MAT 127B-A

Winter 2021

Left Board

Newton 1666 undergrad at Cambridge.

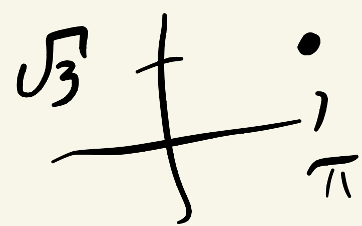
while away from school due to plague.

Computation from MAT 21

Analogous space: plane or $\mathbb{R}^2$

Nice pts

Rational pts $\left(\frac{a}{b}, \frac{c}{d}\right)$



Approx $(\pi, \sqrt{3}) \approx (3.1415\ldots, 3.1622\ldots)$

$(3,3),\ (3.1, 3.1), \ldots$

Both are examples of

Banach Algebras.

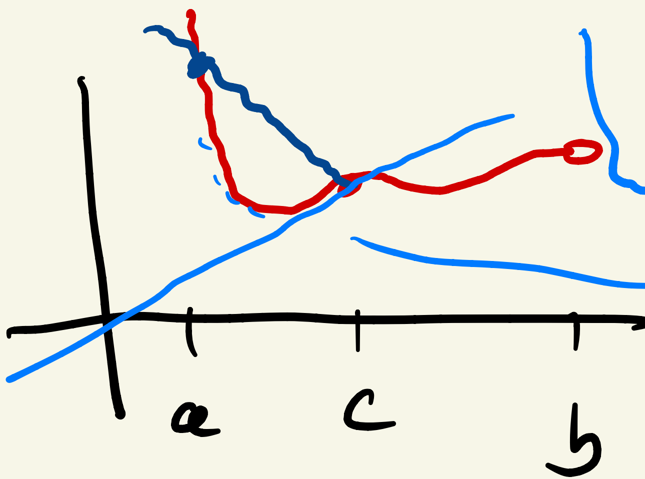
(Ch 13).

(Many more examples in Functional Analysis course).

Def: 8.1 If $a < c < b$ and $f : (a,b) \to \mathbb{R}$ then write

$$f'(c) = (Df)(c) = \lim_{h \to 0} \underbrace{\frac{f(h+c) - f(c)}{h}}_{x = h+c} = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

slope of secant line



slope of tangent

If $f'(c)$ exists say f is differentiable at c.

If f is diff. at every value in (a,b) say f is diff in or on (a,b).

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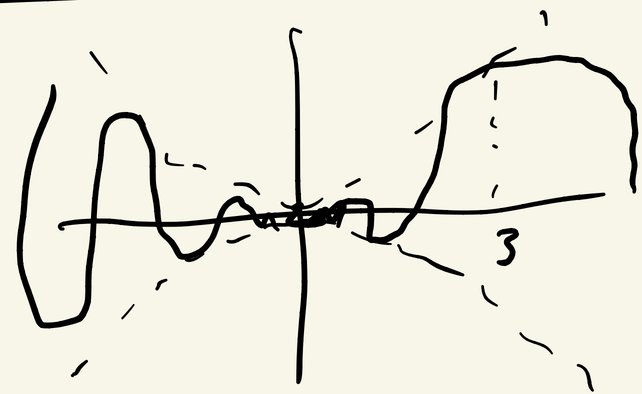
③ $f_3(x) = \begin{cases} x^2 \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$



Compute $f_3'(x)$ using chain and product rules away from $x=0$

and the def. at $x=0$

$$f_3'(x) = \begin{cases} 2x \sin\frac{1}{x} + x^2\left(\frac{-1}{x^2}\cos\frac{1}{x}\right) & x \neq 0 \\ \\ \end{cases}$$

$$f_3'(0) = \lim_{h\to 0} \frac{f_3(h)}{h} = \lim_{h\to 0} \frac{h^{\cancel{2}} \sin\frac{1}{h}}{\cancel{h}}$$

$$= \lim_{h\to 0} h \sin\frac{1}{h} = 0$$

squeeze

or sandwich thm.

Try in break out rm:

$$g_1(x) = \begin{cases} x \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases} \qquad \text{odd}$$

$$g_2(x) = \begin{cases} x^2 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases} \qquad \text{even.}$$

(A) where are the g cts.

(B) where differentiable.

$$g_1'(0) = \lim_{h\to 0} \frac{g_1(h)}{h}$$

$$-h \le g_1(h) \le h$$

$$-1 \le \frac{g_1(h)}{h} \le 1$$

Sandwich technique fails.

If $\lim_{h\to 0} \frac{h \sin \frac{1}{h}}{h}$

$= \lim_{h\to 0} \sin\frac{1}{h}$ exists

then if $x_i = \frac{1}{(2i+\frac{1}{2})\pi}$

then $\sin\frac{1}{x_i} = \sin(2i+\frac{1}{2})\pi$

$= 1$

and $\lim_{h\to 0} \sin\frac{1}{h}$

$= \lim_{i\to\infty} \sin\frac{1}{x_i} = 1$

if $y_i = \frac{1}{(2i-\frac{1}{2})\pi}$

then $\sin \frac{1}{y_i} = \sin\left[\left(2i - \frac{1}{2}\right)\pi\right]$

$= -1$

and $\lim\limits_{n \to 0} \sin \frac{1}{n} = -1$

210106

Def: 6.1 If $f:(a,b)\to\mathbb{R}$ and $a<c<b$ then $\lim_{x\to c} f(x) = L$ if

$\forall \varepsilon>0\ \exists \delta>0\ \forall x\in(a,b)$ with $|x-c|<\delta$ it is true that $|f(x)-f(c)|<\varepsilon$.

Def: If $f:(a,b)\to\mathbb{R}$ and $a<c<b$ then

(7.1) f is <u>continuous</u> (cts) at c if $\lim_{x\to c} f(x) = f(c)$,

(8.1) f is <u>differentiable</u> (diff) at c if $f'(c) = \lim_{x\to c}\frac{f(x)-f(c)}{x-c}$ exists,

New↓

(8.18) f is <u>continuously differentiable</u> (ctsly diff) at c if $f(x)$ is diff. in an interval around c and f' is cts at c.

Notation

Work required to check the examples:

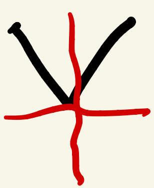
① Check: $|x|$ is cts.

Need $\lim_{x \to c} f(x) = f(c)$

use def. of limit.

ⓐ $x \neq 0$

ⓑ $x = 0$



ⓒ $|x|$ is not diff at $x = 0$

Need $\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{|x|}{x}$ does not exist.

Use the definition.

---

② Check: $f(x)=\begin{cases} x^2 \sin\frac{1}{x} & x\neq 0 \\ 0 & x=0 \end{cases}$ is diff

from last time $f'(x) = \begin{cases} 2x\sin\frac{1}{x} - \cos\frac{1}{x} & x\neq 0 \\ 0 & x=0 \end{cases}$

~~Also~~ $\boxed{\text{② Check: } f'(x) \text{ is not continuous}}$

Details for

Choose $x=0$ and show:

$$\lim_{x\to 0} f'(x) \neq f'(0) = 0$$

Assume for contradiction that $\lim\limits_{x\to 0} f'(x) = 0$

hence: $\forall \varepsilon>0\ \exists \delta>0\ \forall x$ with $|x|<\delta$ it is true that $|f'(x)| < \varepsilon$

Choose $\varepsilon = \frac{1}{2}$

hence by the hypothesis there is a $\delta > 0$

and choose $x = \dfrac{1}{2\pi n}$ with n big enough that $x < \delta$

Now compute: $\varepsilon = \frac{1}{2} > |f'(x)| = \left|2x\sin\frac{1}{x} - \cos\frac{1}{x}\right|$

$$= \left| \frac{1}{2\pi n} \sin(2\pi n) - \cos(2\pi n) \right|$$

$$= | \; 0 \; - 1 \; | = 1$$

oops

which is a contradiction.

Hence $\lim_{x \to 0} f'(x) \neq 0 = f'(0)$

so $f'(x)$ is discont. at 0.

so $f(x) \notin C^1 \mathbb{R}$.

Thm 8.17: If $f'(c)$ exists then f is cts at c.

Proof:

Recall properties of limits:

If $\lim_{x \to a} g(x) = L$ and $\lim_{x \to a} k(x) = M$

then $\lim_{x \to a} (g(x) + k(x)) = L + M$

$\lim_{x \to a} [g(x) \cdot k(x)] = L \cdot M$

Assume: $f'(c) = \lim\limits_{x \to c} \dfrac{f(x) - f(c)}{x - c}$ exists.

want to show $\lim\limits_{x \to c} f(x) \stackrel{?}{=} f(c)$.

Compute: $\lim\limits_{x \to c} (x - c) = 0$

$$f(c) = f(c) + \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \cdot \lim_{x \to c} (x - c)$$

$$= \lim_{x \to c} \left[f(c) + \frac{f(x) - f(c)}{x - c} \cdot x - c \right]$$

$$= \lim_{x \to c} f(x)$$

done.

C^k means $\underbrace{f^{(k)}}_{k^{\text{th}}\text{ deriv.}}$ (exists and) is cts

$\cap$|

D^k means $\overbrace{f^{(k)}}^{k^{\text{th}}\text{ deriv.}}$ exists.

different

C^∞ means for every k $f^{(k)}$ is cts.

$\shortparallel$

D^∞ means for every k $f^{(k)}$ exists

same

eg $\sin(x) = f$ $\quad f^{(1)} = \cos(x)$ $\quad f^{(2)} = -\sin \; \ldots \; f^{(k)}$

2/10/08

Algebraic operations on functions and how they relate to

127A: Limits/Continuity.

Today: Derivatives.

$$\underbrace{Fun(\mathbb{R})}_{\text{all functions } f:\mathbb{R}\to\mathbb{R}} \supsetneq C^0(\mathbb{R}) \supsetneq D^1(\mathbb{R}) \supsetneq C^1(\mathbb{R}) \supsetneq D^2(\mathbb{R}) \cdots$$

$k\cdot, +, \cdot, \circ$

Differentiation:

Thm 8.19 / 8.21: If f and g are diff., $k \in \mathbb{R}$.

(so $f, g \in D^1(\mathbb{R})$). then:

$(kf)'(c) = k(f'(c))$

$(f+g)'(c) = f'(c) + g'(c)$

$(g \cdot f)'(c) = g'(c)f(c) + g(c)f'(c)$

$\left(\frac{f}{g}\right)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{g^2(c)}$

(if $g(c) \neq 0$)

$D(kf) = k(Df)$

$D(f+g) = Df + Dg$

$D(g \cdot f) = (Dg) \cdot f + g \cdot (Df)$

$D\left(\frac{f}{g}\right) = \frac{(Df) \cdot g - f \cdot (Dg)}{g^2}$

$(g \circ f)'(c) = [(g' \circ f)(c)] \cdot f'(c)$ | $D(g \circ f) = [(Dg) \circ f] \cdot (Df)$

Hence: If $f, g \in D^1(\mathbb{R})$

then $k \cdot f,\ f+g,\ f \cdot g,\ f \circ g \in D^1(\mathbb{R})$

by the above rules.

If $f, g \in C^1(\mathbb{R})$ so $f', g' \in C^0(\mathbb{R})$ (both cts.)

$D^2(\mathbb{R})$

$C^2(\mathbb{R})$

$C^\infty(\mathbb{R}) = D^\infty(\mathbb{R})$

$$\mathrm{Fun}(\mathbb{R}) \supseteq C^0(\mathbb{R}) \supseteq D^1(\mathbb{R}) \supseteq C^1(\mathbb{R}) \supseteq D^2(\mathbb{R}) \cdots \supseteq C^\infty(\mathbb{R})$$

$k\cdot, +$

$\bullet, \circ$

Proof of chain rule:

Consider $g, f \in D^1\mathbb{R}$.

Write
$$G_a(x) = \begin{cases} \dfrac{g(x)-g(a)}{x-a} & x \neq a \\ g'(a) & x = a \end{cases}$$

which is cts at a iff g is diff at a

Also $f \in D'(\mathbb{R})$ is diff so f is cts

So by above:

$$\lim_{x \to c} (G_a \circ f)(x) = \lim_{y \to f(c)} G(y)$$

take $a = f(c)$.

Now compute:

$$(g \circ f)'(c) \underset{\text{def.}}{=} \lim_{x \to c} \frac{g(f(x)) - g(f(c))}{x - c}$$

$$= \lim_{x \to c} \frac{g(f(x)) - g(f(c))}{f(x) - f(c)} \frac{f(x) - f(c)}{x - c}$$

$$= \lim_{x \to c} \frac{g(f(x)) - g(f(c))}{f(x) - f(c)} \cdot \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

$$= \lim_{y \to f(c)} \frac{g(y) - g(f(c))}{y - f(c)} \cdot \quad \text{''} \qquad \text{''}$$

$$= g'(f(c)) \cdot f'(c)$$

To show $D^\infty(\mathbb{R}) = C^\infty(\mathbb{R})$

Def: $f \in D^\infty(\mathbb{R})$ if

for every k $f^{(k)}$ exists.

Hence if $f \in D^\infty(\mathbb{R})$: then $f^{(k+1)}$ exists.

so $f^{(k)}$ is diff

so $f^{(k)}$ is cts.

So $f \in C^\infty(\mathbb{R})$

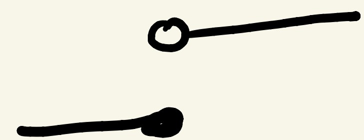
210111 Today: Rules:

Parametric.

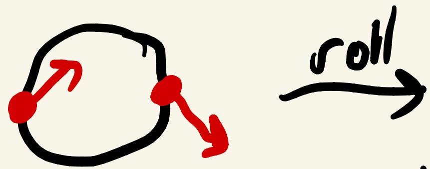
Inverse functions.

Bestiary: which spaces are examples in.

which functions are derivatives?

eg: $f_0(x) = \begin{cases} 1 & x > 0 \\ 0 & x \leq 0 \end{cases} =$ 

Is there any g with $g' = f$?



Ans: slope $= \dfrac{y'(t)}{x'(t)} = \dfrac{\cos(t)}{1-\sin(t)}$

at t with $y(t) = 1 = 1 + \sin(t)$

so $\sin(t) = 0$ or $t = 0$ or π or 2π

$= n\pi$.

so slope $= \dfrac{\cos(n\pi)}{1-\sin(n\pi)} = \cos(n\pi) = \pm 1$

In Br. Rm: Prove ①

Recall quot. rule for limits.

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim f}{\lim g},$$

Try: $\displaystyle \lim_{t \to c} \frac{y(t) - y(c)}{x(t) - x(c)} = \lim_{t \to c} \frac{y(t) - y(c)}{t - c} \frac{t - c}{x(t) - x(c)}$

$$= \lim_{t \to c} \frac{\frac{y(t) - y(c)}{t - c}}{\frac{x(t) - x(c)}{t - c}} = \frac{\displaystyle \lim_{t \to c} \frac{y(t) - y(c)}{t - c}}{\displaystyle \lim_{t \to c} \frac{x(t) - x(c)}{t - c}}$$

$$= \frac{y'}{x'}$$ ✓

Ex:	e^x	x^2	$\tan(x)$	x	$\frac{1}{x}$
f	e^x	x^2	$\tan(x)$	x	$\frac{1}{x}$
f^{-1}	$\ln(x)$	$x^{\frac{1}{2}}$	$\mathrm{Arctan}(x)$	x	$\frac{1}{x}$

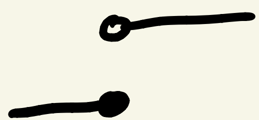
Bestiary:

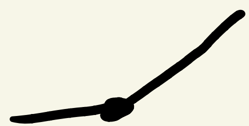
Functions!

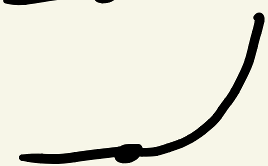
MAT21!

x^k, $\sin(x)$, e^x

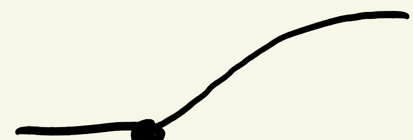
$$f_k(x) = \begin{cases} x^k & x>0 \\ 0 & x\le 0 \end{cases}$$

$f_0(x) =$ 

$f_1(x) =$ 

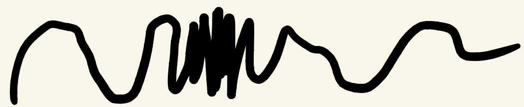
$f_2(x) =$ 

$$F(x) = \begin{cases} e^{-\frac{1}{x}} & x>0 \\ 0 & x<0 \end{cases}$$



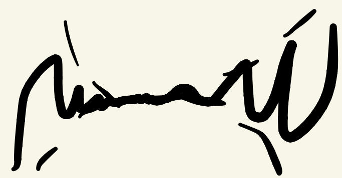
$$g_k(x) = \begin{cases} x^k \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$g_0(x) =$

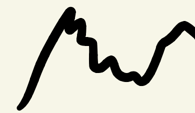


$g_1(x) =$

$g_2(x)$



$W(x) =$ very jagged



Weierstrass fn.

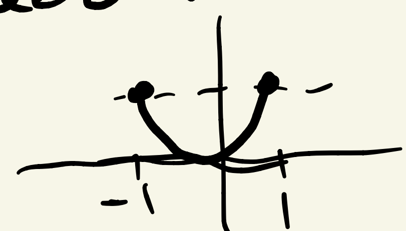
2/01/13 Mean Value Theorem.

(8.33 Hunter or 5.3.2 Abbott).

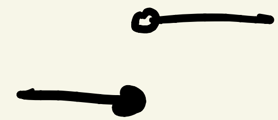
Idea: If $f \in D^1(a,b) \cap C^0[a,b]$

(so $f : [a,b] \to \mathbb{R}$ continuous at every point and diff. at interior points).

then information at a & b give information about some interior point.



Applications (Next time):

Darboux Thm: eg  & D°

Inv. Fn Thm: If $f'(c) \neq 0$ then

f^{-1} exists near c 

Antideriv: If $Dg = Df$

then $g = f + C$

Tay-Lag: How well do polynomials approx a function?

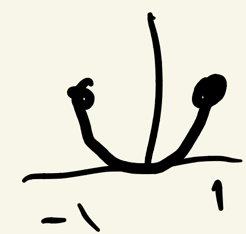
L'Hôspital's Rule.

From 127A:

Thm (7.37 A or 4.4.2 A)

If $f \in C^0[a,b]$ then there are m, M with

$$f(m) = \min \operatorname{Im}(f)$$
$$f(M) = \max \operatorname{Im}(f)$$



$m = 0$

$M = 1$ or -1

$f(x) = x^2$

Thm (Rolle) (8.32 H) (5.3.1 A)

If $f \in C[a,b] \cap D'(a,b)$

and $f(a) = f(b)$ then

$\exists\, c \in (a,b)$ with $\underbrace{f'(c)}_{\text{slope of tangent line}} = \underbrace{0}_{\text{slope of the secant line at } a,b}$.

Proof: By 7.37 choose m, M with

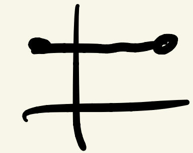
$f(m) = \min \operatorname{Im} f$, $f(M) = \max \operatorname{Im}(f)$

Case 1 either m or M is in (a,b).

then by 8.27 take $c = m$ or M and $f'(c) = 0$

Case 2: If both m and M are endpts:

then $f(m) = f(a) = f(b) = f(M)$



so f is a constant function

so $D(f) = 0$ so any $c \in (a,b)$ has $f'(c) = 0$

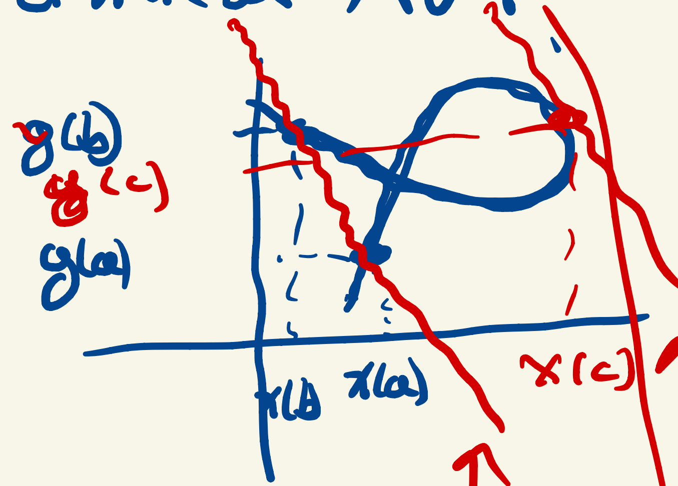
Thm (8.53) (5.3.2)

(Generalized or Parametric or Cauchy) MVT

If $f, g \in D'(a,b) \cap C^0[a,b]$ then

$\exists\, c$ with $[f(b)-f(a)]\,g'(c) = f'(c)[g(b)-g(a)]$

Picture: Parametric version of MVT



$x(t) = g$

$y(t) = f$

then if $x'(c) \neq 0$

and $x(b) - x(a) \neq 0$

then the above conclusion becomes:

$$\underbrace{\frac{y'(c)}{x'(c)}}_{\text{slope of tan line}} = \underbrace{\frac{y(b)-y(a)}{x(b)-x(a)}}_{\text{slope of secant line}}$$

Proof Pleen: Consider:

$$h(x) = [f(b) - f(a)]g(x) + f(x)[g(b) - g(a)]$$

Claim: This sat. the hyp of the MVT.

Next tim: Check: $h \in C^0 \cap D'$

and if $h'(c) = \frac{h(b) - h(a)}{b-a}$ then holds.

210115 No Lecture Monday (MLK).

HW assigned today due wed.

Today: Applications of the MVT.

Recall:

Thm (Gen/Parm/Cauchy MVT) (8.53 H) (5.3.5 A)

$\forall f, g \in C^0[a,b] \cap D^1(a,b)$ $\exists c \in (a,b)$ with $[f(b)-f(a)]\,g'(c) = f'(c)\,[g(b)-g(a)]$

Proof: Take $h(x) = [f(b)-f(a)]\,g(x) - f(x)\,[g(b)-g(a)]$

Apply MUT

In Brk Rm: use to prove Cau MUT.

Check: h is cts and diff.

but: f and g are cts & diff

and both props. are pres by lin comb

(or $C^0[a,b] \cap D^1(a,b)$ is a vect. sp).

Hence: by MUT get $c \in (a,b)$ with

$$h'(c) = \frac{h(b)-h(a)}{b-a} \text{ and compute:}$$

$$\shortparallel$$

$$[f(b)-f(a)]\, g'(x) - f'(x)\,[g(b)-g(a)]$$

and RHS: $\dfrac{h(b)-h(a)}{b-a} = \dfrac{[f(b)-f(a)]g(b)-f(b)[g(b)-g(a)]}{b-a}$

Useful: L'Hôspital's Rule
and the Inv. Fn. Thm

Example:
If $f \in D^2(\mathbb{R})$ and $f(0)=0$ and $f'(0)=0$

and $\forall x,\ f''(x) \le 3$,

Show that $f(2) \le 12$

Ans: Ideas:

Try 2 steps: First bound $f'(x)$ if $x \in [0,2]$.

Claim: $\forall x \in [0,2]$ have $f'(x) \le 6$

similar to before: if $f'(c) > 6$

then $f''(e) = \dfrac{f'(c) - f'(0)}{c - 0} = \dfrac{f'(c)}{c} > 3$.

Similarly since $f'(x) \le 6$

get $f(2) \le 12$

Application:

How close is D to being injective?

Thm (5.3.3) (8.34) $\underbrace{\ker(D)}_{\{f \mid Df = 0 \text{ function}\}} = \{\text{constant functions}\}$.

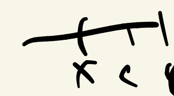
Proof: For contradiction assume

$\exists f \in D'(\mathbb{R})$ with $D(f) = 0$

and some $x < y$ have $f(x) \neq f(y)$

Hence by MVT: $\exists\, c \in (x, y)$

with $f'(c) = \dfrac{f(y) - f(x)}{y - x} \neq 0$

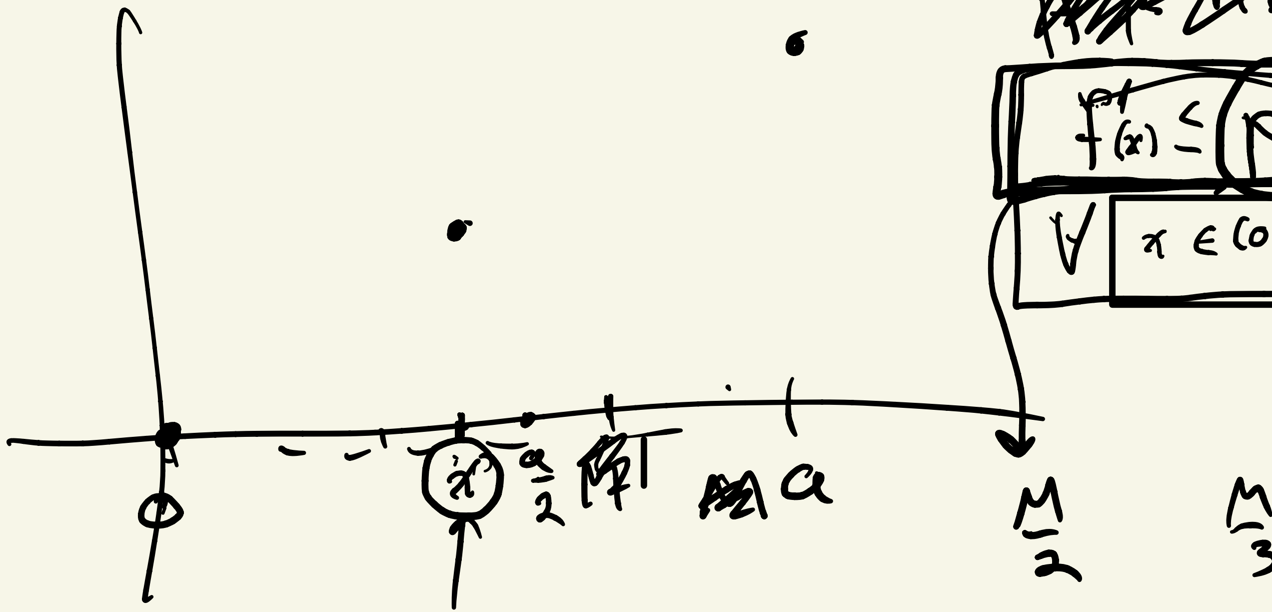


Cor (5.3.4) (8.35)

If $f, g \in \mathcal{D}'(\mathbb{R})$ and $f' = g'$ then $f = g + C$ for some constant C.

Pf: If $f' = g'$ then $\underbrace{f' - g'}_{= (f-g)'} = 0$

so $f - g = C$ is a constant fn.



$f(x) \le M \quad \forall\, x \in (0, M)$

210120 Using the MVT to understand diff. functions (D^1) and derivatives. (D^0)

Compare 127A information about images of continuous fns (C^0) to those of derivatives (D^0)

Recall: $G_2(x) = \begin{cases} x^2 \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases} \in D^1(\mathbb{R}) \subseteq C^0(\mathbb{R})$

$G_2'(x) = \begin{cases} 2x \sin \frac{1}{x} - \cos \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases} \in D^0(\mathbb{R})$

$\notin C^1(\mathbb{R})$

Cor: If $f, g \in D^0(a,b)$ and $\forall x \neq e$ have $f(x) = g(x)$ then $f(e) = g(e)$.

Proof: For contradiction assume $f(e) \neq g(e)$ then. $f - g \in D^0(a,b)$ since $f, g \in D^0(a,b)$ bcuz.

$$(f-g)(x) = \begin{cases} 0 & x \neq e \\ \underbrace{f(e) - g(e)}_{\neq 0} & x = e \end{cases} \notin D^0(a,b)$$

a contradiction

eg: $\underbrace{\begin{Bmatrix} 2x\sin\frac{1}{x} - \cos\frac{1}{x} & x \neq 0 \\ \frac{1}{2} & x = 0 \end{Bmatrix}}$ $\notin D^0(\mathbb{R})$.

even though $\uparrow$ satisfies the IVP.

(connected image).

Proof of Darboux's thm: (c) above:

The thm is equivalent to: If $a < c < d < b$ and $f(x) \in D^0(a,b)$ so $f(x) = F'(x)$ with $F(x) \in D^1(a,b)$ and if α is between $F'(c)$ and $F'(d)$ then there is

$e \in (c,d)$ with $F'(e) = \alpha$

In this case:

either ① $F'(c) < \alpha < F'(d)$] very similar

or ② $\quad '' \quad > \quad '' \quad > \quad ''$] consider this case

Write $G(x) = F(x) - \alpha x$ so $G'(x) = F'(x) - \alpha$

so ② $G'(c) > 0 > G'(d)$

Goal: Find $e \in [c,d]$ with $G'(e) \overset{?}{=} 0$

Since F is diff and hence cts so is G.

and by 7.37 have $M \in [c,d]$ with $G(M) = \max \operatorname{Im} G|_{[c,d]}$.

So case ② $M \in (c,d)$ hence by 8.27 the int. ext. thm have $G'(M) = 0$

so take $e = M$.

ⓒ $M = c$] consider this case

ⓓ $M = d$] similar.

will show thus is not possible by finding x with $G(x) > G(c) = G(M)$. contradicting the choice of M

Taylor Polynomials and Lagrange error term.

(will another type of error later)

Notation: If $f \in D^n(a,b)$ and $c \in (a,b)$

write $P_{n,c}(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots$

$$\cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

and remainder or error is

$$R = R_{n,c,x} = f(x) - P_{n,c}(x).$$

Thm: (8.46): If $f \in D^n(a,b)$, , $c, x \in (a,b)$

then $\exists\ \xi$ between x and c with

$$R_{n,c,x} = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-c)^{n+1}$$

Try: $f(x) = e^x$, $f(x) = \frac{1}{x} = x^{-1}$.

2/01/22 Today: Using Gen MVT.
Start limits of functions,

Recall: (Gen MVT):
Thm (8.53): $\forall\, k, g \in D^1(a,b)$, $a < c < x < b$

$\exists\ \xi \in [c, x]$ and

$$[k(x) - k(c)]\, g'(\xi) = k'(\xi)\,[g(x) - g(c)]$$

That is:

$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall |x - c| < \delta$

have $\left| L - \frac{k(x_0)}{g(x_0)} \right| < \varepsilon$

Back to Taylor-Lagrange Thm.

(another app. of the GMVT),

Thm (8.46) $\forall f \in D^n(a,b)$, $a < c < x < b$

$\exists\, \xi \in [c, x]$ with $\underbrace{f(x) - P_{n,c}(x)}_{R_{n,c,x}} = f^{(n+1)}(\xi)\,\dfrac{(x-c)^{n+1}}{(n+1)!}$

$$P_{n,c}(x) = f(c) + f'(c)(x-c) + \cdots + f^{(n)}(c)\,\frac{(x-c)^n}{n!}$$

Proof: Choose for GMUT:

$k(t) = R_{n,c,x}\,\dfrac{(x-t)^{n+1}}{(x-c)^{n+1}}$ $\qquad$ $g(t) = R_{n,t,x}$

$k(x) - k(c) = -R_{n,c,x}$ $\qquad$ $g(x) - g(c) = (f(x) - f(x)) - R_{n,c,x}$

$$k'(t) = R_{n,c,x}(n+1)(-1)\frac{(x-t)^n}{(x-c)^{n+1}}$$

claim: $$g'(t) = f^{(n+1)}(t)\frac{(x-t)^n}{n!}$$

Now using GMVT: $\exists\ \xi \in [c,x]$

with $[k(x)-k(c)]\,g'(\xi) = k'(\xi)\,[g(x)-g(c)]$

so $$g'(\xi) = k'(\xi)$$

so $$-R_{n,c,t}(n+1)\frac{(x-\xi)^n}{(x-c)^{n+1}} = f^{(n+1)}(\xi)\frac{(x-\xi)^n}{n!}$$

so $$R_{n,c,t} = f^{(n+1)}(\xi)\frac{(x-c)^{n+1}}{(n+1)!}$$

qed.

210125 Ch9: Limits of functions:

Def(3.10) If $(a_n)_{n\in\mathbb{N}} = (a_n)$ is a segence and A a number both in $\mathbb{R}$ then $a_n \xrightarrow[n\to\infty]{\text{in } \mathbb{R}} A$ if $\forall \varepsilon>0\ \exists N\ \forall n>N$ have $\underbrace{|a_n - A| < \varepsilon}$

~~unif~~ ptwise — unif.

How to modify for $\forall x\in(a,b)$ or $\|f_n - F\| < \varepsilon$ for some $\|\cdot\|$ - there are varying choices.

Def If $(f_n(x))$ is a sequence and $F(x)$ a function both in Fun(a,b) then

9.1 $f_n \xrightarrow[n\to\infty]{\text{ptwise}} F$ if $\forall x\in(a,b)$ have $f_n(x) \xrightarrow[n\to\infty]{\text{in } \mathbb{R}} F(x)$

have $|f_n(x) - F(x)| < \varepsilon$

or equiv: $\forall x\in(a,b), \varepsilon>0\ \exists N\ \forall n>N$

9.8 $f_n \xrightarrow[n\to\infty]{\text{uniformly}} F$ if $\forall\ \varepsilon>0\ \exists N\ \forall n>N, x\in(a,b)$ " "

13.45 $f_n \xrightarrow[n\to\infty]{\|\cdot\|\text{-norm}} F$ if $\forall\ \varepsilon>0\ \exists N\ \forall n>N$ have $\|f_n - F\| < \varepsilon$

where: $\|\bullet\|:(\text{some functions}) \to \mathbb{R}_{\geq 0}$
is a norm (Def 13.20)

Examples: $\|g(x)\|_{sup} = \|g(x)\| = \sup_{x\in(a,b)} |g(x)|$

$\to \|g(x)\|_{L^1} = \int_0^b |g(x)|\,dx$

$\|g(x)\|_{C^1} = \|g(x)\|_{sup} + \|g'(x)\|_{sup}$

Note: Two of: (A) pointwise are the same
(B) uniform
(C) $\|\cdot\|_{sup}$-norm
(D) $\|\cdot\|_{c^1}$-norm

Questions: For which intervals does $f_n \xrightarrow[n\to\infty]{} F$ in each of the senses above

	ptwise	unif. $= \|\cdot\|_{\sup}$-norm	$\|\cdot\|_{c^1}$-norm
1	$(-1, 1]$	$[-1+\varepsilon, 1-\varepsilon]$	$[-1+\varepsilon, 1-\varepsilon]$
2	$\mathbb{R}$	$\mathbb{R}$	$(-\infty, -\varepsilon], [\varepsilon, \infty)$
3	$\mathbb{R} - \{0\}$	for any ε $(-\infty, -\varepsilon], [\varepsilon, \infty)$	$\longrightarrow$ same.
4 $n \to \infty$	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}$
4 $n \to -\infty$	$\mathbb{R}$	$(-\infty, 0] \cup [\varepsilon, \infty)$	$\longrightarrow$ same.

5	$\mathbb{R}$	$\mathbb{R}$	nowhere,
6	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}$
7	$\mathbb{R}$	$(-\infty, k]$ for any k	$(-\infty, k]$
8	$(-1, 1]$	$[-1+\varepsilon, 1-\varepsilon]$	$\longrightarrow$ same.

much more study later.

210127 | Fn convergence and Cauchy sequence-

Def:
3.45 A sequence (a_n) in $\mathbb{R}$ is <u>1.1-Cauchy</u> if

$\forall \varepsilon > 0 \; \exists N \in \mathbb{Z} \; \forall n, m > N$ have

$$|a_n - a_m| < \varepsilon$$

9.12: A sequence $(f_n(x))$ in $Fun(a,b)$ is uniformly-Cauchy if

$\forall \varepsilon > 0 \ \exists N \in \mathbb{Z} \ \forall n, m > N$ have
$\forall x \in (a,b)$ have $|f_n(x) - f_m(x)| < \varepsilon$

13.49: A sequence $(f_n(x))$ of functions is $\|\cdot\|$-Cauchy if
$\forall \varepsilon > 0 \ \exists N \in \mathbb{Z} \ \forall n, m > N$ have
$\|f_n - f_m\| < \varepsilon$.

Lemma: If $(f_n(x))$ is a sequence in $\mathrm{Fun}(a,b)$ then it converges unif. to $F(x)$ iff it converges to $F(x)$ in $\|\cdot\|_{sup}$-norm.

Proof: If $f_n \longrightarrow F$ uniformly ~~then~~ iff

$\forall \varepsilon>0\ \exists N\in\mathbb{Z}\ \forall n>N$ have $\forall x\in(a,b)$ have $|f_n(x)-F(x)|<\varepsilon$

iff $\forall \varepsilon>0\ \exists N\in\mathbb{Z}\ \forall n>N$ have $\sup_{x\in(a,b)}|f_n(x)-F(x)|<\varepsilon$

iff $\forall \varepsilon>0\ \exists N\in\mathbb{Z}\ \forall n>N$ have $\|f_n(x)-F(x)\|_{sup}<\varepsilon$

iff $f_n \longrightarrow F$ in $\|\cdot\|_{sup}$-norm.

Note: Similarly:

$(f_n(x))$ in $\mathrm{Fun}[a,b]$ is $\|\cdot\|_{\sup}$-norm-Cauchy iff it is uniformly-Cauchy.

Brk. Rm: (1) Find sequences of functions that are uniformly-Cauchy.

(2) Find a seq. which conv. ptwise but is not unif-Cauchy.

Examples:

① $\frac{1}{n}\sin\sqrt{n}\,x$ is Unif Cauchy? ✓

② $\frac{1}{x+\frac{1}{n}}$ not unif. Cauchy? ✓

Proof of Thm 9.13

One direction:

If $(f_n(x))$ converges unif. to $F(x)$.

then $\forall \varepsilon>0 \ \exists N_{\frac{\varepsilon}{2}} \ \forall n>N_{\frac{\varepsilon}{2}}, x$ have $|f_n(x)-F(x)|<\frac{\varepsilon}{2}$

hence $\forall \varepsilon>0 \ \exists M=N_{\frac{\varepsilon}{2}} \ \forall n,m>M, x$

$|f_n(x)-f_m(x)|=|f_n(x)-F(x)+F(x)-f_m(x)|$

$\leq |f_n(x) - F(x)| + |F(x) - f_m(x)| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

other dir!

If $(f_n(x))$ is unif-Cauchy:

then $\forall \varepsilon > 0 \; \exists M_{\frac{\varepsilon}{2}} \; \forall n, m > M_{\frac{\varepsilon}{2}}$ have $|f_n(x) - f_m(x)| < \frac{\varepsilon}{2}$

So $\forall x$ have $(f_n(x))$ is $|\cdot|$-Cauchy in $\mathbb{R}$

so by thm 3.46 converges. to L.

Define $F(x) = L$.

Compare: $(f_n(x)) \xrightarrow{\text{unif.}} F(x)$.

[Finish next time.

210129

Thm 9.13 (Now HW 6.2.5 ed2)

If (f_n) is a segence in $\mathrm{Fun}(a,b)$

then (f_n) is uniformly - Cauchy

iff it is unifmly - convergent,

Equivalently: $\mathrm{Fun}(a,b)$ is $\|\cdot\|_{sup}$-complete

---

Break Out Room:

① Find a bound $\|W\|_{sup} < ?$

define next.

② Find (f_n) conv. ptwise to f, with each $\|f_n\|_{sup} < \infty$ but $\|f\|_{sup} = \infty$.

The Weierstrass function(s).

Def: 9.24:

$$f_n(x) = \sum_{j=1}^{n} 2^{-j} \cos(3^j x)$$

Claim: $f_n(x)$ converges in $\|\cdot\|_{sup}$-norm.

Call the limit. $W(x)$.

Pf: By 9.13 it suffices to check that (f_n) is $\|\cdot\|_{sup}$-Cauchy.

that is we need:

$\forall \varepsilon > 0 \ \exists M_\varepsilon \ \forall n, m \geq M_\varepsilon$ here $\|f_n - f_m\|_{sup} < \varepsilon$.

To do this: choose M_ε with $2^{-M_\varepsilon} < \varepsilon$

and compute:

$$\|f_n - f_m\|_{sup} = \left\| \sum_{j=n+1}^{m} 2^{-j} \cos(3^j x) \right\|_{sup}$$

$$\leq \left| \sum_{j=n+1}^{m} 2^{-j} \right| < \sum_{j=n+1}^{\infty} 2^{-j} = 2^{-n} < \varepsilon$$

qed.

Thm 9.16:

If (f_n) is a sequence in $C^0(a,b)$ conv. unif. to $f \in Fun(a,b)$ then $f \in C^0(a,b)$.

Equivalently: $C^0(a,b)$ is $\|\cdot\|_{sup}$ - complete.

Proof: Note: $f \in C^0(a,b)$ iff $\forall c \in (a,b), \varepsilon > 0 \ \exists \delta > 0 \ \forall |x-c| < \delta$ have $|f(x) - f(c)| < \varepsilon$.

Assume f_n are cts so $\forall c, n, \varepsilon \ \exists \delta_{c,n,\varepsilon/4} \ \forall |x-c| < \delta_{c,n,\varepsilon/4}$ have

$|f_n(x) - f_n(c)| < \varepsilon/4$

also $f_n \xrightarrow{\text{unif.}} f$ so

$\forall \varepsilon \; \exists M_{\varepsilon/4} \; \forall n \geq M_{\varepsilon/4}$ have

$\|f_n - f\|_\infty < \varepsilon/4$

Combining!

$\forall c \in (a,b), \varepsilon > 0 \; \exists \delta_{c,\varepsilon} = \delta_{n,c,\frac{\varepsilon}{4}} \; \forall |x-c| < \delta_{c,\varepsilon}$

have $|f(x) - f(c)| \leq |f(x) - f_{M_{\frac{\varepsilon}{4}}}(x)| + |f_{M_{\frac{\varepsilon}{4}}}(x) - f_{M_{\frac{\varepsilon}{4}}}(c)|$

$$+ \left| f_{M_{\frac{\varepsilon}{4}}}(c) - f(c) \right|$$

$$< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} < \varepsilon .$$

210201

Properties	pointwise	uniform	$\|\cdot\|_{C^1}$-norm
Bounded.	X f_n	9.13	✓
Continuity.	X g_n	9.16	✓
Differentiability	X	X h_n	9.18 (tedious)

$$\underbrace{f_n(x) = \frac{1}{x+\frac{1}{n}} \text{ in } \mathrm{Fun}(0,1)}_{\text{bdd}} \xrightarrow[\text{not unif.}]{\text{ptwise}} \underbrace{f(x) = \frac{1}{x}}_{\text{not bdd.}}$$

$$\underbrace{g_n(x) = x^n \text{ in } \mathrm{Fun}[0,1]}_{\text{cts}} \xrightarrow[\text{not unif}]{\text{ptwise}} \underbrace{g(x) = \begin{cases} 1 & x = 1 \\ 0 & x < 1 \end{cases}}_{\text{not cts.}}$$

$$\underbrace{h_n(x) = \sqrt{x^2 + \tfrac{1}{n}}}_{\text{diff}} \xrightarrow[\text{not in } \|\cdot\|_{C^1}]{\text{unif}} \underbrace{h(x) = |x|}_{\text{not diff.}}$$

③ $f_n(x)$ is differentiable so $f_n - f_m$ is also

so by MVT have

$\underbrace{\forall n, m}_{\text{in } \mathbb{N}}, \underbrace{x, p}_{\text{in } (a,b)}$ $\exists \underbrace{\rho_{n,m,x,p}}_{\text{between } x \text{ and } p}$ with

$$\frac{(f_n - f_m)(x) - (f_n - f_m)(p)}{x - p} = (f_n - f_m)'(\rho)$$

④ Again $f_n(x)$ is differentiable

so $\forall \underbrace{n}_{\text{in } \mathbb{N}}, \underbrace{p}_{\text{in } (a,b)}, \varepsilon > 0 \ \exists \delta_{n,p,\varepsilon} \ \forall |x-p| < \delta$

have $\left| \dfrac{f_n(x) - f_n(p)}{x-p} - f_n'(p) \right| < \dfrac{\varepsilon}{5}$

Use ① → ④ to show: $f' = g$ on

$\forall p \in (a,b), \varepsilon > 0 \ \exists \delta \ \forall |x-p| < \delta$ have

☆ $\left| \dfrac{f(x) - f(p)}{x-p} - g(p) \right| \overset{?}{<} \varepsilon$.

Recall: $W(x)$ = Weierstrass fn.

is a unif. limit of

$$g_n(x) = \sum_{j=0}^{n} 2^{-j} \cos(3^j x)$$

Claim: $W(x) \in D^0(\mathbb{R})$.

Fact (Prove later): Cts fns are all derivatives and $W(x)$ is cts by 9.16

Use 9.18 to show $W(x)$ is a deriv.

using: $f_n(x) = \sum_{j=0}^{n} 2^{-j} \frac{1}{3^j} \sin(3^j x)$

$$= \sum_{j=0}^{n} 6^{-j} \sin(3^j x)$$

so $g_n(x) = f_n'(x)$

by 9.18 $f_n \xrightarrow{\text{unif}} \tilde{w}$ and $\tilde{w}' = w$.

20202 03 Notation: (Ch 13), § 13.2

Def: 13.20: A <u>norm</u> on a $\mathbb{R}$-vector space X is $\|\cdot\| : X \longrightarrow \mathbb{R}_{\geq 0}$ with

1. $\|x\| = 0$ iff $x = 0$
2. $\|kx\| = |k|\,\|x\|$ if $k \in \mathbb{R}$.
3. $\|x+y\| \leq \|x\| + \|y\|$ (triangle ineq),

Ex: $|\cdot|$ is a norm on $\mathbb{R}$.

$\|\cdot\|_{sup}$ is a norm on $Bdd[a,b]$

(but not on $Fun[a,b]$ since:

eg $\| \frac{1}{x} \|_{sup}$ is not defined).

$\| \cdot \|_{C^1}$ is a norm on $C^1[a,b]$.

(where $\| f \|_{C^1} = \| f \|_{sup} + \| f' \|_{sup}$ which is defined since: f' is cts on $[a,b]$ so by a thm in Ch 3 f' has a max & min so $\| f' \|_{sup}$ exists

and f is differentiable so f is cts. and again achieves its max and min so $\| f \|_{sup}$ exists.

so $\| f \|_{C^1} = \| f \|_{sup} + \| f' \|_{sup}$ exists)

Def: 13.45
Prop 13.21
Def: 3.10

A sequence (x_n) in a normed vector space $(X, \|\cdot\|)$ is

ⓐ $\|\cdot\|$-convergent to $x \in X$ if
$\forall \varepsilon > 0 \ \exists N \ \forall n \geq N$ have $\|x_n - x\| < \varepsilon$

ⓑ $\|\cdot\|$-Cauchy if
$\forall \varepsilon > 0 \ \exists M \ \forall n, m \geq M$ have $\|x_n - x_m\| < \varepsilon$

Def: 13.54 A normed vector space $(X, \|\cdot\|)$ is a <u>Banach space</u> if every $\|\cdot\|$-Cauchy sequence is $\|\cdot\|$-convergent.

Thm: $(D^0[a,b] \cap Bdd[a,b], \|\cdot\|_{sup})$ is a Banach space.

Thm: $(C^1[a,b], \|\cdot\|_{C^1})$ is a Banach space,

Example: Construct a function $u(x) \in D^0[0,1] \cap B[0,1]$ as a limit of

$$g_n(x) = \sum_{j=0}^{n} 4^{-j} \left((x-a_j)^2 \cos \frac{1}{x-a_j} \right)'$$

and check: $U(x)$ is discts at every a_j.

Choose $a_1 = \frac{1}{2}$ $\quad a_4 = \frac{1}{4}$

$a_2 = \frac{1}{3}$ $\quad a_5 = \frac{3}{4}$

$a_3 = \frac{2}{3}$ $\quad \ddots$

a_j runs through all rationals in $(0,1)$.

Check: U is dicont. at $\frac{1}{2} = a_1$

210205 Midterm next week;

see 2020 exams on web page.

Example for Banach

Then power series:

Recall: $(C^0[0,1], \|\cdot\|_{sup})$ is a Banach space.

Notation: $\|\cdot\|_{sup} = \|\cdot\|_{unif} = \|\cdot\|_{C^0} = \|\cdot\|_\infty$

Thm: $(C^2[0,1], \|\cdot\|_{C^2})$ is a Banach space.

if $\|f\|_{C^2} = \|f\|_{sup} + \|f'\|_{sup} + \frac{1}{2}\|f''\|_{sup}$

Fun note (Banach algebras):

Note: If $f, g \in C^0[0,1]$
then $\|fg\|_{sup} \leq \|f\|_{sup} \|g\|_{sup}$

If $f \in C^1[0,1]$
then $\|fg\|_{C^1} = \|fg\|_{sup} + \|f'g + fg'\|_{sup}$

$$\leq \|f\|_{sup}\|g\|_{sup} + \|f'\|_{sup}\|g\|_{sup} + \|f\|_{sup}\|g'\|_{sup}$$

using above and triangle ineq.

$$\leq (\|f\|_{sup} + \|f'\|_{sup})(\|g\|_{sup} + \|g'\|_{sup})$$

$= \|f\|_{C^1} \|g\|_{C^1}$

Claim: If $f, g \in C^2[0,1]$

A then $\|fg\|_{C^2} \leq \|f\|_{C^2} \|g\|_{C^2}$

Also if $\|f\|_{\tilde{C}^2} = \|f\|_{\sup} + \|f'\|_{\sup} + \frac{1}{2} \|f''\|_{\sup}$

B then if $f = g = x$ then $\|fg\|_{\tilde{C}^2} \not\leq \|f\|_{\tilde{C}^2} \|g\|_{\tilde{C}^2}$

Break
Oat
Rm
⋮
check

B If $f = x$

$\|x\|_{C^2} = \|x\|_{\sup} + \|1\|_{\sup} + \frac{1}{2}\|0\|_{\sup} = 1+1$

$\|x\|_{\tilde{C}^2} = \|x\|_{\sup} + \|1\|_{\sup} + \|0\|_{\sup} = 1+1$

$$\|x^2\|_{C^2} = \|x^2\|_{\sup} + \|2x\|_{\sup} + \tfrac{1}{2}\|2\|_{\sup} = 4$$

$$\|x^2\|_{\tilde{C}^2} = \quad '' \qquad '' \quad + \|2\|_{\sup} = 5$$

so
$$\|x^2\|_{C^2} = \|x\|_{C^2}^2$$
$$\|x^2\|_{\tilde{C}^2} > \|x\|_{C^2}^2$$

Notation:

A power series about c with coefficients a_n is written

$$\underbrace{\sum_{n=0}^{\infty} a_n (x-c)^n}_{\text{means } \lim_{N\to\infty} \sum_{n=0}^{N}}$$

Examples: (A) $\sum_{n=0}^{\infty} x^n$ related to $\frac{1}{1-x}$

(B) $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ related to e^x

210208

Midterm Friday:

Old exams on web page,

1 sheet notes,

Covering through today (Ch 10).

In regular zoom session.

Use cameras.

Instruction sheet on webpage

by tomorrow.

Computing Radii of convergence:

Recall from Ch 4.

4.19 (Comparison) $\left|\sum \frac{\sin(n)}{2^n}\right| \leq \sum \frac{1}{2^n} = \frac{1}{1-\frac{1}{2}} = 2$.

4.24 (Ratio) $\sum a_n$ converges if

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$$

(Integral) If $f(x)$ is inc. and $\int_1^\infty f(x)\,dx < \infty$ then $\sum_{n=1}^{\infty} f(n)$ converges.

eg $\sum_{n=1}^{\infty} \frac{1}{n^2}$ conv. since $\int_1^\infty \frac{dx}{x^2} = \frac{-2}{x}\Big|_1^\infty = 2 < \infty$.

Example power series: Find R.

① $\sum_{n=0}^{\infty} x^n \overset{\text{ptwise}}{=\!=} \frac{1}{1-x}$ if $x \in (-1,1) = (-R,R)$

by Tay/Lag

② $\sum \frac{x^n}{n!} = e^x$ if $x \in \mathbb{R}$ $(R=\infty)$

③ $\sum n!\, x^n = 1$ if $x=0$ and undef else. $(R=0)$

④ $\sum (n+1) x^n \overset{\text{der.}}{=\!=} \frac{1}{(1-x)^2}$ if $x \in (-1,1)$, $R=1$

$\overset{\text{of ①}}{}$

⑤ $\sum \frac{1}{n} x^n \overset{\text{antider.}}{=} -\ln(1-x)$ if $x \in (-1,1)$ $R=1$

⑥ $\sum \left(\frac{x}{2}\right)^n = \frac{1}{1-\frac{x}{2}} = \frac{2}{2-x}$ if $x \in (-2,2)$, $R=2$

⑦ $\sum \frac{1}{2}\left(\frac{x-1}{2}\right)^n = \frac{1}{2\left(1-\frac{x-1}{2}\right)} = \frac{1}{1-x}$ $\underbrace{\text{if } x \in (-1, 3)}_{\text{center at } c=1}$ $R=2$

⑧ $\sum 2^n n^2 x^n$

$R = \frac{1}{2}$

⑨ $\sum 2^n \sin(n) x^n$

$\underbrace{R \geq \frac{1}{2}}_{\text{comparison + ratio test}}$

210210 Will post practice answers.

More power series:

Thm 10.22

If $S(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$ is a limit of a power series with radius of conv. $R_S > 0$

then $S'(x) = \sum_{n=1}^{\infty} a_n n (x-c)^{n-1} = \sum_{n=0}^{\infty} a_{n+1} (n+1) (x-c)^n$ and $R_{S'} = R_S$.

Proof: By thm 10.3

$\left(S_N(x) = \sum_{n=0}^{N} a_n (x-c)^n \right) \xrightarrow{\text{unif}} S_N(x)$

in $[c-r, c+r]$ for any $r < R_S$.

and compute (will skip)!

ch4.

if $\sum a_n(x-c)^n$ converges and $|y-c|<|x-c|$

then $\sum a_n n (y-c)^{n-1}$ converges.

hence $\sum a_n n (x-c)^{n-1}$ conv. unif. by 10.3

to some fn $T(x)$.

in $[c-r, c+r]$.

So $(S_N) \xrightarrow{\text{ptwise}} S$ and $(S_N') \xrightarrow{\text{unif}} T$

in $[c-r, c+r]$

So by 9.18 $S'(x) = T(x)$.

Cor: Functions which are limits of power series are completely determined by any neighborhood of c.

This will show many functions are not limits of power series.

Thm 10.25: If $f(0)=0$ and $f'(x) = f(x)$.

then $f(x) = Ce^x$

Proof: $e^x = \sum \frac{x^n}{n!}$

Compute $(e^x)' = \sum \frac{n\, x^{n-1}}{n!} = \sum \frac{x^{n-1}}{(n-1)!} = \sum \frac{x^n}{n!}$

$= e^x$

Note $e^x \neq 0$.

Compute: $\left(\frac{f(x)}{e^x}\right)' = \frac{f'(x)e^x - f(x)(e^x)'}{(e^x)^2}$

$= \frac{0}{e^{2x}} = 0$

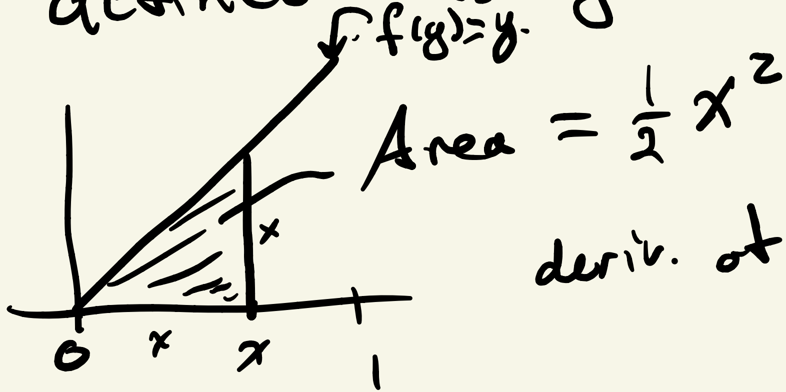
So by Thm 8.34 have $\frac{f(x)}{e^x} = C$ a const.

so $f(x) = Ce^x$.

started with derivative:
defined as a limit of difference quotient
or geometrically slope.

Next! Inverse of derivative:
Integral.
defined using area:

Eg!



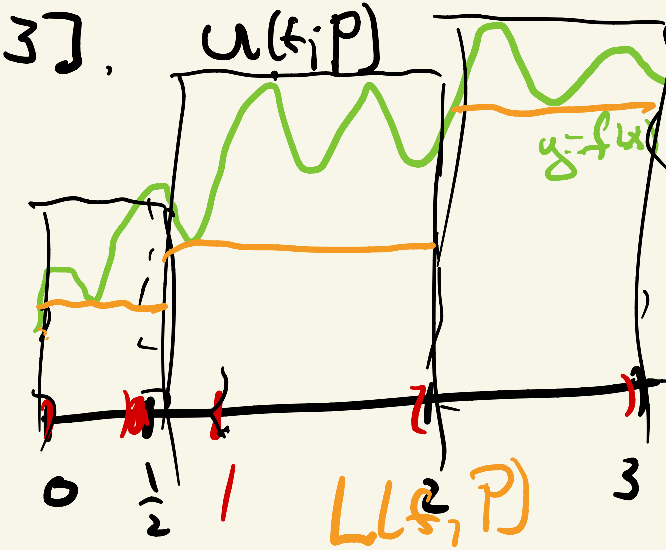
deriv. of Area = original fn.

Explore next: which functions can be integrated.
(not looking for a formulae).

Look at §11.1 for basic properties of $\sup_{a \in I} f(a)$ and $\inf_{a \in I} f(a)$

Example: Partitions of $[0,3]$.

$$P = \{\underset{x_0}{\overset{\shortparallel}{0}},\ \underset{x_1}{\overset{\shortparallel}{\tfrac{1}{2}}},\ \underset{x_2}{\overset{\shortparallel}{2}},\ \underset{x_3}{\overset{\shortparallel}{3}}\}$$



$I_1 = [0, \tfrac{1}{2}]$ $\qquad \ell_1 = \tfrac{1}{2}$

$I_2 = [\tfrac{1}{2}, 2]$ $\qquad \ell_2 = \tfrac{3}{2}$

$I_3 = [2, 3]$ $\qquad \ell_3 = 1$

$$Q = \{0, \tfrac{1}{2}, 1, 2, 3\} \supseteq P$$

Q is a refinement of P.

Def: If $f \in Bdd[a,b]$

then (a) $U(f) = \inf_{P \in \Pi[a,b]} U(f;P) = \overline{\int_{x=a}^{b}} f(x)\,dx$

(b) $L(f) = \sup_{P \in \Pi[a,b]} L(f;P) = \underline{\int_{x=a}^{b}} f(x)\,dx$

Def: If $f \in Bdd[a,b]$ and $U(f) = L(f)$

then f is call <u>(Riemann)</u> integrable.

and write $f \in RInt[a,b]$

and $U(f) = L(f) = \overline{\int} f = \underline{\int} f = \int_{x=a}^{b} f(x)\,dx$

Break out Rooms:

Consider $[a,b] = [0,1]$

$$f(x) = x$$

$$P_n = \{0, \tfrac{1}{n}, \tfrac{2}{n}, \dots, \tfrac{n-1}{n}, 1\}$$

Find:

ⓐ $I_i = \left[\frac{i-1}{n}, \frac{i}{n}\right]$

ⓑ $l_i = \frac{1}{n}$

ⓒ $U(f, P_n) = \sum_{i=1}^{n} \frac{1}{n} f\left(\frac{i}{n}\right) = \sum_{i=1}^{n} \frac{1}{n} \frac{i}{n} = \frac{1}{n^2} \sum_{i=1}^{n} i$

ⓓ $L(f, P_n) = \sum_{i=1}^{n} \frac{1}{n} f\left(\frac{i-1}{n}\right) = \sum_{i=1}^{n} \frac{1}{n} \frac{(i-1)}{n}$

$= \frac{1}{n^2} \sum_{i=0}^{n-1} i = \frac{1}{2} - \frac{1}{2n}$

ⓔ $u(f) = \frac{1}{2}$

$\frac{1}{2} + \frac{1}{2n}$

$\shortparallel$

$\frac{1}{n^2} \frac{n(n+1)}{2}$

$\shortparallel$

n is a multiple of m

(f) For which n & m is P_n a refinement of P_m?

Helpful fact: $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

$$\sum_{i=1}^{3} i = 1+2+3 = 6$$

Thm 11.20: If $f \in \text{Bdd}[a,b]$ and $P \subseteq Q \in \Pi[a,b]$ then

(a) $U(f;Q) \leq U(f;P)$

(b) $L(f;Q) \geq L(f;P)$.

Thm 11.21: If $f \in \text{Bdd}[a,b]$ and $P, Q \in \Pi[a,b]$ are any two partitions.

then $L(f;P) \leq U(f;Q)$

Prop 11.22: If $f \in \text{Bdd}[a,b]$ then $L(f) \leq U(f)$.

Proof of 11.21 using 11.20.

Consider $R = P \cup Q$

which is a refinement of both P and Q.

hence $L(f; P) \underset{11.20}{\leq} L(f; R) \underset{\text{def}}{\leq} U(f; R) \underset{11.20}{\leq} U(f; Q)$

Proof of 11.22 using 11.21. Next time.

210219

11.28

Thm: $C^0[a,b] \subseteq R\,Int[a,b]$

(A step toward the fact that $C^0[a,b] \subseteq D^0[a,b]$)

Thm: (Next week)

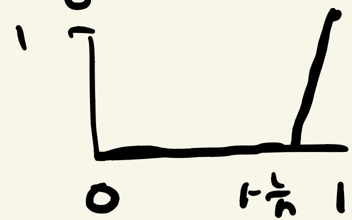
If $\|f\|_{L^1} = \int_a^b |f(x)|\,dx$

then $(C^0[a,b], \|\cdot\|_{L^1})$ is a normed linear space but not Banach:

Recall: This proof requires:

- If $f \in C^0[a,b]$ and $\int |f|dx = 0$ then $f=0$.
- $\int |cf|dx = |c| \int |f| dx$
- $\int |f+g| dx \le \int |f| dx + \int |g| dx$
- Find (f_n) which is $\|\cdot\|_{L^1}$-Cauchy but

not $\|\cdot\|_{L^1}$-convergent.

eg $f_n(x) =$ 

Approach to proving 11.28:

Cauchy criteria:

Def: $R\mathrm{Int}[a,b] \overset{\text{def}}{=\!=} \{f \in \mathrm{Bdd}[a,b] \mid U(f) = L(f)\}$.

$\overset{11.23}{=\!=} \{f \in \mathrm{Bdd}[a,b] \mid \forall \varepsilon > 0\ \exists P_\varepsilon \in \Pi[a,b] \text{ with } U(f;P) - L(f;P) < \varepsilon\}$

$\overset{11.26}{=\!=} \{f \in \mathrm{Bdd}[a,b] \mid \exists (Q_n)_{n \in \mathbb{N}},\ Q_n \in \Pi[a,b] \text{ with } \lim_{n\to\infty}[U(f;Q_n) - L(f;Q_n)] = 0\}$

Proof of 11.28 using 11.23

Recall Thm 7.42: If $f \in C^0[a,b]$ then f is <u>uniformly</u> continuous.

cts: $\forall \varepsilon > 0,\ x \in [a,b]\ \exists \delta_{\varepsilon, x} > 0\ \forall\ |y-x| < \delta$ have $|f(y)-f(x)| < \varepsilon$

unif. cts: $\forall \varepsilon > 0\ \exists \delta_\varepsilon > 0\ \forall x \in [a,b]$, " " .

Ass. f is unif. cts in $[a,b]$ and $\forall \varepsilon > 0$ ~~and~~ take $m > \dfrac{b-a}{\delta_{\frac{\varepsilon}{b-a}}}$ and

compute: $U(f; R_m) - L(f; R_m) < \varepsilon$.

Computing: For $R_m = \cancel{\{0, \frac{1}{m}, \frac{2}{m}, \ldots, 1\}}$
$\{a, a+\frac{b-a}{m}, a+2\frac{b-a}{m}, \ldots, b\}$

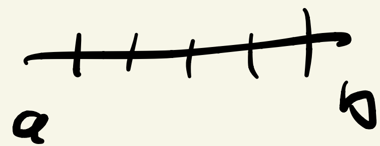
each I_i has $l_i =$

$$\frac{b-a}{m} < \delta_{\frac{\varepsilon}{b-a}}$$

so $\forall\, x,y \in I_i$ have $|x-y| < \delta_{\frac{\varepsilon}{b-a}}$

so $|f(x)-f(y)| < \frac{\varepsilon}{b-a}$

so $\sup\limits_{x\in I_i} f(x) - \inf\limits_{y\in I_i} f(y) < \frac{\varepsilon}{b-a}$



m equal part.

so $U(f;R_m) - L(f;R_m)$

$$= \sum_{i=1}^{m} l_i \left(\sup_{x\in I_i} f(x) - \inf_{y\in I_i} f(y)\right) < m \cdot \frac{b-a}{m} \cdot \frac{\varepsilon}{b-a} = \varepsilon.$$

210222) Algebraic Properties of integrals

$$C[0,1] \subseteq \underbrace{RInt[0,1]}_{\text{vector space}} \xrightarrow[\int_0^1 \cdot\, dx]{} \mathbb{R}$$

linear

monotone

Recall: Definitions:

If $f \in Bdd[0,1]$

and $P \in \pi[0,1]$

$$U(f;P) = \sum_{i=1}^{n} l_i \sup_{x \in I_i} f(x)$$

$$L(f;P) = \text{" " "} \inf \text{" "}$$

$$U(f) = \inf_P U(f;P) = \overline{\int} f\,dx$$

$$L(f) = \sup_P L(f;P) = \underline{\int} f\,dx$$

so

$$L(f;P) \le L(f) < L(f;P) + \varepsilon$$

$\forall P$ $\qquad$ $\forall \varepsilon > 0 \ \exists P$

$$U(f;P) \ge U(f) > U(f;P) - \varepsilon$$

$\bigstar$

Prop 11.21: If $P, Q \in \Pi[a,b]$, $f \in Bdd[a,b]$ then $U(f;P) \geq L(f;Q)$

Proof: $U(f;P) \underset{\substack{11.20 \\ (a)}}{\geq} U(f;P\cup Q) \geq L(f;P\cup Q) \underset{\substack{11.20 \\ (b)}}{\geq} L(f;Q)$

Prop 11.22 If $f \in Bdd[a,b]$

then $U(f) \geq L(f)$.

Proof: Assume for contradiction $\varepsilon > 0$.

$$L(f) = U(f) + 2\varepsilon$$

Compute: $L(f) \underset{\substack{\bigstar \\ \exists P}}{<} L(f;P) + \varepsilon \underset{\substack{11.21 \\ \forall P,Q}}{\leq} U(f;Q) + \varepsilon \underset{\substack{\bigstar \\ \exists Q}}{<} U(f) + 2\varepsilon \overset{L(f)}{\underset{}{=}}$

a contradiction,

Thm 11.33: If $f, g \in RInt[a,b]$

then $f+g \in RInt[a,b]$

and $\int_0^1 (f(x)+g(x))dx = \int_0^1 f(x)dx + \int_0^1 g(x)dx$

Proof: Compute:

$$\overline{\int_0^1}(f+g)dx = \inf_P U(f+g;P) = \inf_P \sum_{i=1}^n l_i \sup_{x\in I_i}(f(x)+g(x))$$

$$\leq \inf_P \sum_{i=1}^n l_i \Big[\sup_{x\in I_i} f(x) + \sup_{y\in I_i} g(y)\Big]$$

$$= \overline{\int_0^1} f(x)dx + \overline{\int_0^1} g(y)dy \overset{f,g \in RInt}{=\!=} \underline{\int_0^1} f(x)dx + \underline{\int_0^1} g(y)dy$$

Similarly $\underline{\int_0^1}(f+g)\,dx \geq \checkmark$

so $U(f+g) \leq L(f+g) \underset{11.22}{\leq} U(f+g)$

so $U(f+g) = L(f+g)$ and $f+g \in RInt[a,b]$

and both inequalities above are equalities.

so $\int_0^1 (f+g)\,dx = \int_0^1 f\,dx + \int_0^1 g\,dx$.

210224) More about the space $RInt[a,b]$ of Riemann-integrable functions.

Already found

$RInt[a,b]$ is a vector space 11.32 / 11.33

$C^0[a,b] \subseteq RInt[a,b]$ 11.28.

$Inc[a,b] \subseteq$ " " 11.30

$f(x) = \begin{Bmatrix} 1 & x = \frac{1}{2} \\ 0 & \text{else} \end{Bmatrix} \subseteq RInt[0,1]$ not listed above.

$\div$ $\left(\int_0^1 f(x)\,dx = 0 \right)$

Def: If $f \in RInt[a,b]$ then

$$\|f\|_{L^1} = \int_a^b |f(x)|\,dx$$

Recall: $(X, \|\cdot\|)$ is a normed lin sp (Def 13.20).

if X is a $\mathbb{R}$- vect. sp.

zero : If $\|f\|=0$ then $f=0$. fails but holds for $C^0[a,b]$

scal : If $c \in \mathbb{R}$ then $\|cf\| = |c|\,\|f\|$. ✓ holds

triangle : If $f, g \in X$ then $\|f+g\| \leq \|f\| + \|g\|$. ✓ holds

Note: $(RInt[a,b], \|\cdot\|_{L^1})$ is not a normed lin sp.

since eg: $f(x) = \begin{cases} 1 & x = \frac{1}{2} \\ 0 & \text{else} \end{cases} \in RInt[0,1]$ above.

has $\|f\|_{L^1} = 0$ but $f \neq 0$.

so zero property does not hold

Find more integrable functions.

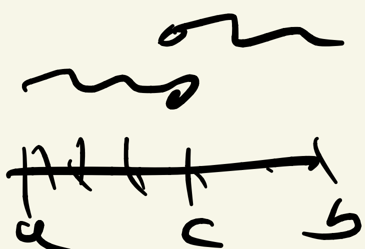
Already know:

$RInt[a,b]$

- is a vector space
- contains $C^0[a,b]$
- contains $Inc[a,b]$ and $Dec[a,b]$
- contains
- contains

Claim 11.44: A bounded fn $f \in Bdd[a,b]$ is R. Integrable iff f (for $c \in [a,b]$) $f|_{[a,c]} \in RInt[a,c]$ and $f|_{[c,b]} \in RInt[c,b]$.

(and $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$)



Proof: $\Leftarrow$: Assume $f|_{[a,c]}$ & $f|_{[c,b]}$ are Integ.

and show: $L(f) = U(f) = \int_a^c f\,dx + \int_c^b f\,dx$

Compute: for every $\varepsilon > 0$

$$L(f|_{[a,c]}) \leq L(f|_{[a,c]}; Q_1) + \varepsilon$$

$$L(f|_{[c,b]}) \leq L(f|_{[c,b]}; Q_2) + \varepsilon$$

so $L(f) \geq L(f; Q_1 \cup Q_2) = L(f|_{[a,c]}; Q_1) + L(f|_{[c,b]}; Q_2)$

$$\geq \int f|_{[a,c]}\,dx + \int f|_{[c,b]}\,dx - 2\varepsilon$$

Similarly $U(f) \leq$ " " $+ 2\varepsilon$

so $U(f) - L(f) \le 4\varepsilon$ so $U(f) = L(f)$

$$= \int f|_{[a,c]}\,dx + \int f_{[c,b]}\,dx.$$

$\Longrightarrow$ is similar

210226

More R. Integrable functions. And Fund. Thm. of Calc.

Know: 11.28 $C^0[a,b] \subseteq R\,Int[a,b]$

11.30 $Inc[a,b] \subseteq R\,Int[a,b]$

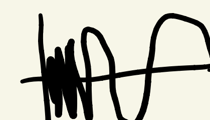
$$\chi_{\mathbb{Q}}(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & \text{else} \end{cases} \notin R\,Int[a,b].$$

Thomae fn: $T(x) = \begin{cases} \frac{1}{n} & \text{if } x \ \frac{a}{n} \\ 0 & \text{else} \end{cases} \in R\,Int[a,b]$

$\mathrm{disc}(f) = \{x \mid f \text{ is not continuous at } x\}.$

$\mathrm{disc}(\chi_{\mathbb{Q}}) = [0,1]$

$\mathrm{disc}(T) = \mathbb{Q} \cap [0,1]$



Question: Is $f(x) = \{\sin(\frac{1}{x}) \quad x \neq 0 \}$

$$\left.\begin{matrix} 0 & x=0 \end{matrix}\right\}$$

$$\operatorname{disc}(f) = \{0\}$$

in $RInt[0,1]$?

Is $$g(x) = \begin{cases} \sin\left(\dfrac{1}{\sin\frac{1}{x}}\right) & \text{if } \sin\frac{1}{x} \neq 0 \text{ and } x \neq 0 \\ 0 & \text{else.} \end{cases}$$

$$\operatorname{disc}(g) = \{x \mid f(x) = 0\}$$

$$= \left\{\frac{1}{n\pi}\right\} \cup \{0\}.$$

in $RInt[0,1]$?

Cor 11.53: If $f \in Bdd[a,b]$ and $\operatorname{disc}(f)$ is a finite set then $f \in RInt[a,b]$.

Skip Proof: (Uses 11.50 and 11.44).

Br. O Rm: Show $g(x) \in RInt[0,1]$.

using 11.50 and 11.53

Proof: Use thm: 11.50:

Consider $g|_{[\varepsilon,1]}$ but $\mathrm{disc}(g|_{[\varepsilon,1]}) = \{\frac{1}{n\pi} \mid \frac{1}{n\pi} > \varepsilon\}$

$= \{\frac{1}{n\pi} \mid n \leq \frac{1}{\pi\varepsilon}\}$.

which is finite.

Hence by Cor 11.53 $g|_{[\varepsilon,1]} \in RInt[\varepsilon,1]$

Therefore by Thm 11.50 $g \in RInt[0,1]$.

Thm 11.61: There is a collection Z of subsets of $[a,b]$ so that a bounded function f is in $RInt[a,b]$ iff

$\text{disc}(f) \in Z$.

Call: elements of Z <u>sets of measure 0</u>.

Thm 12.6: If $f \in RInt[a,b]$ then $Jf \in C^0[a,b]$.

Proof: Check $\lim_{x \to c^+} Jf(x) \overset{?}{=} Jf(c)$

$$\lim_{x \to c^-} Jf(x) \stackrel{?}{=} Jf(c)$$

Compute: $\displaystyle \lim_{x \to c^+} Jf(x) = \lim_{x \to c^+} \int_a^x f(y)\,dy$

so $\displaystyle \lim_{x \to c^+} Jf(x) - Jf(c)$

$$= \lim_{x \to c^+} \left(\int_a^x f - \int_a^c f \right) = \lim_{x \to c^+} \int_c^x f(y)\,dy.$$

$$\leq \lim_{x \to c^+} (x-c)\,\|f\|_{\sup} = 0$$

210361 Recall:

$$\begin{tikzcd}
C^0[a,b] \arrow[d, phantom, "\cap" description] & & & \{f\in C^1[a,b] \mid f(a)=0\} \arrow[d, phantom, "\text{g.35 incl} \; \shortparallel"] \\
R\,Int[a,b] \arrow[r, "12.6", "J"'] & C^0[a,b] \arrow[d, phantom, "\cap\; 12.4"] & & C^1_0[a,b] \arrow[ll, "D", "\text{bijection}"'] \arrow[d, phantom, "\cap"] \\
& D^0[a,b] & & D^1_0[a,b] \arrow[ll, "D", "\text{onto by dcf.}"']
\end{tikzcd}$$

$$(Jf)(x) = \int_a^x f(y)\,dy$$

so $\quad Jf(a) = 0$

Cor of 12.4: $C^0[a,b] \subseteq D^0[a,b]$.

Proof: If $f \in C^0[a,b]$ then $f = F'$

where $F(x) = J_a^x f(x) = \int_a^x f(y)\,dy$.

hence f is a derivative and in $D^0[a,b]$.

Cor of 12.4: $D: C_0^1[a,b] \longrightarrow C^0[a,b]$

is a bijection

Pf: From 8.35 if $Df = Dg$ then

$f = g + C$ but if $f(a) = g(a) = 0$

and $f(a) = g(a) + C$

then $C = 0$ so $f = g$ and D is injective.

By 12.4 D is onto since $DJf = f$,

210303

Product rule 8.19 $\Longrightarrow$ Integration by parts 12.10
for deriv. used later (next class)
for weak derivatives.

$\downarrow$

Chain rule 8.21 $\Longrightarrow$ Integ. by substitution.
12.12

Recall:

Thm 12.1: If f is diff in (a,b)
cts in $[a,b]$
and f' is R. Int. in $[a,b]$

then $\int_a^b f'(t)dt = f(b)-f(a)$.

Thm 12.10:

If $f, g \in D^1(a,b) \cap C^0[a,b]$
and $f', g' \in RInt[a,b]$ hyp for 12.1

then $\int_a^b f'(t)g(t)\,dt = -\int_a^b f(t)g'(t)\,dt + \left[f(b)g(b) - f(a)g(a)\right].$

Proof: Recall: $(fg)' = f'g + fg'$ (8.19)

Check fg satisfies hyp. for 12.1

$fg \in D^1$ by 8.19, $fg \in C^0$ from Ch 7.

$f'g \in RI$ since: f' and g are R.Int. by hyp.
since g is cts (11.28).

and by HW 7.4.6
get $f'g$ R.Int.

similarly fg' is R.Int.

and R.Int $[a,b]$ is a vector space so
$f'g + fg'$ is R.Int.

Apply 12.1: $\int_a^b (fg)' dt = \int_a^b (f'g + fg') dt$

$$= \int_a^b f'g \, dt + \int_a^b fg' \, dt$$

$f(b)g(b) - f(a)g(a)$

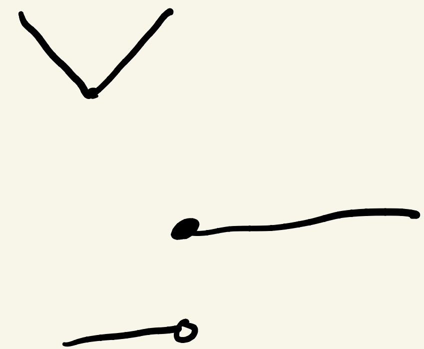
so $\int_a^b f'(t)g(t) \, dt = -\int_a^b f(t)\, g'(t) \, dt + [f(b)g(b) - f(a)g(a)]$

$fg\Big|_a^b$

Try:

Find a weak der. for (a) $f(x) = |x|$

or (b) $h(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$



Only one exists.

Ans:

For g as above.

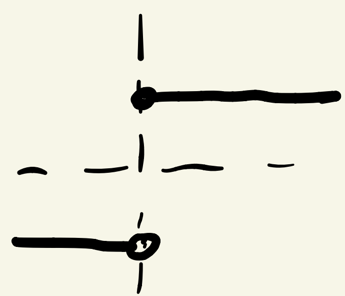
(a) $\int_{-1}^{1} |t|\, g'(t)\,dt = -\int_{-1}^{0} t\, g'(t)\,dt + \int_{0}^{1} t\, g'(t)\,dt$

$\overset{\text{parts}}{=} + \int_{-1}^{0} 1\cdot g(t)\,dt - \int_{0}^{1} 1\cdot g(t)\,dt$

$= -\int_{-1}^{1} k(t)\, g(t)\,dt \quad \text{if} \quad k(t) = \begin{cases} 1 & x>0 \\ -1 & x<0 \end{cases}.$

(b) $\int_{-1}^{1} h(t)\, g'(t)\,dt = \int_{0}^{1} g'(t)\,dt = g(1) - g(0)$

$= -g(0)$

(a): $k(t)$  is a wk deriv. of $|t|$.

Thm 12.12: If g is diff in (a,b)
cts in $[a,b]$
and g' is R. Int in $[a,b]$
and f is cts in Image(g).

then
$$\left.\begin{aligned}\int_a^b f(g(t))\,g'(t)\,dt \\ = \int_{g(a)}^{g(b)} f(y)\,dy\end{aligned}\right]$$
substitute $y = g(t)$.

Proof: Similar to 12.10.

Integrals and sequences of functions, 12.3 and 12.4 (add more fns to integrate),

Recall: A sequence of R. Int. fns. f_n in $[a,b]$ is L^1-Cauchy if

$\forall \varepsilon > 0 \ \exists M \ \forall n, m \geq M$ have

$$\|f_n - f_m\|_{L^1} = \int_a^b |f_n(t) - f_m(t)| \, dt < \varepsilon$$

and L^1-convergent to $f \in RInt[a,b]$

if $\forall \varepsilon > 0 \ \exists N \ \forall n \geq N$ have

$$\|f_n - f\|_{L^1} = \int_a^b |f_n(t) - f(t)|\,dt < \varepsilon.$$

Note: If (f_n) L^1-converges to f

$$\left((f_n) \xrightarrow{L^1} f \right)$$

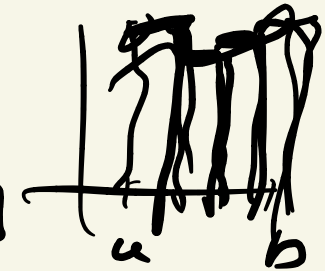
then $\lim_{n\to\infty} \int_a^b f_n(t)\,dt = \int_a^b f(t)\,dt$

210305)

Example: If $f \in RInt[a,b]$

and $P \in \pi[a,b]$

write $f_{P,u}(x) = \begin{cases} \sup_{x \in I_i} f(x) & \text{if } x \in (x_{i-1}, x_i] \\ f(a) & \text{if } x = a \end{cases}$



Note: $\int_a^b f_{P,u}(x) = \sum l_i \sup_{x \in I_i} f(x) = U(f;P)$

and $f_{P,u}(x) \geq f(x)$

so $\| f_{P,u} - f \|_{L^1} = \int | f_{P,u}(x) - f(x) | dx$

$$= \int f_{P,u}(x)\,dx - \int f(x)\,dx$$

$$= U(f;P) - U(f)$$

Hence: If (P_n) is a sequence in $\Pi[a,b]$ with $\lim_{n\to\infty} U(f;P_n) = U(f)$

then $(f_{P_n,u})$ is L^1 convergent to f.

Similarly for $f_{P,L}$.

Thm 12.17:

If (f_n) conv. unif to f
then (f_n) conv. in L^1 to f.

(Also if (f_n) is unif-Cauchy
then (f_n) is L^1-Cauchy).

Proof: If $(f_n) \xrightarrow{\text{unif}} f \in \text{Bdd}[a,b]$ then

$\forall \varepsilon > 0 \ \exists N \ \forall n \geq N$ have $\| f_n - f \|_{\sup} < \dfrac{\varepsilon}{b-a}$

so $\forall \varepsilon > 0 \ \exists N \ \forall n \geq N$ have

$$\| f_n - f \|_{L^1} = \int_a^b | f_n(x) - f(x) | \, dx \leq U(f_n - f; \{a,b\})$$

$$= (b-a) \sup |f_n - f| = (b-a) \|f_n - f\|_{\sup}$$

$$< \varepsilon$$

Cauchy proof is similar.

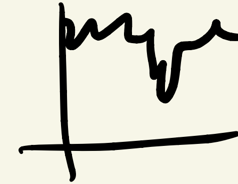
Recall:

(9.24) (Weierstrass fn):

$$W(x) = \sum_{n=0}^{\infty} 2^{n} \cos(3^n x)$$

converges uniformly to

a cts, nowhere diff. fn



Hence $\displaystyle\int_0^{\frac{\pi}{3}} W(x)\,dx = \sum_{n=0}^{\infty} 2^{-n} \int_0^{\frac{\pi}{3}} \cos(3^n x)\,dx$

since the sum conv unif and hence in L^1

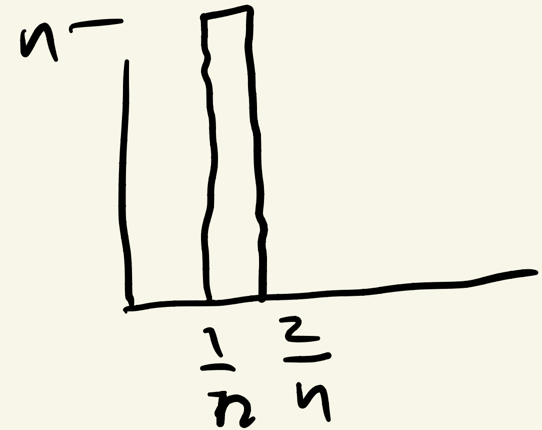
$$= \sum_{n=0}^{\infty} 2^{-n} \frac{1}{3^n} \sin(3^n x)\Big|_0^{\frac{\pi}{3}}$$

$$= 2^0 \frac{1}{3^0}\left[\sin\left(\tfrac{\pi}{3}\right) - \sin(0)\right] + \sum_{n=1}^{\infty} 0 = \sin\frac{\pi}{3}$$

$$= \frac{\sqrt{3}}{2}.$$

Ex: Pointwise conv. to 0 but not L^1-conv:

$$f_n(x) = \begin{cases} n & \frac{1}{n} \le x \le \frac{2}{n} \\ 0 & \text{else} \end{cases} \in \mathrm{Bdd}[0,1]$$



This is not L^1-Cauchy.
so not L^1 convergent to any function.

since:

L^1-Cauchy means:

$\forall \varepsilon \ \exists M \ \forall n,m \geq M$ have $\|f_n - f_m\|_{L^1} < \varepsilon$

so not L^1-Cauchy means

$\exists \varepsilon \ \forall M \ \exists n,m \geq M$ with $\|f_n - f_m\|_{L^1} \geq \varepsilon$

← I get to choose →

Choose $\varepsilon = 1$ for any M ~~take~~ $n = 2M$, $m = M$

and compute $\|f_n - f_m\|_{L^1} = \int_0^1 |f_n(x) - f_m(x)|\,dx$

$$= \int_0^1 \begin{Bmatrix} 2M & \frac{1}{2M} \leq x \leq \frac{2}{2M} \\ M & \frac{1}{M} \leq x \leq \frac{2}{M} \\ 0 & \text{else} \end{Bmatrix} dx = 2 > 1 = \varepsilon$$

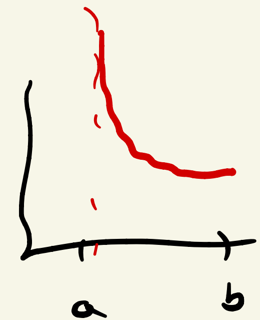
210308 Integrals of non-integrable functions.

Improper integrals:

Def 12.24

If $f \in \mathrm{Fun}(a, b]$ and $\forall \varepsilon > 0$ have $f \in \mathrm{RInt}[a+\varepsilon, b]$

then write $\int_a^b f(x)dx = \lim_{\varepsilon \to 0^+} \int_{a+\varepsilon}^b f(x)dx$



and call this the <u>improper integral</u> of f on $[a,b]$.

(This limit may or may not exist.)

Def: 12.26:

If $f \in \mathrm{Fun}[a, \infty)$ and $\forall n$

have $f \in \mathrm{RInt}[a, n]$

then write $\int_a^\infty f(x)dx = \lim_{n \to \infty} \int_a^n f(x)dx$

and call this the <u>improper int</u>

of f on $[a, \infty)$.

Def: 12.32: If $f \in \text{Fun}(a,b]$ and $\forall \varepsilon > 0$ have $f \in \text{RInt}[a+\varepsilon, b]$

and $\lim_{\varepsilon \to 0^+} \int_{a+\varepsilon}^{b} |f(x)|\,dx = \lim_{\varepsilon \to 0^+} \| f|_{[a+\varepsilon, b]} \|_{L^1}$

exists then call f

<u>absolutely</u> <u>improperly</u> <u>integrable</u> on $(a,b]$.

and similarly (analogous to 12.26) for $[a, \infty)$.

Example: $\frac{1}{x^{1-\alpha}}$ is abs. imp. int. on $[0,1]$ if $\alpha > 0$ and on $[1, \infty)$ if $\alpha < 0$.

Example: $\int_0^1 \frac{\sin \frac{1}{x}}{x^{1-\alpha}} \, dx$ with $\alpha > 0$.

This improper integral exists since

$$\left| \frac{\sin \frac{1}{x}}{x^{1-\alpha}} \right| \leq \frac{1}{x^{1-\alpha}} = \left| \frac{1}{x^{1-\alpha}} \right|$$

which is abs. imp. int. in $[0,1]$ by 12.25.

so $\frac{\sin\frac{1}{x}}{x^{1-\delta}}$ is imp. int. on $[0,1]$ by 12.33,

Similarly: $\int_1^\infty \frac{\sin(x)}{x^{1+\alpha}}\,dx$ with $\alpha > 0$

exists. (eg change variables $y = \frac{1}{x}$
from previous example).

(Dirichlet),
Example: 12.36: $\int_0^\infty \frac{\sin(x)}{x}\,dx = \frac{\pi}{2}$

exists but $\int_0^\infty \left|\frac{\sin(x)}{x}\right| dx$ does not.

so call $\frac{\sin(x)}{x}$ conditionally

improperly int. on $[0, \infty)$.

Example: $\text{p.v.} \int_{-\infty}^{1} \frac{e^t}{t}\, dt$

$$= \int_{-\infty}^{-1} \frac{e^t}{t}\, dt + \text{p.v.} \int_{-1}^{1} \frac{e^t}{t}\, dt$$

$$= \lim_{n \to \infty} \int_{-n}^{-1} \frac{e^t}{t}\, dt + \lim_{\varepsilon \to 0} \left[\int_{-1}^{-\varepsilon} \frac{e^t}{t}\, dt + \int_{\varepsilon}^{1} \frac{e^t}{t}\, dt \right]$$

Claim: both limits exist.

$\underline{210310}$ Practice final on web page.

Recall Ex 12.41:

Show $\underbrace{\text{p.v.}}_{\text{at } t=0} \displaystyle\int_{\underbrace{-\infty}_{\text{improper}}}^{1} \frac{e^t}{t}\,dt$ $\qquad f(t) = \dfrac{e^t}{t}$

$$= \int_{-\infty}^{-1} f(t)\,dt + \text{p.v.} \int_{-1}^{1} f(t)\,dt$$

$$\int_{-\infty}^{-1} \frac{e^t}{t}\,dt = \lim_{n\to\infty} \int_{-n}^{-1} \frac{e^t}{t}\,dt \leq \lim_{n\to\infty} \int_{-n}^{-1} e^t\,dt$$

$$= \frac{1}{e} - \lim_{n\to\infty} e^{-n}$$

$$= \frac{1}{e}$$

Note: $\dfrac{e^t}{t}$ is cts in $[-n,-1]$

$$\left|\frac{e^t}{t}\right| \leq e^t \quad \text{if} \quad t \in [-n, -1]$$

For p.v. $\int_{-1}^{1} \frac{e^t}{t}\, dt$.

Recall: $e^t = \sum_{n=0}^{\infty} t^n \frac{1}{n!}$ $= 1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \cdots$ conv. with $R = \infty$

so $$\frac{e^t}{t} = \frac{1}{t} + \underbrace{\sum_{n=0}^{\infty} \frac{t^n}{(n+1)!}}_{\substack{\text{power series with } R=\infty \\ \text{so cts in } [-1,1].}}$$

Write $\tilde{f}(t) = \begin{cases} \frac{e^t}{t} - \frac{1}{t} & t \neq 0 \\ 1 & t = 0 \end{cases}$ is cts (or compute: $\lim_{t\to 0}\left(\frac{e^t}{t}-\frac{1}{t}\right)$

$= \lim_{t\to 0} \frac{e^t-1}{t} \underset{\text{L'Hô}}{=} \lim_{t\to 0}\frac{e^t}{1} = 1$)

So $\text{p.v.}\int_{-1}^{1}\frac{e^t}{t}dt = \lim_{\varepsilon\to 0}\left[\int_{-1}^{-\varepsilon}\frac{e^t}{t}dt + \int_{\varepsilon}^{1}\frac{e^t}{t}dt\right]$

$= \lim_{\varepsilon\to 0}\left[\int_{-1}^{-\varepsilon}\frac{e^t}{t}dt + \int_{\varepsilon}^{1}\frac{e^t}{t}dt - \underbrace{\left[\int_{-1}^{-\varepsilon}\frac{dt}{t} + \int_{\varepsilon}^{1}\frac{dt}{t}\right]}_{0 \text{ since } \frac{1}{t} \text{ is odd.}}\right]$

$$= \lim_{\varepsilon \to 0} \left[\int_{-1}^{-\varepsilon} \tilde{f}(t)\,dt + \int_{\varepsilon}^{1} \tilde{f}(t)\,dt \right]$$

$$= \int_{-1}^{1} \tilde{f}(t)\,dt$$

which exists since $\tilde{f}$ is cts hence integrable on $[-1, 1]$.

Example of Taylor Errors:

$f(x) = e^x$ and $c = 0$

$R_n(x) = e^x - 1 - x - \cdots - \frac{x^n}{n!}$

so $P_n(x) = \sum_{r=0}^{n} \frac{x^r}{r!}$

and 8.46: $R_n(10) = \frac{e^{\xi}}{(n+1)!} 10^{n+1} \leq \frac{e^{10}\, 10^{n+1}}{(n+1)!}$

and 12.48: $R_n(10) = \frac{1}{n!} \int_0^{10} e^t (10-t)^n \, dt$

$= \frac{1}{n!} \int_0^{10} e^{10-s} s^n \, ds$

$s = 10 - t$

(

$$\leq \frac{1}{n!}\int_0^{10} e^{10} s^n \, ds = \underbrace{\frac{e^{10}\,10^{n+1}}{(n+1)!}}_{\text{sharpr if } n>9}$$

also

$$\leq \frac{1}{n!}\int_0^{10} e^{10-s}\, 10^n \, ds$$

$$= \underbrace{\frac{10^n}{n!}\left(e^{10}-1\right)}_{\text{sharper if } n<9}$$

Base case: $n=0$

$$R_0(x) = f(x) - P_0(x) = f(x) - f(c)$$

$$= f\Big|_c^x \underset{\substack{\text{since} \\ f^{(1)} \in RInt[c,x] \\ \text{and FTC.}}}{=} \int_c^x f^{(1)}(t)\,dt$$

$$= \frac{1}{0!} \int_c^x f^{(1)}(t)(x-t)^0\,dt$$

so done.

Induction step:

Assume $R_n(x) = \frac{1}{n!}\int_c^x f^{(n+1)}(t)\,(x-t)^n\,dt$

and compute:

$$R_{n+1}(x) = f(x) - P_{n+1}(x)$$

$$= \underbrace{f(x) - [P_n(x)}_{} + \frac{f^{(n+1)}(c)}{(n+1)!}(x-c)^{n+1}]$$

$$= R_n(x) - \text{''} \quad \text{''}.$$

$$= + \int f^{(n+2)}(t)\,(x-t)^{n+1}\,dt\,\frac{1}{(n+1)!}$$

use $u = f^{(n+1)}(x)$

$v = (x-c)^{n+1}$

and parts

$\underbrace{+ \quad -}_{\text{cancel}}$

= R.H.S. from Thm.

q.e.d.

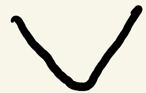
210312 | Office Hrs: Class time Monday.
(or email)

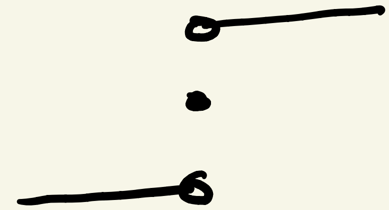
Studied spaces:

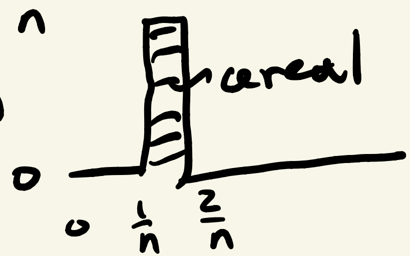
$$
\begin{tikzcd}
D^0[0,1] \arrow[dr, phantom, "\supseteq" sloped] & & D^1[0,1] \arrow[ll, "D"'] \arrow[dl, phantom, "\supseteq" sloped] \arrow[dr, phantom, "\supseteq" sloped] & & & \\
& C^0[0,1] \arrow[dl, phantom, "\subseteq" sloped] \arrow[dr, phantom, "\supseteq" sloped] & & C^1[0,1] \arrow[r, phantom, "\supseteq \cdots"] & \supseteq C^\infty[0,1] \arrow[d, phantom, "\cup\!\!|"] & \\
R\,Int[0,1] \arrow[rr, "J"', "\text{bijection}"] & & \underbrace{C^0_0[0,1]}_{f(0)=0} & & \underbrace{C^\omega[0,1]}_{\substack{\text{power series} \\ \text{real analytic.}}} &
\end{tikzcd}
$$

Useful examples:

sequences of functions $\rightarrow$ functions,

9.6 $f_n(x) = \dfrac{x^2}{\sqrt{x^2+\frac{1}{n}}}$  $\longrightarrow$ $|x|$ 

13.89 $f_n(x)$  $\longrightarrow$ 

12.22 $f_n(x)$ 

$\longrightarrow$ 0

11.15 $f_n(x) = \begin{cases} 1 & \text{fin many values} \\ 0 & \text{else} \end{cases}$ $\longrightarrow$  Dirichlet $\begin{cases} 1 & x \in \mathbb{Q} \\ 0 & \text{else} \end{cases}$

11.16 similar, $\longrightarrow$ Thomae $\begin{cases} \frac{1}{n} & x = \frac{a}{n} \\ 0 & \text{else} \end{cases}$

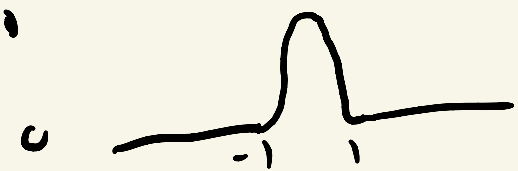
10.1 $\sum a_n x^n$, e^x, $\sin(x)$, $\frac{1}{1-x}$] analytic.

9.24 $\omega(x) = \sum_{n=0}^{\infty} 2^{-n}\cos(8^n x) \longrightarrow \omega(x)$

8.9/8.10 $x^4 \sin\frac{1}{x}$

10.33 smooth, not analytic, cptly supp.

$$f(x) = \begin{cases} e^{\frac{1}{x^2-1}} & |x| \leq 1 \\ 0 & \text{else} \end{cases}$$



12.25 $\frac{1}{x^p}$ which have impropr integrals? $\int_0^1$ and $\int_1^\infty$

Problem: Find all functions $f(x)$ with: $f'''(x) = ke^x$ for some constant k.

and $f(0) = f'(0) = 0$.

Ans: Integrate:

$$(f''(x))' = ke^x$$

so $f''(x) = ke^x + a$ for a constant a,

and $f'(x) = ke^x + ax + b$ " " a, b

and $f(x) = ke^x + ax^2 + bx + c$ " " a, b, c

hence $\quad f(0) = k + c = 0 \quad$ so $\quad c = -k$

$\quad\quad\quad f'(0) = k + b = 0 \quad$ so $\quad b = -k$

so $\quad f(x) = ke^x + ax^2 - kx - k$

(a 2-dim. space of functions:
all lin. comb. of x^2 and $e^x - kx - k$).