Math 127BB Midterm Exam Answers May 4, 2020- 1:10-2:00

1. (8 pts: Derivative)
   Consider the following function:
   $$f(x) = \begin{cases} \sin(x)\sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}.$$
   (a) Show that $f(x)$ is continuous at 0.
   ANS: $-|\sin(x)| \leq f(x) \leq |\sin(x)|$ and $\sin(x)$ is continuous at 0 with $\sin(0) = 0$ so by the squeeze lemma $\lim_{x\to 0} f(x) = 0 = f(0)$.

   (b) Show that $f(x)$ is not differentiable at 0.
   ANS: $f'(0) = lim_{x\to 0}\frac{f(x)}{x}$ which does not exist since $lim_{x\to 0}\frac{\sin(x)}{x} = 1$ by L'Hospital's rule so for sufficiently large n and $x = \frac{2}{(4n+1)\pi}$ we get $\frac{f(x)}{x} \geq \frac{1}{2}$ while for every n and $y = \frac{2}{(4n)\pi}$ we get $\frac{f(x)}{x} = 0$.

2. (10 pts: Convergence)
   Consider the following sequence of differentiable functions on $\mathbb{R}$:
   $$f_n(x) = \begin{cases} -2(n^{-1}) & x < -n^{-1} \\ 3x - n^2x^3 & |x| \leq n^{-1} \\ 2(n^{-1}) & x > n^{-1} \end{cases}.$$
   (a) Find the pointwise limit of the sequence $(f_n(x))$.
   ANS: $f(x) = 0$.

   (b) Find the pointwise limit of the derivative sequence $(f_n'(x))$.
   ANS: $g(x) = 0$ if $x \neq 0$ and $g(0) = 3$.

   (c) Determine which one (a or b) converges uniformly and which only pointwise.
   ANS: (a) is uniform.

3. (8 pts: Mean Value)
   Consider a power series $F(x) = \sum_{k=0}^{\infty} a_k x^k$.
   Assume that $a_0 = 0$, $\sum_{k=1}^{\infty} a_k = 1$ and $\sum_{k=1}^{\infty} 2^k a_k = 100$.

   Show that there is some c with $F'(c) = 1$.

   ANS: Since the second sum converges $R \geq 2$ so F is continuous and differentiable in $(-1, 1]$ with $F(1) = 1$ and $F(0) = a_0 = 0$. Thus by the MVT the result follows.

4. (8 pts: Taylor Lagrange)
   Find a number ϵ so that if $|x| < \epsilon$ then $|\sin(2x) - 2x| < \frac{|x|}{100}$.

   ANS: Take $\epsilon = \frac{1}{200}$. Taking $f(x) = \sin(2x)$ the LHS is $|R_1(x)|$ (also $|R_2(x)|$) so by Tay-Lag for some $c \in (0, x)$ it equals $|\sin^{(2)}(2c)\frac{x^2}{2}| \leq |\frac{4x^2}{2}| < 2\epsilon|x| < \frac{|x|}{100}$.

5. (8 pts: M-Test)
   Find the radius of convergence for the power series

$$\sum_{n=0}^{\infty} \frac{n! x^{2n}}{3^n n^5}.$$

   ANS: 0.

6. (8 pts: Smooth)
   Find a sequence a_k so that if $F(x) = \sum_{k=0}^{\infty} a_k x^k$ is the associated power series then for every $n > 0$ we have

$$2n > F^{(n)}(0) > n.$$

   Be sure to justify the fact that $F^{(n)}(0)$ exists.

   ANS: Take $a_k = \frac{3k}{2k!}$. By the M-test the radius of convergence is positive (infinite) so F is real analytic with $F^{(n)}(0) = n! a_n = \frac{3n}{2}$.