7.5.6 (Integration-by-parts)

(a) Assume $h(x)$ and $k(x)$ have continuous derivatives on $[a, b]$ and derive the familiar integration-by-parts formula

$$\int_a^b h(t)k'(t)dt = h(b)k(b) - h(a)k(a) - \int_a^b h'(t)k(t)dt.$$

Let $F(x) = h(x)k(x)$ on $[a, b]$. Because $h, k \in C^1[a, b]$. we know $F'(x) = h'(x)k(x) + h(x)k'(x) \in C[a, b]$ and are both integrable on $[a, b]$. Apply FTC (Theorem 7.5.1(i)), with the linearity of integrals, we get

$$\int_a^b h' \cdot k + \int_a^b h \cdot k' = \int_a^b (h' \cdot k + h \cdot k') = \int_a^b F' = F(b) - F(a) = h(b)k(b) - h(a)k(a)$$

□

(b) Explain how the result in Exercise 7.4.6 can be used to slightly weaken the hypothesis in part (a).

In part (a), we need $h', k' \in C[a, b]$ to show $h'k$ and hk' are integrable. With the result from Exercise 7.4.6, we just need $h', k' \in D^0[a, b]$. So we can weaken the hypothesis in part (a) by assuming $h, k \in D[a, b]$ instead of $h, k \in C^1[a, b]$.

7.5.7 Use part (ii) of Theorem 7.5.1 to construct another proof of part (i) of Theorem 7.5.1 under the stronger hypothesis that f is continuous. (To get started, set $G(x) = \int_a^x f$.)

Under the stronger hypothesis, we know f is integrable and conitinuous on $[a, b]$. Define $G(x) = \int_a^b f$. By part (ii) of Theorem 7.5.1, we know G is differentiable on $[a, b]$ with $G'(x) = f(x)$. Because $F'(x) = f(x) = G'(x)$ for all $x \in [a, b]$, by Corollary 5.3.4 we have $F(x) = G(x) + c$ for some constant $c \in \mathbb{R}$ for all $x \in [a, b]$. We then have

$$F(b) - F(a) = G(b) + c - G(a) - c = G(b) - G(a) = \int_a^b f - \underbrace{\int_a^a f}_{=0} = \int_a^b f$$

□

7.5.10 (Change-of-variable Formula). Let $g : [a, b] \to \mathbb{R}$ be differentiable and assume g' is continuous. Let $f : [c, d] \to \mathbb{R}$ be continuous, and assume that the range of g is contained in $[c, d]$ so that the composition $f \circ g$ is properly defined.

(a) Why are we sure f is the derivative of some function? How about $(f \circ g)g'$?

We know f is integrable on $[c, d]$ because $f \in C[c, d]$. Let $G(x) = \int_c^x f$ for $x \in [c, d]$. By FTC (Theorem 7.5.1(ii)), we know $G \in D[c, d]$ with $G'(x) = f(x)$ for $x \in [c, d]$. We just showed that any continuous function $f : [c, d] \to \mathbb{R}$ is a derivative function of $\int_c^x f : [c, d] \to \mathbb{R}$. Similarly, $(f \circ g)g'$ is also a derivative function because $(f \circ g)g' \in C[a, b]$.

(b) Prove the change-of-variable formula

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(t)dt.$$

We know the LHS of RHS of the above equation make sense because both intergrands are derivative functions on their corresponding domains. It left to show that the equality holds. Let $G(x)$ be defined as in part (a), we then have $G'(x) = f(x)$ for all $x \in [c, d]$ and $f(g(x))g'(x) = G'(g(x))g'(x) = (G \circ g)'(x)$ for all $x \in [a, b]$, where the second equality following from chain rule (Theorem 5.2.5). Apply FTC (Theorem 7.5.1(i)) gives us

$$\int_a^b (G \circ g)'(x)dx = (G \circ g)(b) - (G \circ g)(a) = G(g(b)) - G(g(a)) = \int_{g(a)}^{g(b)} G'(t)dt$$

which together with the two equalities we proved before proves the statement. $\square$