g(x_1) = 0 = g(x_2).$$

By Mean Value Theorem there exists a $'c' \in A$ between x_1 and x_2 such that

$$0 = [g(x_1) - g(x_2)] = g'(c)[x_1 - x_2].$$

This implies $g'(c) = 0$ which is a contradiction to the fact that $g'(x) \neq 0$ for $x \in A$.

Hence there can be atmost 1 fixed point of f in A.

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