**7.3.2** Recall the Thomae's function

$$t(x) = \begin{cases} 1 & \text{if } x = 0 \\ \frac{1}{n} & \text{if } x = \frac{m}{n} \in \mathbb{Q} \setminus \{0\} \text{ is in lowest terms with } n > 0 \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

has a countable set of discontinuities occurring at precisely every rational number. Follow these steps to prove $t(x)$ is integrable on $[0,1]$ with $\int_0^1 t = 0$.

(a) First argue that $L(t, P) = 0$ for any partition $P$ of $[0,1]$.

It suffices to show $\inf\{f(x) : x \in [a,b]\} = 0$ for any $0 \le a < b \le 1$. Fix arbitrary pair of $a, b \in [0,1]$ with $a < b$, we know there exists $q \in [0,1] \setminus \mathbb{Q}$ such that $a < q < b$. Then $\inf\{f(x) : x \in [a,b]\} = f(q) = 0$. □

(b) Let $\epsilon > 0$, and consider the set of points $D_{\frac{\epsilon}{2}} = \{x \in [0,1] : t(x) \ge \frac{\epsilon}{2}\}$. How big is $D_{\frac{\epsilon}{2}}$?

For the case $\epsilon > 2$, we have $|D_{\frac{\epsilon}{2}}| = 0$ because $\sup t(x) = 1$. For $0 < \epsilon \le 2$, we know from the Archimedean property that there exists $N \in \mathbb{N}$ such that $\frac{1}{n} \le \frac{\epsilon}{2}$ for all $n > N$ and $\frac{1}{n} > \frac{\epsilon}{2}$ for $n = 1, \cdots, N$. Together with $t(0) = 1$, we obtain $|G_{\frac{\epsilon}{2}}| \le \frac{(1+N)N}{2} + 1$, where the inequality follows from the fact that we might count fractions that are not of the lowest terms. (More precisely, we have $|G_{\frac{\epsilon}{2}}| = \sum_{n=1}^{N} \phi(n) + 1$, where $\phi(n)$ is the Euler's toitent function.) □

(c) To complete the argument, explain how to construct a partition $P_\epsilon$ of $[0,1]$ so that $U(t, P_\epsilon) < \epsilon$.

With the same notations as in part (b), we see that $|G_{\frac{\epsilon}{2}}| \le 2N^2$. We construct a partition such that the length of any subinterval containing a point in $D_{\frac{\epsilon}{2}}$ is $\frac{\epsilon}{4N^2}$. One possible partition is

$$P_\epsilon = \bigcup_{1 \le m < n \le N} \left\{ \frac{m}{n} - \frac{\epsilon}{8N^2}, \frac{m}{n} + \frac{\epsilon}{8N^2} \right\} \bigcup \{0, 1\}$$

Denote the interval $[\frac{m}{n} - \frac{\epsilon}{8N^2}, \frac{m}{n} + \frac{\epsilon}{8N^2}]$ by $I_{mn}$. It is easy to check that $I_{mn}$'s are disjoint for $1 \le m < n \le N$. Computing $U(t, P_\epsilon)$ by summing over all $I_{mn}$'s first and then over all other subintervals, we obtain

$$U(t, P_\epsilon) \le \sum_{n=1}^{N} \sum_{m=1}^{n} \underbrace{\frac{|I_{mn}|}{N}}_{\le |I_{mn}|} + (1-0)\frac{\epsilon}{2} \le |G_{\frac{\epsilon}{2}}||I_{mn}| + \frac{\epsilon}{2} \le 2N^2 \frac{\epsilon}{4N^2} + \frac{\epsilon}{2} < \epsilon$$

□

**7.3.3** Let

$$f(x) = \begin{cases} 1 & \text{if } x = \frac{1}{n} \text{ for some } n \in \mathbb{N} \\ 0 & \text{othewise.} \end{cases}$$

Show that $f$ is integrable on $[0,1]$ and compute $\int_0^1 f$.

It is easy to see that $L(f, P) = 0$ for any partition. Now we show $U(f, P) = 0$. For any $\epsilon > 0$, consider the partition $P_\epsilon = \cup_{n=1}^{\infty} \{\frac{1}{n} - \frac{\epsilon}{4 \cdot 2^n}, \frac{1}{n} + \frac{\epsilon}{4 \cdot 2^n}\} \cup \{0, 1\}$ and denote the interval $[\frac{1}{n} - \frac{\epsilon}{4 \cdot 2^n}, \frac{1}{n} + \frac{\epsilon}{4 \cdot 2^n}]$ by $I_n$. We then have

$$U(f, P_\epsilon) = \sum_{n=1}^{\infty} 1 \cdot |I_n| + 0 \cdot \left|[0,1] \setminus \cup_{n=1}^{\infty} I_n\right| = \sum_{n=1}^{\infty} \frac{\epsilon}{2 \cdot 2^n} = \frac{\epsilon}{2} \sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{\epsilon}{2} < \epsilon$$

Therefore, $f$ in integrable on $[0,1]$ and $\int_0^1 f = 0$ □

**Remark.** We can also pick partitions for which the subintervals containing $\frac{1}{n}$ has length $\frac{1}{cn^\alpha}$ for $\alpha > 1$ and for some appropriate constant $c$. Note here we need $\alpha > 1$ because the harmonic series diverges.

**7.3.4** Let $f$ and $g$ be functions defined on (possibly different) closed intervals, and assume the range of $f$ is contained in the domain of $g$ so that the composition $g \circ f$ is properly defined.

(a) Show, by example, that it is not the case that $f$ and $g$ are integrable, then $g \circ f$ is integrable.

Consider the Thomae's function restricted to $[0,1]$ from exercise 7.3.2 and denote it by $f(x)$. Consider $g(x) : [0,1] \to \mathbb{R}$ given by $g(0) = 0$ and $g(x) = 1$ otherwise. We then have

$$(g \circ f)(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \cap [0,1] \\ 0 & \text{if } x \in [0,1] \setminus \mathbb{Q} \end{cases}$$

By the density of $\mathbb{Q}$ in $\mathbb{R}$ and its consequences, we can show that $U(g \circ f) = 1$ and $L(g \circ f) = 0$, so $g \circ f$ is not integrable on $[0,1]$. However, $f$ and $g$ are both integrable on $[0,1]$. (For $g$, consider the partition $\{0, \frac{\epsilon}{2}, 1\}$ for the corresponding $\epsilon > 0$).

Now decide the validity of each of the following conjecture, supplying a proof or conterexample as appropriate.

(b) If $f$ is increasing and $g$ is integrable, then $g \circ f$ is integrable.

We prove the above statement. Fix $\epsilon > 0$. Because $g$ is integrable, we can find a partition $P_g = \{x_0 < \cdots < x_n\}$ of $\text{Rng}(f) \subseteq \text{Dom}(g)$ such that $U(g|_{\text{Rng}(f)}, P_g) - L(g|_{\text{Rng}(f)}, P_g) < \epsilon$. Because $f$ is increasing, it is one-to-one so $f^{-1} : \text{Rng}(f) \to \text{Dom}(f)$ is well-defined and is also increasing. This means that the set $P = \{f^{-1}(x_0), \cdots, f^{-1}(x_n)\}$ is a partition of $\text{Dom}(f)$. We then have

$$\begin{aligned} U(g \circ f, P) - L(g \circ f, P) &= \sum_{k=1}^{n} (\underbrace{\sup_{[x_{k-1},x_k]} g \circ f}_{\substack{=(g\circ f)(f^{-1}(x_k)) \\ =g(x_k)}} - \underbrace{\inf_{[x_{k-1},x_k]} g \circ f}_{\substack{=(g\circ f)(f^{-1}(x_{k-1})) \\ =g(x_{k-1})}})[x_k - x_{k-1}] \\ &= U(g|_{\text{Rng}(f)}, P_g) - L(g|_{\text{Rng}(f)}, P_g) < \epsilon \end{aligned}$$

Therefore, $g$ is integrable. □

(c) If $f$ is integrable and $g$ is increasing, then $g \circ f$ is integrable.

The example in part (a) is an counterexample for this statement.