Math 127BA Midterm Exam Answers May 4, 2020- 9:00-9:50

1. (8 pts: Derivative)
   Consider the following function:

$$f(x) = \begin{cases} (e^x - 1)\sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases} .$$

   (a) Show that $f(x)$ is continuous at 0.
   ANS: $-|e^x - 1| \leq f(x) \leq |e^x - 1|$ and $e^x - 1$ is continuous at 0 with $e^0 - 1 = 0$ so by the squeeze lemma $\lim_{x\to 0} f(x) = 0 = f(0)$.

   (b) Show that $f(x)$ is not differentiable at 0.
   ANS: $f'(0) = lim_{x\to 0}\frac{f(x)}{x}$ which does not exist since $lim_{x\to 0}\frac{e^x-1}{x} = 1$ by L'Hospital's rule so for sufficiently large n and $x = \frac{2}{(4n+1)\pi}$ we get $\frac{f(x)}{x} \geq \frac{1}{2}$ while for every n and $y = \frac{2}{(4n)\pi}$ we get $\frac{f(x)}{x} = 0$.

2. Consider the following sequence of differentiable functions on $\mathbb{R}$:

$$f_n(x) = \begin{cases} \frac{-1}{n} & x < \frac{-\pi}{2n} \\ \frac{sin(nx)}{n} & |x| \leq \frac{\pi}{2n} \\ \frac{1}{n} & x > \frac{\pi}{2n} \end{cases} .$$

   (a) Find the pointwise limit of the sequence $(f_n(x))$.
   ANS: $f(x) = 0$.

   (b) Find the pointwise limit of the derivative sequence $(f_n'(x))$.
   ANS: $g(x) = 0$ if $x \neq 0$ and $g(0) = 1$.

   (c) Determine which one (a or b) converges uniformly and which only pointwise.
   ANS: (a) is uniform.

3. (8 pts: Mean Value)
   Consider a power series $F(x) = \sum_{k=0}^{\infty} a_k x^k$.
   Assume that $a_0 = 0$ and

$$\lim_{x\to 1^-} F(x) = \sum_{k=1}^{\infty} a_k = 1.$$

   Show that there is some c with $F'(c) = 1$.

   ANS: Since the sum converges $R \geq 1$ so F is differentiable in $(-1, 1)$ and the limit shows F is continuous in $(-1, 1]$ with $F(1) = 1$ and $F(0) = a_0 = 0$. Thus by the MVT the result follows.

4. (8 pts: Taylor Lagrange)
   Find a number ϵ so that if $|x| < \epsilon$ then $|\cos(x) - 1 + \frac{x^2}{2}| < \frac{x^2}{100}$.

   ANS: Take $\epsilon = \frac{6}{100}$. Taking $f(x) = \cos(x)$ the LHS is $|R_2(x)|$ (also $|R_3(x)|$) so by Tay-Lag for some $c \in (0, x)$ it equals $|\cos^{(3)}(c)\frac{x^3}{6}| \leq |\frac{x^3}{6}| < \frac{\epsilon x^2}{6} = \frac{x^2}{100}$.

5. (8 pts: M-Test)
   Find the radius of convergence for the power series

$$\sum_{k=0}^{\infty} \frac{3^n x^{2n}}{(4n)! n^5}.$$

   ANS: ∞.

6. (8 pts: Smooth)
   Find a sequence a_k so that if $F(x) = \sum_{k=0}^{\infty} a_k x^k$ is the associated power series then

$$\lim_{n\to\infty} F^{(n)}(0) = 3.$$

   Be sure to justify the fact that $F^{(n)}(0)$ exists.

   ANS: Take $a_k = \frac{3}{k!}$. By the M-test the radius of convergence is positive (infinite) so F is real analytic with $F^{(n)}(0) = n!a_n = 3$.