MAT 127B
HW 10 Solutions(6.2.3/6.2.5/6.2.13)

Exercise 1 (6.2.3)

For each $n \in \mathbb{N}$ and $x \in [0, \infty)$, let

$$g_n(x) = \frac{x}{1 + x^n} \qquad \text{and} \qquad h_n(x) = \begin{cases} 1, & \text{if } x \geq \frac{1}{n} \\ nx, & \text{if } 0 \leq x < \frac{1}{n} \end{cases}$$

Answer the following questions for the sequences (g_n) and (h_n):

a) Find the pointwise limit on $[0, \infty)$.

b) Explain how we know that the convergence cannot be uniform on $[0, \infty)$.

c) Choose a smaller set over which the convergence is uniform and supply an argument to show that this is indeed the case.

Proof.

a) Pointwise limit of g_n.

- If $0 \leq x < 1$,

$$\lim_{n\to\infty} x^n = 0 \implies \lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} \frac{x}{1 + x^n} = x.$$

- If $x = 1$

$$\lim_{n\to\infty} 1^n = 1 \implies \lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} \frac{1}{1 + 1^n} = \frac{1}{2}.$$

- If $x > 1$

$$\lim_{n\to\infty} x^n = \infty \implies \lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} \frac{x}{1 + x^n} = 0.$$

Hence let $g(x)$ be the pointwise limit of $g_n(x)$

$$g(x) = \begin{cases} x, & \text{if } 0 \leq x < 1 \\ \frac{1}{2}, & \text{if } x = 1 \\ 0, & \text{if } x > 1 \end{cases}$$

Pointwise Convergence of $h_n(x)$

- If x=0

$$\lim_{n \to \infty} h_n(x) = \lim_{n \to \infty} 0 = 0.$$

- If $x > 1$,
  Since $\frac{1}{n} \longrightarrow 0$ as $n \longrightarrow \infty$.
  Hence for $x > 1$, there exists $N \in \mathbb{N}$(depends on x) such that for all $n \geq N$,

$$\frac{1}{n} \leq x.$$

  Hence

$$\lim_{n \to \infty} h_n(x) = \lim_{n \to \infty; n \geq N} h_n(x) = \lim_{n \to \infty; n \geq N} 1 = 1$$

Hence let $h(x)$ be the pointwise limit of $h_n(x)$

$$h(x) = \begin{cases} 0, & \text{if } x = 0 \\ 1, & \text{if } x > 0 \end{cases}$$

b) We know by a Theorem that if $f_n(x)$ be a sequence of continuous functions defined on $A \subset \mathbb{R}$ which converges uniformly to a function $f(x)$, then $f(x)$ is continuous in A.

We see that $A = [0, \infty)$, and g_n are continuous functions in A. If the sequence g_n converges uniformly to g, then the function $g(x)$ should be continuous in $[0\infty)$.

Since that is not the case, hence the convergence is not uniform.

Similarly for $h_n(x)$.

We see that $A = [0, \infty)$, and h_n are continuous functions in A.

This is because

$$\lim_{x \to \frac{1}{n}^+} h_n(x) = \lim_{x \to \frac{1}{n}^+} 1 = 1$$

$$\lim_{x \to \frac{1}{n}^-} h_n(x) = \lim_{x \to \frac{1}{n}^+} nx = n\frac{1}{n} = 1.$$

If the sequence h_n converges uniformly to h, then the function $h(x)$ should be continuous in $[0, \infty)$.

Since that is not the case, hence the convergence is not uniform.

c) Let us choose the interval,

$$A = \left[0, \frac{1}{2}\right].$$

Given $\epsilon > 0$, choose $N \in \mathbb{N}$ such that

$$N > \max\left\{\frac{-\ln(\epsilon)}{\ln 2}, 1\right\}.$$

Let $n \geq N$.

For any $x \in [0, \frac{1}{2}]$

$$|g_n(x) - g(x)| = \left|\frac{x}{1 + x^n} - x\right| = \left|\frac{x^{n+1}}{1 + x^n}\right|.$$

Since $x \geq 0$,

Hence

$$1 + x^n \geq 1 \implies \frac{1}{1+x^n} \leq 1 \qquad \text{and} \qquad x^{n+1} \leq \frac{1}{2^{n+1}}$$

$$\implies \left| \frac{x^{n+1}}{1+x^n} \right| \leq \frac{1}{2^{n+1}}.$$

Now, for the choice of n,

$$n \geq N > \frac{-\ln(\epsilon)}{\ln 2}$$
$$\implies 2^{n+1} > 2^n > \frac{1}{\epsilon}$$
$$\implies \frac{1}{2^{n+1}} < \epsilon.$$

Hence we have,

$$|g_n(x) - g(x)| < \epsilon.$$

for any $n \geq N$ and for any $x \in \left[0, \frac{1}{2}\right]$.
Hence in $A = [0, \frac{1}{2}]$, $g_n(x)$ converges uniformly to $g(x)$.

For $h_n(x)$ choose

$$A = [2, 3].$$

Since $n \geq 1$, hence for $x \in [2, 3]$, $h_n(x) = 1$ for every $x \in [0, 1]$.

Given $\epsilon > 0$, choose $N = 1 \in \mathbb{N}$, such that for any $n \geq N$,

$$|h_n(x) - h(x)| = 0 < \epsilon$$

for any $x \in [2, 3]$
Hence in $A = [2, 3]$, $h_n(x)$ converges uniformly to $h(x)$.

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Exercise 2 (6.2.5)

Using the Cauchy Criterion for convergent sequences of real numbers, supply a proof for Theorem 6.2,5 (First, define a candidate for f(x), and then argue that $f_n \longrightarrow f$ uniformly.)

Proof.

Theorem 6.2.5

A sequence of functions (f_n) defined on a set $A \subset \mathbb{R}$ converges uniformly on A if and only if for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $|f_n(x) - f_m(x)| < \epsilon$ whenever $m, n \geq N$ and $x \in A$.

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Let (f_n) be a sequence of functions which converge uniformly on A to f(say).

Given $\epsilon > 0$,
there exists $N' \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \frac{\epsilon}{2}$$

for $n \geq N$ and for all $x \in A$.

Choose $N = N' \in \mathbb{N}$.
For $n, m \geq N$, and for all $x \in A$ we have

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f(x) - f_m(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

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Let for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $|f_n(x) - f_m(x)| < \epsilon$ whenever $m, n \geq N$ and $x \in A$.

Given $\epsilon > 0$,
there exists $N \in \mathbb{N}$ such that for every $x \in A$ and for any $m, n \geq N, |f_n(x) - f_m(x)| < \epsilon$.
Let $x \in A$ be any element.
Consider the sequence $\{f_n(x)\} \subset \mathbb{R}$.

Since we know that the sequence $\{f_n(x)\}$ is a Cauchy sequence for every $x \in A$ in $\mathbb{R}$, hence we know that this sequence converges to a real number. We define,

$$f(x) = \lim_{n\to\infty} f_n(x).$$

for every $x \in A$.

We will show that f_n converges uniformly to f.

Given $\epsilon > 0$,
choose the $N \in \mathbb{N}$ such that $|f_n(x) - f_m(x)| < \frac{\epsilon}{2}$
for every $n, m \geq N$ and for every $x \in A$. [From the assumption]

We will show,

$$|f_n(x) - f(x)| < \epsilon$$

for all $n \geq N$ and for every $x \in A$.

Let $x \in A$ be any element.
Since the sequence $f_n(x)$ converges to $f(x)$, hence given the $\epsilon > 0$, choose $N_x \in \mathbb{N}$(may depend on x) such that $|f_m(x) - f(x)| < \frac{\epsilon}{2}$. for all $m \geq N_x$.

Let $n \geq N$.
Choose $m > \max\{N, N_x\}$.
Then

$$|f_n(x) - f(x)| < |f_n(x) - f_m(x)| + |f_m(x) - f(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Given $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for any $n \geq N$ and for any $x \in A$, we have

$$|f_n(x) - f(x)| < \epsilon.$$

This proves that f_n converges uniformly to f on A.

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Exercise 3 (6.2.13)

Recall that the Bolzano–Weierstrass Theorem (Theorem 2.5.5) states that every bounded sequence of real numbers has a convergent subsequence. An analogous statement for bounded sequences of functions is not true in general, but under stronger hypotheses several different

conclusions are possible. One avenue is to assume the common domain for all of the functions in the sequence is countable. (Another is explored in the next two exercises.)
Let $A = x_1, x_2, x_3, \dots$ be a countable set. For each $n \in N$, let f_n be defined on A and assume there exists an $M > 0$ such that $|f_n(x)| \leq M$ for all $n \in N$ and $x \in A$. Follow these steps to show that there exists a subsequence of (f_n) that converges pointwise on A.

a) Why does the sequence of real numbers $f_n(x_1)$ necessarily contain a convergent subsequence (f_{n_k})? To indicate that the subsequence of functions (f_{n_k}) is generated by considering the values of the functions at x_1, we will use the notation $f_{n_k} = f_{1,k}$.

b) Now, explain why the sequence $f_{1,k}(x_2)$ contains a convergent subsequence.

c) Carefully construct a nested family of subsequences $(f_{m,k})$, and show how this can be used to produce a single subsequence of (f_n) that converges at every point of A.

Proof.

a) Consider the sequence

$$\{f_n(x_1)\}_{n\in\mathbb{N}}.$$

We know by Bolzano-Weierstrass Theorem states that every bounded sequence of real numbers has a convergent subsequence.

$|f_n(x_1)| \leq M$ for every $n \in \mathbb{N}$ implies that there is a convergent subsequence in $\{f_n(x_1)\}$.

Let $\{f_{n_k}(x_1)\}_{k\in\mathbb{N}}$ be the subsequence which converges. We call $f_{n_k} = f_{1,k}$.

$$\{f_{1,k}\}_{k\in\mathbb{N}} \quad \text{such that } f_{1,k}(x_1) \text{ is a convergent sequence.}$$

b) Consider the sequence in $\mathbb{R}$,

$$\{f_{1,k}(x_2)\}.$$

Since $|f_{1,k}(x_2)| \leq M$ is a bounded sequence implies there is a convergent subsequence in $\mathbb{R}$, call it $f_{2,k}(x_2)$.

We have

$$\{f_n\} \supset \{f_{1,k}\} \supset \{f_{2,k}\} \text{ such that } \{f_{2,k}(x_2)\}, \{f_{1,k}(x_1)\} \text{ converges in } \mathbb{R}.$$

c) We will continue the process. After we obtain $\{f_{i,k}\}$ such that $f_{i,k}(x_i)$ converges, we consider the sequence $\{f_{i,k}(x_{i+1})\}$.

Since $|f_{i,k}(x_{i+1})| \leq M$ is a bounded sequence in $\mathbb{R}$, hence there exists a convergent subsequence $\{f_{i+1,k}(x_{i+1})\}$.

Hence we obtain

$$\{f_n\} \supset \{f_{1,k}\} \supset \{f_{2,k}\} \supset \cdots \supset \{f_{i,k}\} \supset \{f_{i+1,k}\} \supset \ldots \text{ such that } \{f_{j,k}(x_j)\} \text{ converges.}$$

Consider the subsequence

$$\{g_n\} = \{f_{1,1}, f_{2,1}, f_{2,2}, f_{3,1}, f_{3,2}, f_{3,3}, f_{4,1} \ldots\}.$$

OR

$$\{g_n\} = \{f_{i,j} | i \geq j\}$$

with dictionary order.

If,

$f_{i,j} = g_k$ and $f_{i'.j'} = g_{k'}$ then

$k < k'$ if $(i, j) < (i', j')$ in dictionary order.

Need to show that the sequence $\{g_n\}$ converges pointwise on A.

Let x_i be any point on A.

Consider the sequence $\{f_{i,k(x_i)}\}$.

We know that this sequence $\{f_i(x_i)\}$ converges hence every subsequence of it.

$$\{g_n\} = \{f_{1,1}, f_{2,1} \ldots f_{i-1,i-1}, f_{i,1}, f_{i,2}, \ldots f_{i,i}, f_{i+1,1} \ldots\}$$

Since we have the nested family of subsequence, hence $\{f_{j,k}\} \subset \{f_{i,k}\}$ for every $j \geq i$, hence the sequence

$$\{f_{i,1}, f_{i,2}, \ldots f_{i,i}, f_{i+1,1} \ldots f_{j,t} \ldots\} \subset \{g_n\}; \qquad i \leq j; t \leq j$$

is a subsequence $\{f_{i,k}\}$, and we know $f_{i,k}(x_i)$ converges

hence

$$\{f_{i,1}(x_i), f_{i,2}(x_i), \ldots f_{i,i}(x_i), f_{i+1,1}(x_i) \ldots f_{j,t}(x_i)\} \ldots\} \subset \{f_{i,k}(x_i); \qquad i \leq j; t \leq j$$

converges.

Hence

$$\lim_{n\to\infty} g_n(x_i) = \lim_{l\to\infty;k\leq l} f_{l,k}(x_i) = \lim_{l\to\infty;k\leq l;l\geq i} f_{l,k}(x_i)$$

which is exactly the limit of the sequence

$$\{f_{i,1}(x_i), f_{i,2}(x_i), \dots f_{i,i}(x_i), f_{i+1,1}(x_i) \dots f_{j,t}(x_i)\} \dots\} \subset \{f_{i,k}(x_i); \qquad i \leq j; t \leq j$$

which converges.

This implies

$\{g_n(x_i)\}$ converges and since x_i was an arbitrary element of A, hence the subsequence $\{g_n\} \subset \{f_n\}$ converges pointwise on A.

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