6.2.1 Let

$$f_n(x) = \frac{nx}{1+nx^2}$$

(a) Find the pointwise limit of (f_n) for all $x \in (0, \infty)$.

Fix $x > 0$. Let $f_1(n) = nx$ and $f_2(n) = 1 + nx^2$. We know $f_1, f_2 \in D(\mathbb{R})$ and $f_1(n) \to \infty$ and $f_2(n) \to \infty$ as $n \to \infty$. Because $f_2'(n) = x^2 \neq 0$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} f_2(n) = \infty$, a similar proof as in Theorem 5.3.8 in Abbott shows that $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{f_1(n)}{f_2(n)} = \lim_{n\to\infty} \frac{f_1'(n)}{f_2'(n)} = \frac{1}{x}$, which is the pointwise limit of (f_n) for all $x \in (0, \infty)$.

Alternatively, we can also observe first that the pointwise limit might be $\frac{1}{x}$ and then show $|f_n(x) - \frac{1}{x}| \to 0$ as $n \to \infty$ for each $x \in (0, \infty)$. □

(b) Is the convergence uniform on $0, \infty)$?

No. Denote the pointwise limit of (f_n) by $f(x)$ and observe that for fixed $n \in \mathbb{N}$,

$$|f_n(x) - f(x)| = \left| \frac{nx}{1+nx^2} - \frac{1}{x} \right| = \frac{1}{x+nx^2} \to \infty \quad \text{as } x \to 0^+$$

This means that for any $n \in \mathbb{N}$, we can find $\epsilon > 0$ (i.e. any positive real number) and $x > 0$ small enough such that $|f_n(x) - f(x)| \geq \epsilon$. □

(c) Is the convergence uniform on $(0, 1)$?

No with the same argument as in part (b).

(d) Is the convergence uniform on $(1, \infty)$?

Yes. Observe that for $x \in (1, \infty)$,

$$|f_n(x) - f(x)| = \frac{1}{x+nx^2} \leq \frac{1}{1+n} \to 0 \quad \text{as } n \to \infty$$

This means that for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ independent of x such that $\frac{1}{1+n} < \epsilon$ for all $n > N$. □

6.2.2

(a) Define a sequence of functions on $\mathbb{R}$ by

$$f_n(x) = \begin{cases} 1 & \text{if } x = 1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}$$

and let f be the pointwise limit of f_n.
Is each f_n continuous at zero? Does $f_n \to f$ uniformly on $\mathbb{R}$? Is f continuous at zero?

(i) Yes. We have $\lim_{x\to 0} f_n(x) = 0$ because there are only finitely many nonzero terms for each $n \in \mathbb{N}$. (ii) No. We have $f(x) = \lim_{n\to\infty} f_n(x) = 1$ for $x = \frac{1}{n}, n \in \mathbb{N}$ and $f(x) = 0$ otherwise. For each $n \in \mathbb{N}$, consider $x = \frac{1}{n+1}$ and we obtain $f_n(x) - f(x) = 0 - 1 = -1$. Pick $\epsilon = \frac{1}{2}$ and we are done. (iii) No. Because $f(0) = 0$ but $\lim_{n\to\infty}(\frac{1}{n}) = 1$.

(b) Repeat the exercise using the sequence of functions

$$g_n(x) = \begin{cases} x & \text{if } x = 1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}$$

(i) Yes. Same reason as in part (a). (ii) Yes. Note that $g(x) = \lim_{n\to\infty} g_n(x) = x$ if $x = \frac{1}{n}, n \in \mathbb{N}$. Then $g_n \to g$ uniformly because for all $x \in \mathbb{R}$ we have that $|g_n(x) - g(x)| \leq \frac{1}{n+1} \to 0$ as $n \to \infty$. (iii) Yes. Because $g(0) = 0$ and $|g(x) - 0| \leq x \to 0$ as $x \to 0$.

(c) Repeat the exercise once more with the sequence

$$h_n(x) = \begin{cases} 1 & \text{if } x = \frac{1}{n} \\ x & \text{if } x = 1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{n-1} \\ 0 & \text{otherwise} \end{cases}$$

(i) Yes. Same reason as above. (ii) No. Note that $h(x) = \lim_{n\to\infty} h_n(x) = g(x)$. For each $n \in \mathbb{N}$ and $n > 1$, consider $x = \frac{1}{n}$ and we get $h_n(x) - h(x) = \frac{n-1}{n}$. Pick $\epsilon = \frac{n-1}{2n}$ and we are done. (iii) Yes. Same reason as part (b).

In each case, explain how the results are consistent with the content of the Continuous Limit Theorem (Theorem 6.2.6).

In (a) we have each f_n is continuous at 0 but f is not, which implies that f_n is not uniform convergent. In (b) the uniform convergent of g_n preserves the continuity of g at 0 because each g_n is continuous at 0. In (c) we have h_n continuous at 0 but not uniform convergent, which does not say anything about the continuity of h.

6.2.9 Assume (f_n) and (g_n) are uniformly convergent sequences of functions.

(a) Show that $(f_n + g_n)$ is uniformly convergent sequence of functions.

Let $A = \text{Dom}(f) \cap \text{Dom}(g)$. Because (f_n) and (g_n) are uniformly convergent sequence of functions, we know there exist two limit functions f and g. And that for all $x \in A$ and for all $\epsilon > 0$, there exist $N_{f,\epsilon}, N_{g,\epsilon} \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \epsilon$ and $|g_n(x) - g(x)| < \epsilon$ for $n > N_{f,\epsilon}$ and $n > N_{g,\epsilon}$, respectively. Therefore, for each $\epsilon > 0$, pick $N_\epsilon := \max(N_{f,\epsilon}, N_{g,\epsilon})$. We then have for all $x \in A$ and for all $n > N_\epsilon$ that

$$|(f_n(x) + g_n(x)) - (f + g)| \leq |f_n(x) - f(x)| + |g_n(x) - g(x)| \leq 2\epsilon$$

Therefore, $(f_n + g_n) \to (f + g)$ uniformly. □

(b) Give an example to show that the product $(f_n g_n)$ may not converge uniformly.

Consider $f_n(x) = g_n(x) = \frac{1}{x} + \frac{1}{n}$ with $\text{Dom}(f_n) = \text{Dom}(g_n) = (0, \infty)$ and $n \in \mathbb{N}$. We know both sequences of functions converge uniformly to the limit function $\frac{1}{x}$ because $f_n(x) - \frac{1}{x} = \frac{1}{n}$ for all $x \in \text{Dom}(f)$ and we can pick n large enough so that $\frac{1}{n} < \epsilon$ for any $\epsilon > 0$. However, consider the sequence of functions $(f_n g_n)$ with its pointwise limit $\frac{1}{x^2}$. We have for any $n \in \mathbb{N}$ that

$$\left| f_n(x) g_n(x) - \frac{1}{x^2} \right| = \left(\frac{1}{x} + \frac{1}{n} \right)^2 - \frac{1}{x^2} = \frac{1}{n^2} + \frac{2}{xn} \to \infty \quad \text{as } x \to 0^+$$

which implies that $(f_n g_n)$ does not converge uniformly. □

(c) Prove that if there exists an $M > 0$ such that $|f_n| \leq M$ and $|g_n| \leq M$ for all $n \in \mathbb{N}$, then $(f_n g_n)$ does converge uniformly.

With the same notations as in part (a). For any $\epsilon > 0$, choose $N_\epsilon \in \mathbb{N}$ and we have for all $x \in D$ for any $n > N_\epsilon$ that

$$\begin{aligned} |f_n(x)g_n(x) - f(x)g(x)| &= |f_n(x)g_n(x) - f_n(x)g(x) + f_n(x)g(x) - f(x)g(x)| \\ &\leq \underbrace{|f_n(x)|}_{\leq M} \underbrace{|g_n(x) - g(x)|}_{<\epsilon} + \underbrace{|f_n(x) - f(x)|}_{<\epsilon} \underbrace{|g(x)|}_{\leq M} \\ &< 2M\epsilon \end{aligned}$$