7.4.3 Decide which of the following conjectures is true and supply a short proof. For those that are not true, give a counterexample.

(a) If $|f|$ is integrable on $[a,b]$, then f is also integrable on this set.

Consider the counterexample where $f(x)=1$ for $x\in\mathbb{Q}\cap[a,b]$ and $f(x)=-1$ for $x\in[a,b]\setminus\mathbb{Q}$.

(b) Assume g is not integrable and $g(x)\geq 0$ on $[a,b]$. If $g(x)>0$ for infinite number of points $x\in[a,b]$, then $\int_a^b g>0$.

Consider the functions in Exercise 7.3.2 and 7.3.3 as counterexamples.

(c) If g is continuous on $[a,b]$ and $g(x)\geq 0$ with $g(y_0)>0$ for at least one point $y_0\in[a,b]$, then $\int_a^b g>0$.

We prove this statement. Because $g\in C[a,b]$, there exists $\delta(y_0)>0$ such that for any $x\in[a,b]$ with $|x-y_0|<\delta(y_0)$, we have $|g(x)-g(y_0)|<\frac{g(y_0)}{2}$. This means that

$$\int_a^b g=\underbrace{\int_a^{y_0-\delta(y_0)} g}_{\geq 0}+\int_{y_0-\delta(y_0)}^{y_0+\delta(y_0)} g+\underbrace{\int_{y_0+\delta(y_0)}^{b} g}_{\geq 0}\geq\int_{y_0-\delta(y_0)}^{y_0+\delta(y_0)} g\geq 2\delta(y_0)\frac{g(y_0)}{2}>0$$

□

7.4.5 Let f and g be integrable functions on $[a,b]$.

(a) Show that if P is any partition of $[a,b]$, then

$$U(f+g,P)\leq U(f,P)+U(g,P).$$

Provide a specific example where the inequality is strict. What does the corresponding inequality for lower sums look like?

Consider any $a\leq c<d\leq b$, we have

$$\sup_{x\in[c,d]}[f(x)+g(x)]\leq\sup_{x\in[c,d]}\left[\sup_{y\in[c,d]}f(y)+\sup_{y\in[c,d]}g(y)\right]\leq\sup_{y\in[c,d]}f(y)+\sup_{y\in[c,d]}g(y)=\sup_{x\in[c,d]}f(x)+\sup_{x\in[c,d]}g(x)$$

Because the choice of c and d is arbitrary, the above inequality holds for any subinterval in any partition of $[a,b]$. This proves the statement.

Example: Consider $f:[0,1]\to\mathbb{R}$ with $f(x)=1$ for $x\in[0,a]\cap\mathbb{Q}$ and $f(x)=1$ for $x\in[0,1]\setminus\mathbb{Q}$. Consider $g(x)=-f(x)$. We have for any partition of $[a,b]$ that $U(f+g,P)=0$ and $U(f,P)=U(g,P)=1$

The corresponding inequality for lower sums is $L(f+g,P)\geq L(f,P)+L(g,P)$.

(b) Review the proof of Theorem 7.4.2 (ii), and provide an argument for part (i) of this theorem.

Let $\mathcal{P}[a,b]$ denote all possible partitions of $[a,b]$. We have from part (a) that

$$\underbrace{\inf_{\mathbf{P}[a,b]}U(f+g,P)}_{U(f+g)}\leq U(f+g,P)\leq U(f,P)+U(g,P),\quad\forall P\in\mathcal{P}[a,b]$$

$$\Rightarrow\quad U(f+g)\leq\inf_{\mathbf{P}[a,b]}\big(U(f,P)+U(g,P)\big)\leq\underbrace{\inf_{\mathbf{P}[a,b]}U(f,P)}_{=U(f)}+\underbrace{\inf_{\mathbf{P}[a,b]}U(g,P)}_{=U(g)}$$

Similarly, we obtain $L(f+g)\geq L(f)+L(g)$. Putting these together,

$$L(f)+L(g)\leq L(f+g)\leq U(f+g)\leq U(f)+U(g)$$

Because f and g are both integrable on $[a,b]$, we have $\int_a^b f = U(f) = L(f)$ and $\int_a^b g = U(g) = L(g)$. This means that

$$L(f+g) = U(f+g) = \int_a^b f + \int_a^b g$$

Therefore, $f+g$ is integrable on $[a,b]$ with $\int_a^b (f+g) = \int_a^b f + \int_a^b g$ □

7.4.6 Although not part of Theorem 7.4.2, it is true that the product of integrable functions is integrable. Provide the details for each step in the following proof of this fact:

(a) If f satisfies $|f(x)| \leq M$ on $[a,b]$, show

$$|(f(x))^2 - (f(y))^2| \leq 2M|f(x) - f(y)|.$$

We have for any $x, y \in [a,b]$,

$$|(f(x))^2 - (f(y))^2| = \underbrace{|f(x)+f(y)|}_{\leq |f(x)|+|f(y)| \leq 2M} |f(x) - f(y))| \leq 2M|f(x) - f(y))|$$

□

(b) Prove that if f is integrable on $[a,b]$, then so is f^2.

We show f^2 satisfies the integrability criterion. Fix $\epsilon > 0$. Because f is integrable on $[a,b]$, there exists $P_\epsilon = \{x_0 < \cdots < x_n\}$ of $[a,b]$ such that $U(f, P_\epsilon) - L(f, P_\epsilon) < \frac{\epsilon}{2M}$. Recall from part (a) that

$$\begin{aligned}
& (f(x))^2 - (f(y))^2 \leq 2M|f(x) - f(y)|, \quad \forall x, y \in [a,b] \\
\Rightarrow \quad & \underbrace{\sup_{x \in [x_{k-1}, x_k]} (f(x))^2}_{=M_{k,f^2}} - (f(y))^2 \leq 2M \underbrace{\sup_{x \in [x_{k-1}, x_k]} |f(x) - f(y)|}_{= \underbrace{\sup_{x \in [x_{k-1}, x_k]} f(x)}_{=M_{k,f}} - f(y)}, \quad \forall y \in [a,b] \\
\Rightarrow \quad & M_{k,f^2} - \underbrace{\inf_{y \in [x_{k-1}, x_k]} (f(y))^2}_{=m_{k,f^2}} \leq 2M\big(M_{k,f} - \underbrace{\inf_{y \in [x_{k-1}, x_k]} f(y)}_{=m_{k,f}}\big)
\end{aligned}$$

We then have

$$U(f^2, P_\epsilon) - L(f^2, P_\epsilon) = \sum_{k=1}^n (M_{k,f^2} - m_{k,f^2})\Delta x_k \leq 2M \sum_{k=1}^n (M_{k,f} - m_{k,f})\Delta x_k < 2M \frac{\epsilon}{2M} = \epsilon$$

□

(c) Now show that if f and g are integrable, then fg is integrable. (Consider $(f+g)^2$.)

Note that $fg = \frac{(f+g)^2 - f^2 - g^2}{2}$. Because f and g are integrable, we know from part (a), (b) and exercise 7.4.5 that f^2, g^2 and $(f+g)^2$ are all integrable, and so are their linear combinations. □