MAT 127B - B

Winter 2021

Left Board

Newton invent calc in 1666
undergrad. at Cambridge
while univ. closed for the plague.

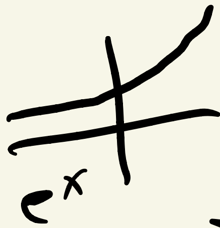
Computation in 21

Focus on spaces of functions
approx functions using
- polynomials
- trig polynomials

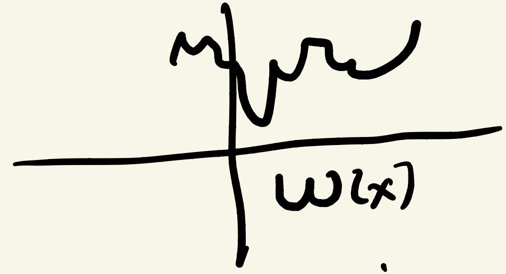
eg $$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$$

eg: $W(x) = \cos(x) + \frac{1}{2}\cos(3x) + \frac{1}{4}\cos(9x) + \cdots$

graphs:



exponential growth



very jagged.

similar to motion of a molecule in a liquid.

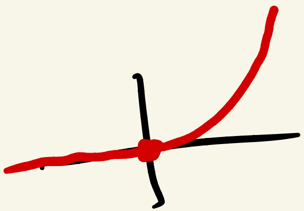
Def 8.1 If $a < c < b$ and $f:(a,b) \to \mathbb{R}$ write:

$$f'(c) = (Df)(c) = \lim_{h \to 0} \frac{f(h+c)-f(c)}{h} \underset{x=h+c}{=} \lim_{x \to c} \overbrace{\underbrace{\frac{f(x)-f(c)}{x-c}}_{\text{secant slope}}}^{\text{slope of tangt.}}$$

and if $f'(c)$ exists say f is differentiable at c

and if f is diff. at all pts in (a,b) it is diff on/in (a,b).

Ex ②: $f_2(x) = \begin{cases} x^2 & x>0 \\ 0 & x=0 \\ 0 & x<0 \end{cases}$



$f_2'(3) = f_1'(3) = 6$

since derivatives are local.

so $f_2'(x) = \begin{cases} 2x & x>0 \\ 0 & x=0 \\ 0 & x<0 \end{cases}$

and $f_2'(0) = \lim_{h\to 0} \dfrac{f_2(h)}{h}$

Note: $\lim\limits_{x\to a} F(x) = L$ iff

$$\lim_{x\to a^+} F(x) = L = \lim_{x\to a^-} F(x)$$

Computing above:

$$\lim_{h\to 0^+} \frac{f_2(h)}{h} = \lim_{h\to 0^+} \frac{h^2}{h} = \lim_{h\to 0^+} h = 0$$

$$\|$$

$$\lim_{h\to 0^-} \frac{f_2(h)}{h} = \lim_{h\to 0^-} \frac{0}{h} = \lim_{h\to 0^-} 0 = 0$$

(B) derivative:

$$f_3'(x) = \begin{cases} 2x \sin\frac{1}{x} + x^2\left(-\frac{1}{x^2}\cos\left(\frac{1}{x}\right)\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

by def:

$$f_3'(0) = \lim_{h\to 0} \frac{f_3(h)}{h} = \lim_{h\to 0} h \sin\frac{1}{h}$$

by squeeze again

$$\boxed{\lim_{h\to 0} \frac{\sin\frac{1}{h}}{\frac{1}{h}} = \lim_{H\to\infty} \frac{\sin(H)}{H} = 0}$$

$$-|h| \leq h \sin\frac{1}{h} \leq |h|$$

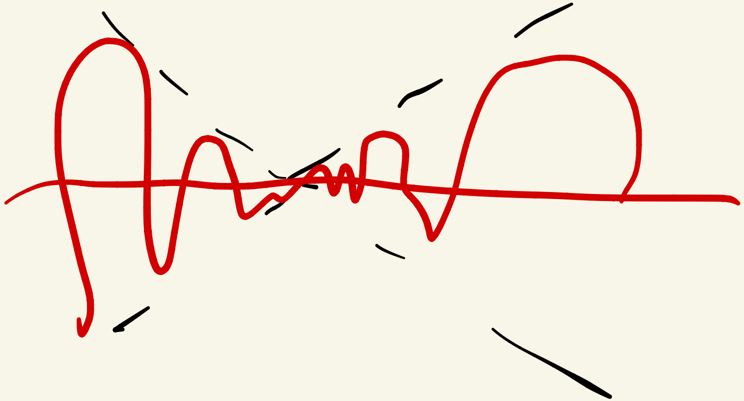
$$\lim_{h\to 0} -|h| = 0 \leq \lim_{h\to} h\sin\frac{1}{h} \leq \lim_{h\to} |h| = 0$$

$$\quad\quad\quad\quad\quad\quad \| $$

$$\quad\quad\quad\quad\quad\quad 0 \quad \text{by squee}$$

guess

g_1



g_2

210106 (Numbering from Hunter)

Def: If $f:(a,b)\to\mathbb{R}$ and $a<c<b$

6.1 $\lim_{x\to c} f(x)=L$ if

$\forall \varepsilon>0\ \exists \delta>0\ \forall x\in(a,b)$ with $|x-c|<\delta$

it is true that $|f(x)-L|<\varepsilon$.

7.1 f is ~~cts~~ continuous (cts) at c if

$\lim_{x\to c} f(x) = f(c)$

8.1 f is differentiable (diff) at c if

$$f'(c) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c} \text{ exists.}$$

New

8.18 f is <u>continuously differentiable</u> (cts diff) at c if f is diff in a nbhd of c and f' is cts at c.

Show last time: ② [Ⓐ diff. away from 0 with rules
Ⓑ diff at 0 with def.

$$f'(x) = \begin{cases} 2x\sin\frac{1}{x} - \cos\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

] rules.

] squeeze thm helped

hence $f(x) \in D^1\mathbb{R}$

Compute: f' is not cts at 0.

hence $f(x) \notin C^1\mathbb{R}$.

Need to show $\lim_{x\to 0} f'(x) \neq f'(0) = 0$

Assume for contradiction $\lim_{x\to 0} f'(x) = 0$

so by def. of lim;

$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall |x| < \delta$ have $|f'(x)| < \varepsilon$.

Choose $\varepsilon = \frac{1}{2}$

by the hyp there is $\delta > 0$

choose $x = \frac{1}{2n\pi}$ with n large

enough that $x < \delta$

Compute: $\varepsilon = \frac{1}{2} > |f'(x)| = \left| 2\frac{1}{2\pi n}\sin(2\pi n) - \cos(2\pi n) \right|$

$= |0-1| = 1$

oops

This is a contradiction so

so $\lim_{x\to 0} f'(x) \neq 0$ so f' is dicont. at 0 so $f' \notin C^1 \mathbb{R}$

Thm 8.17 If f is diff at c
then f is cts at c.

Proof: Assume f is diff at c

so $\displaystyle\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$ exists.

Want to show $\displaystyle\lim_{x \to c} f(x) \overset{?}{=} f(c)$

Asside: If $\lim_{x \to c} F(x) = L$

and $\lim_{x \to c} G(x) = M$

then $\lim_{x \to c} \left(F(x) + G(x)\right) = L + M$

$\lim_{x \to c} \left(F(x) \cdot G(x)\right) = L \cdot M$

$\lim_{x \to c} (x - c) = 0$

210108

Algebraic operations of functions, and continuity, diff.

Write $\mathrm{Fun}(\mathbb{R})$ for all functions

$$f : \mathbb{R} \to \mathbb{R}$$

If $f, g \in \mathrm{Fun}(\mathbb{R})$ and $k \in \mathbb{R}$ write:

$k \cdot f \in \mathrm{Fun}$.

$f + g \in \mathrm{Fun}$

$f \cdot g \in \mathrm{Fun}$

$g \circ f \in \text{Fun}$

Recall: $\text{Fun}(\mathbb{R}) \supsetneq C^0(\mathbb{R}) \supsetneq D^1(\mathbb{R}) \supsetneq C^1(\mathbb{R})$
$\supsetneq D^2(\mathbb{R}) \supseteq \cdots \supseteq C^\infty(\mathbb{R})$
$C^\infty(\mathbb{R}) = D^\infty(\mathbb{R})$

Thm: If $f, g \in C^0(\mathbb{R})$ (both cts).
then $kf,\ f+g,\ f \cdot g,\ g \circ f \in C^0(\mathbb{R})$
(the new ones ~~are~~ also cts).

Ex: If G is cts at $f(c)$ and f is cts at c
then $G \circ f$ is cts at c so

$$\lim_{x\to c} (G\circ f)(x) = (G\circ f)(c) = G(f(c)) = \lim_{y\to f(c)} G(y)$$

Derivatives! Thms 8.19 / 8.21

If $f, g \in D'(\mathbb{R})$ (diff) and $k \in \mathbb{R}$

$(k\cdot f)' = kf'$ — $D(kf) = k(Df)$

$(f+g)' = f' + g'$ — $D(f+g) = Df + Dg$.

(product)

$(f\cdot g)' = f'\cdot g + f\cdot g'$ — $D(f\cdot g) = (Df)\cdot g + f\cdot(Dg)$

$\left(\frac{f}{g}\right)' = \frac{f'\cdot g - f\cdot g'}{g^2}$ if $g \neq 0$ — $D\left(\frac{f}{g}\right) = \frac{(Df)\cdot g - f\cdot(Dg)}{g^2}$

(chain)

$(g \circ f)' = (g' \circ f) \cdot f'$ | $D(g \circ f) = [(Dg) \circ f] \cdot (Df)$

Cor: If $f, g \in D'(\mathbb{R})$ and $k \in \mathbb{R}$.

then $kf, f+g, f \cdot g, g \circ f \in D'(\mathbb{R})$.

Proof idea for chain Rule.

Given $f, g \in D'(\mathbb{R})$.

Compute: $(g \circ f)'(c) \overset{\text{Def}}{=} \lim\limits_{x \to c} \dfrac{(g \circ f)(x) - (g \circ f)(c)}{x - c}$

$$= \lim_{x \to c} \frac{g(f(x)) - g(f(c))}{x - c}$$

be careful about: $f(x) - f(c) = 0$ (or $f(x) = f(c)$).

$$= \lim_{x \to c} \frac{f(x) - f(c)}{f(x) - f(c)} \cdot \frac{g(f(x)) - g'(f(c))}{x - c}$$

$$= \lim_{x \to c} \frac{g(f(x)) - g(f(c))}{f(x) - f(c)} \cdot \frac{f(x) - f(c)}{x - c}$$

if both new limits exist.

$$= \lim_{x \to c} \text{"} \quad \text{"} \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

$$= \lim_{y \to f(c)} \frac{g(y) - g(f(c))}{y - f(c)} \cdot f'(c)$$

$$= g'(f(c)) \cdot f'(c).$$

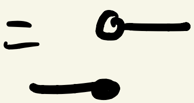
2/10/11

Today: Parametric Curves

- ex
- pf.

Inverse fn. derivs.

Bestiary: functions and where they are.

ex: Is $f_0(x) = \begin{cases} 1 & x > 0 \\ 0 & x \le 0 \end{cases} =$ 

a derivative? $f_0 \stackrel{?}{=} g'$??

Prove ① in Bcb. Rms.

Hint: Limit quotient rule

$$\lim \frac{f}{g} = \frac{\lim f}{\lim g}.$$

Proof:

$$\cancel{}\ y'(c) = \lim_{t \to c} \frac{y(t) - y(c)}{t - c}$$

$$x'(c) = \lim_{t \to c} \frac{x(t) - x(c)}{t - c}$$

hence $\dfrac{y'(c)}{x'(c)} = \lim\limits_{t \to c} \dfrac{y(t) - y(c)}{t - c}$

$$\lim_{t \to c} \frac{x(t) - x(c)}{t - c}$$

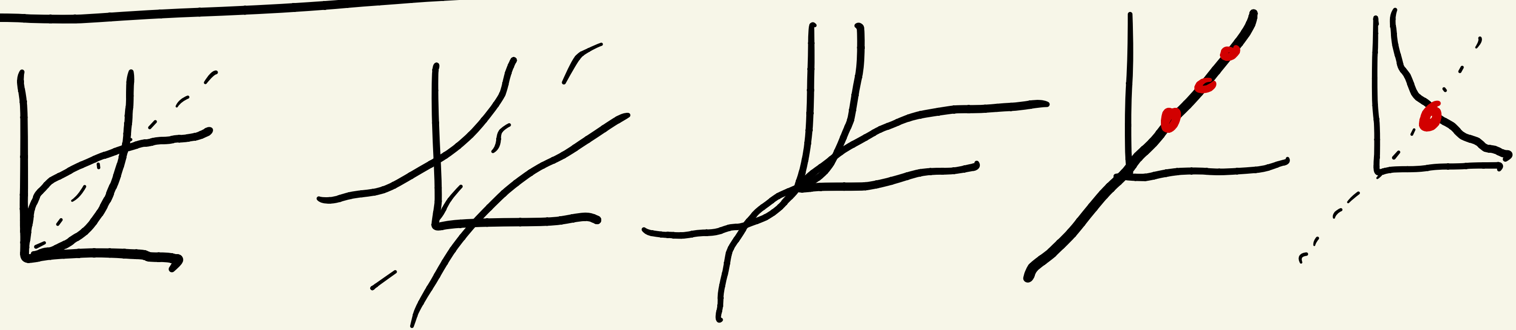
$$= \lim_{t \to c} \frac{\frac{y(t) - y(c)}{t - c}}{\frac{x(t) - x(c)}{t - c}}$$

since
$y'(c)$ & $x'(c)$
exist
and $x'(c) \neq 0$

$$= \lim_{t \to c} \frac{y(t) - y(c)}{x(t) - x(c)},$$

Ex of inv fns:

$f(x)$	x^2	$x^{\frac{1}{2}}$	e^x	$\tan(x)$	x	$\frac{1}{x}$
$f^{-1}(x)$	$x^{\frac{1}{2}}$	x^2	$\ln(x)$	$\operatorname{Arctan}(x)$	x	$\frac{1}{x}$



eg: $D(\ln(x)) = \dfrac{1}{D(e^x) \circ \ln(x)} = \dfrac{1}{e^x \circ \ln(x)} = \dfrac{1}{e^{\ln(x)}} = \dfrac{1}{x}$

Problem! If $f = f^{-1}$ and $f(c) = c$ what are the possibilities for $f'(c)$?

Ans: 1 or -1

$$f'(c) = (f^{-1})'(c) = \frac{1}{f'(f^{-1}(c))}$$

$$= \frac{1}{f'(f(c))} = \frac{1}{f'(c)}$$

so $[f'(c)]^2 = 1$ so $f'(c) = 1$ or -1

New def:

$$D^0(\mathbb{R}) = \{ g'(x) \mid g(x) \in D'(\mathbb{R}) \}.$$

eg $(x^2)' = 2x \in D^0(\mathbb{R})$.

$$\left.\begin{cases} x^2 \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}\right\}' = \begin{cases} 2x\sin\frac{1}{x} - \cos\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$$\underset{g_2'}{\shortparallel} \qquad \in D^0(\mathbb{R}).$$

210113 # Mean Value Theorem:

Idea: If $f \in C^0[a,b] \cap D^1(a,b)$

that is $f: [a,b] \longrightarrow \mathbb{R}$ and

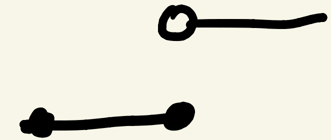
f is cts at every point

and diff. at every interrer pt.

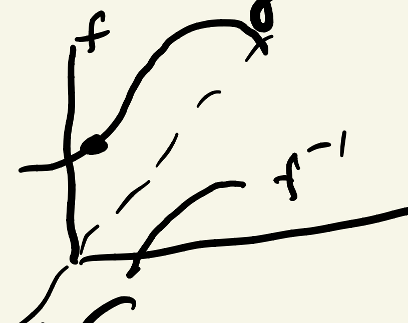
eg: $f(x) = x^2 \in C^0[-1,1] \cap D^1(-1,1)$.

Use facts about the behavior at a & b to get facts in the interior.

Applications:

Darboux Thm: eg:  $\not\subseteq D^\circ$

Inv. Fn. Thm: If $f'(c) \neq 0$ then f^{-1} exists locally



Antiderivs:

If $Df = Dg$ then $f = g + C$

Tay-Lag: How well will a polynomial approximate a function?

L'Hôspital's Limint Rule

Thm (7.37 H) (4.4.2 A)

If $f \in C^0[a,b]$ then

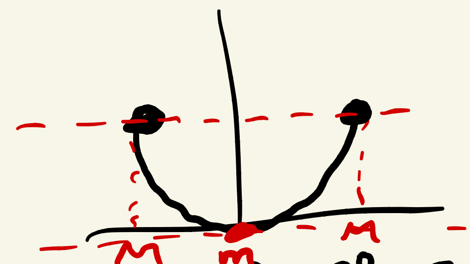
$\exists\ m, M$ with $f(m) = \min(\operatorname{Im} f)$

and $f(M) = \operatorname{Max}(\operatorname{Im} f)$

Thm (Rolle) (8.32 H) (5.3.1 A)

If $f \in C^0[a,b] \cap D^1(a,b)$

and $f(a) = f(b)$ then



$f(x) = x^2 \in C^0[-1,1]$

$\exists\, c \in (a,b)$ with $\underbrace{f'(c)}_{\text{slope of tang at } c} = \underbrace{0}_{\text{slope of secant line at } a \,\&\, b}$

Proof: By 7.37 choose $m, M \in [a,b]$

with $f(m) = \min(\mathrm{Im}(f))$

$f(M) = \mathrm{Max}(\mathrm{Im}(f))$

Case 1: If m or $M \in (a,b)$ then

by 8.27 have $f'(m) = 0$ or $f'(M) = 0$.

Case 2: If m and M are both endpoints.

then $f(m) = f(a) = f(b) = f(M)$

so f is a constant fn

and $D(f) = 0$

so take c any pt in (a,b)

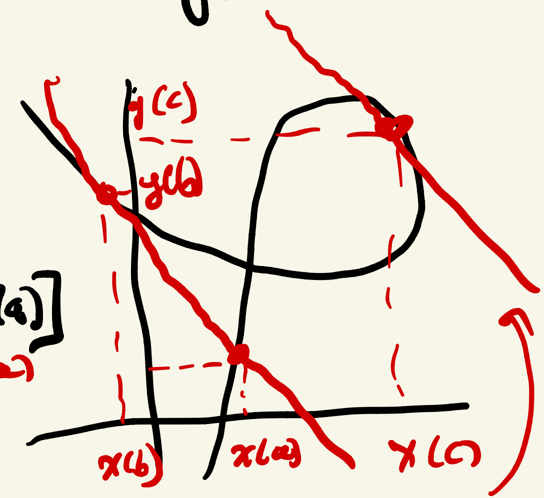
and $f'(c) = 0$.

Thm: (Generalized or Parametric or Cauchy) MVT

(8.5.3 H)(5.3.2 A)

If $f, g \in C^0[a,b] \cap D^1(a,b)$

$\exists\, c$ with $[f(b) - f(a)]\,g'(c) = f'(c)\,[g(b) - g(a)]$



Picture: $g = x(t)$ and $f = y(t)$

if $x'(c) \neq 0$ and $x(b) - x(a) \neq 0$

then the above c for is the same as!

$$\underbrace{\frac{y(b)-y(a)}{x(b)-x(a)}}_{\text{slope of the secant line at } a \,\&\, b} = \underbrace{\frac{y'(c)}{x'(c)}}_{\text{limit from last time}}$$

or slope of the tangent line at $(x(c), y(c))$.

Proof Plan: Use MVT on $h(x)$ with

set $h(x) = [f(b) - f(a)]\, g(x) + f(x)\, [g(b) - g(a)]$

numbers,

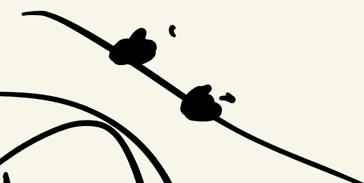
Check: $h(x) \overset{?}{\in} C^0[a,b] \cap D^1(a,b)$

and $h(c) = \dfrac{h(b) - h(a)}{b - a}$ gaurantees f is not increasing on any interval containing ~~around~~ 0.

For contradiction assume f is inc. on some interval $(-\delta, \varepsilon)$

find $0 < \underline{a < b} < \boxed{\varepsilon}$

with $f(a) > f(b)$



$a = \frac{1}{n^2}$

2/10/15 Monday (MLK Day)

HW will be due Wed.

Thm (8.53 H)(5.3.5 A)

(Gen or Param or Cauchy) MVT

If $f, g \in C^0[a,b] \cap D^1(a,b)$ then $\exists c \in (a,b)$

with $[f(b)-f(a)]g'(c) = f'(c)[g(b)-g(a)]$

Proof: Consider

$$h(x) = [f(b)-f(a)]g(x) - f(x)[g(b)-g(a)]$$

Plan: Use h in MVT for the proof

Need: $h \in \underbrace{C^0[a,b] \cap D^1(a,b)}$ which is true since $\downarrow$ is a linear space and h is a lin. comb. of f & g.

Hence can apply MVT and get $c \in (a,b)$ with

$$h'(c) = \frac{h(b)-h(a)}{b-a}$$

$$[f(b)-f(a)]g'(x) - f'(x)[g(b)-g(a)] \stackrel{??}{=} 0$$

What is the kernel of

$$D : D^1(\mathbb{R}) \to D^0(\mathbb{R}).$$?

That is for which f is $f' = \underbrace{0}_{\text{the } 0 \text{ function}}$?

Thm (5.3.3 A) (8.34 H):
If $f \in D^1(\mathbb{R})$ and $D(f) = 0$] the 0 fn.
then $f(x) = C$] a const. fn.

Proof: Use contradiction and MVT.

Assume for contradiction $\exists f \in D'(\mathbb{R})$ with $f'(x) = 0$ but $f(x) \neq f(y)$

hence $f|_{[x,y]} \in D'(x,y) \cap C^0[x,y]$

so by MVT get $c \in (x,y)$

with $f'(c) = \frac{f(y)-f(x)}{y-x} \neq 0$ a contradict.

Cor: (5.3.5 A)(8.35 H)

$\forall f, g \in D'(\mathbb{R})$ with $f' = g'$

have $f(x) = g(x) + C$

Proof: If $f' = g'$ then $f' - g' = 0$

or $(f-g)' = 0$ so $f(x) - g(x) = C$

by the Thm above and $f(x) = g(x) + C$.

Ex: If $f \in D^2(\mathbb{R})$

and $f(0) = 0$ and $f'(0) = 0$

and $f''(x) \leq 3$ $\forall x$.

Show that $f(2) \leq 12$

Ans: Try contradiction:

By the previous example: $f'(2) \le 6$

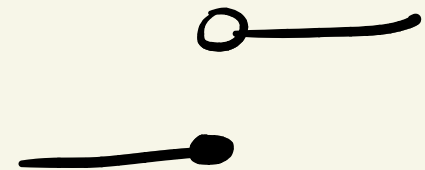
and similarly get $\forall x \in [0,2]$ have $f'(x) \le 6$

Now repeat and get $f(2) \le 12$

(Sketch)

Ex: By Darboux's Thm:

$$f_0(x) = \left\{ \begin{matrix} 1 & x>0 \\ 0 & x\le 0 \end{matrix} \right\} =$$



is not a derivative.

since if $f = F'$ then $F'(-1)=0$

and $F'(1)=1$

and $\alpha = \frac{1}{2}$ is between 0 & 1

but $\nexists\, x$ with $F'(x)=\frac{1}{2}$

Ex: Claim:

$$-4x^2+\left(\begin{Bmatrix} x^2\sin\frac{1}{x} & x\neq 0 \\ 0 & x=0 \end{Bmatrix}'\right)^2+\left(\begin{Bmatrix} x^2\cos\frac{1}{x} & x\neq 0 \\ 0 & x=0 \end{Bmatrix}'\right)^2$$

$$= \begin{Bmatrix} 1 & x\neq 0 \\ 0 & x=0 \end{Bmatrix}$$

Not in $\underbrace{D^0(\mathbb{R})}_{\text{a deriv.}}$

by Darboux

computation you can do.

hence there is $f\in D^0(\mathbb{R})$
with $f^2\notin D^0(\mathbb{R})$,

210120 Using the MVT to understand functions.

Compare functions which are derivatives (D^0).
to continuous functions (127A) (C^0).

Consider $a < c < d < b$ and if $f \in \mathrm{Fun}(a,b)$
$(\quad [\quad] \quad)$

write $\mathrm{Im}(f|_{[c,d]}) = \{ f(x) \mid c \le x \le d \}$

(old)
Thm: (7.35) If $f \in C^0(a,b)$ then
$\mathrm{Im}(f|_{[c,d]})$ is closed and bounded.

(7.44) If $f \in C^0(a,b)$ then

$\mathrm{Im}(f|_{[c,d]})$ is connected.

Upshot: If f is cts then $\mathrm{Im}\, f|_{[c,d]}$

$= [f(m), f(M)]$. a closed interval or pt.

(new)

Thm: ~~If $f \in D^0(a,b)$ &&~~

ⓐ There is $f \in D^0(a,b)$ with $\mathrm{Im}\, f|_{[c,d]}$ not bounded.

ⓑ " " " not closed.

ⓒ (Darboux): If $f \in D^0(a,b)$ then $\mathrm{Im}\, f|_{[c,d]}$ is connected.

Proof of Darboux's Thm:

The thm is equivalent to:

$\forall F \in D^1(a,b)$ with $a<c<d<b$ and

α between $F'(c)$ and $F'(d)$

there is $e \in [c,d]$ with $F'(e) = \alpha$.

Proving this form:

Taylor's Polynomials with Lagrange error term.

(another error estimate will come later)

Def: If $f \in D^n(a,b)$ and $c \in (a,b)$

write $P_{n,c}(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots$

$$\cdots \frac{f^{(n)}(c)}{n!}(x-c)^n$$

and $R_{n,c,x} = f(x) - P_{n,c}(x)$

= error or remainder,

Thm: If $f \in D^n(a,b)$ and $c, x \in (a,b)$

then there is ξ between c and x

with $R_{n,c,x} = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x+c)^{n+1}$

Hardest part about using this is the lack of information about ρ.

Examples: $f(x) = e^{x}$

$f(x) = \frac{1}{x} = x^{-1}$

behave differently.

210122 Today: Gen MVT applications

Recall: GMVT

Thm(8.53) $\forall\, k, g \in D^1(a,b)$, $a<c<x<b$

$\exists\, \xi \in [c,x]$ with $[k(x)-k(c)]\,g'(\xi) = k'(\xi)\,[g(x)-g(c)]$

Thm (Taylor-Lagrange) (8.46)

$\forall f \in D^{n+1}(a,b)$, $a<c<x<b$ $\exists\, \xi \in [c,x]$

with $\underbrace{f(x) - P_{n,c}(x)}_{R_{n,c,x}} = f^{(n+1)}(\xi)\,\frac{(x-c)^{n+1}}{(n+1)!}$

Recall: $P_{n,c}(x) = f(c) + f'(c)(x-c) + \dots + f^{(n)}(c)\frac{(x-c)^n}{n!}$

Proof: For GMVT choose:

$$k(t) = R_{n,c,x}\frac{(x-t)^{n+1}}{(x-c)^{n+1}} \qquad g(t) = R_{n,t,x}$$

$$k(x) - k(c) = -R_{n,c,x} = g(x) - g(c) = \left(\cancelto{0}{f(x) - f(x)}\right) - R_{n,c,x}$$

$$k'(t) = R_{n,c,x}\frac{(n+1)(-1)(x-t)^n}{(x-c)^{n+1}}$$

Claim: $g'(t) = -f^{(n+1)}(t)\frac{(x-t)^n}{n!}$

By GMVT: $\exists\ \xi \in [c,x]$ with $[k(x)-\cancel{k(c)}]\,g'(\xi) = k'(\xi)[g(x)-\cancel{g(c)}]$

so $$g'(\xi) = k'(\xi)$$

cancel

or $$-f^{(n+1)}(\xi)\,\frac{(x-\xi)^n}{n!} = -R_{n,c,x}\,\frac{(n+1)\,(x-\xi)^n}{(x-c)^{n+1}}$$

so $$f^{(n+1)}(\xi)\,\frac{(x-c)^{n+1}}{(n+1)!} = R_{n,c,x}$$

Next: Limits of functions,

Ch 9:

2/10/25

$(a_n)_{n\in\mathbb{N}} = (a_1, a_2, a_3 \dots)$

Recall!

Def (3.10) If (a_n) is a sequence and A a number both in $\mathbb{R}$ then

$$a_n \xrightarrow[n\to\infty]{\text{in } \mathbb{R}} A \text{ if}$$

$$\forall \varepsilon > 0\ \exists N \in \mathbb{Z}\ \forall n > N \text{ have } \underbrace{|a_n - A|}_{\|f_n - F\|} < \varepsilon$$

ptwise $\quad \forall x \in [a,b] \quad$ uniform

$\|f_n - F\| < \varepsilon$ for some norm $\|\cdot\|$. (13.20) many possible norms.

How to modify for functions:

Def: If $(f_n(x))$ is a sequence and $F(x)$

a function both in Fun[a,b] then

9.1 $f_n \xrightarrow[n\to\infty]{\text{ptwise}} F$ if

$\forall x \in [a,b], \varepsilon > 0 \ \exists N \in \mathbb{Z} \ \forall n > N$ have $|f_n(x) - F(x)| < \varepsilon$

9.8 $f_n \xrightarrow[n\to\infty]{\text{uniformly}} F$ if

$\forall \varepsilon > 0 \ \exists N \in \mathbb{Z} \ \forall n > N, x \in [a,b]$ have $|f_n(x) - F(x)| < \varepsilon$

13.45 $f_n \xrightarrow[n\to\infty]{\|\cdot\|\text{-norm}} F$ if

$\forall \varepsilon > 0 \ \exists N \in \mathbb{Z} \ \forall n > N$ have $\|f_n - F\| < \varepsilon$

where $\|\cdot\| : (\text{Some functions}) \longrightarrow \mathbb{R}_{\geq 0}$

is some norm!

Example norms:

sup: $\|g(x)\|_{sup} = \sup_{x\in[a,b]} |g(x)|$

L^1: $\|g(x)\|_{L^1} = \int_a^b |g(x)|\,dx$

C^1: $\|g(x)\|_{C^1} = \|g(x)\|_{sup} + \|g'(x)\|_{sup}$

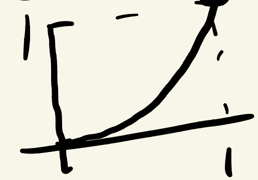
Two of these are the same:

(A) $\xrightarrow{\text{ptwise}}$ (B) $\xrightarrow{\text{unif.}}$ (C) $\xrightarrow{\|\cdot\|_{sup}\text{-norm}}$ (D) $\xrightarrow{\|\cdot\|_{C^1}\text{-norm}}$

In which intervals do each of the

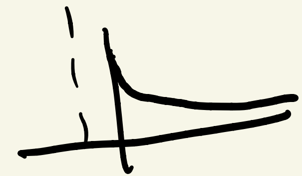
following converge in the senses A B C D ?

① $f_n(x) = x^n$, $F(x) = \begin{cases} 1 & x \geq 1 \\ 0 & x < 1 \end{cases}$

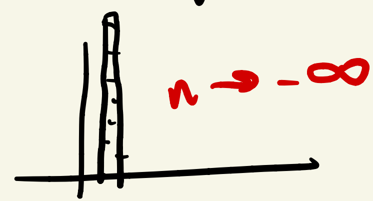


② $f_n(x) = \sqrt{x^2 + \frac{1}{n}}$, $F(x) = |x|$

③ $f_n(x) = \frac{1}{x + \frac{1}{n}}$, $F(x) = \frac{1}{x}$



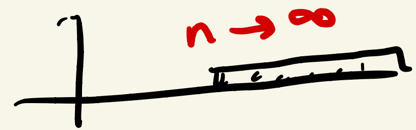
④ $f_n(x) = \begin{cases} 2^{-n} & 2^n \leq x \leq 2 \cdot 2^n \\ 0 \end{cases}$



$n \to \infty$

$n \to -\infty$

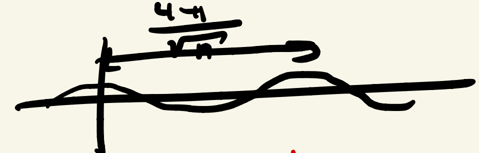
$F(x) = 0$



⑤ $f_n(x) = \frac{1}{n} \sin(nx)$, $F(x) = 0$

$\updownarrow \frac{2}{n}$

⑥ $f_n(x) = \frac{1}{n}\sin(\sqrt{n}x)$. $F(x)=0$



$\updownarrow \frac{2}{n}$

⑦ $1+x+\frac{x^2}{2}+\dots+\frac{x^n}{n!}=f_n(x)$, $F(x)=e^x$ (Taylor) $R=e^{\xi}\frac{x^{n+1}}{(n+1)!}$

⑧ $1+x+\dots+x^n=f_n(x)$, $F(x)=\frac{1}{1-x}$ (Taylor).

Lemma: If $(f_n(x))$ is a sequence in $\mathrm{Fun}(a,b)$ then $f_n \xrightarrow{\text{unif}} F$ iff $f_n \xrightarrow{\|\cdot\|_{\sup}} F$

Proof: $f_n \xrightarrow{\text{unif}} F$

iff $\forall \varepsilon>0\ \exists N\in\mathbb{Z}\ \forall n>N$ have $\forall x\in(a,b)$

have $|f_n(x)-F(x)|<\varepsilon$

$\boxed{210127}$

iff $\forall \varepsilon>0\ \exists N\in\mathbb{Z}\ \forall n>N$ have $\sup_{x\in(a,b)} |f_n(x)-F(x)|<\varepsilon$

iff $\forall \varepsilon>0\ \exists N\in\mathbb{Z}\ \forall n>N$ have $\|f_n(x)-F(x)\|_{\sup}<\varepsilon$

iff $f_n \xrightarrow{\|\cdot\|_{\sup}} F$

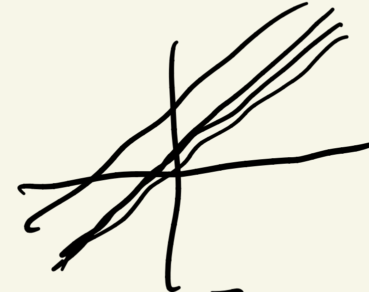
Br. Out Rm:

Find: A Example of a unif Cauchy seq of fns.

B Example of a ptwise-conv seq. of fns which is not unif-Cauchy.

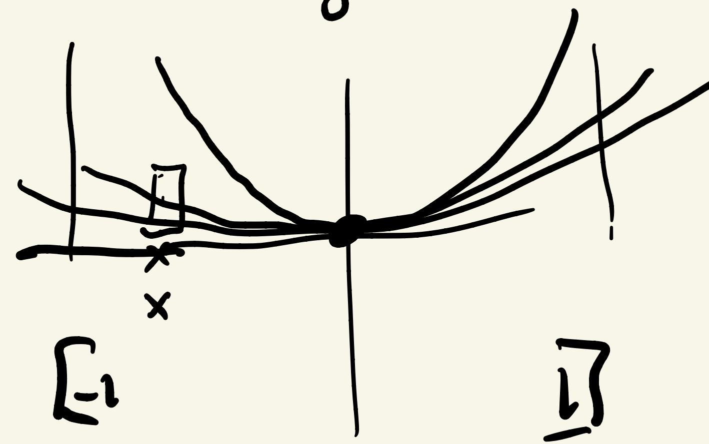
Examples:

(A) $x + \frac{1}{n}$] unif-Cauchy in $\mathbb{R}$



(B) $\frac{1}{n} x^2$] pt.wise conv. and ptwise $|\cdot|$-Cauchy] in $\mathbb{R}$
not ~~not~~-unif-Cauchy in $\mathbb{R}$

$\frac{1}{n} x^2$



Thm 3.46: A seq. (a_n) in $\mathbb{R}$ is conv. iff it is $|\cdot|$-Cauchy.

Thm 9.13: A seq. $(f_n(x))$ in $\mathrm{Fun}(a,b)$ is unif-conv. iff it is unif-Cauchy.

210129

Def: $Bdd(a,b)$ = bounded functions on (a,b)

$= \{ f \in Fun(a,b) \mid \exists B \in \mathbb{R}_{>0} \ \forall x \in (a,b)$ have $|f(x)| \leq B \}$.

$= \{ f \in Fun(a,b) \mid \|f\|_{sup}$ exists (or is finite) (or $< \infty$) $\}$.

Br. Rm. Quest.

① Find a bound for $\|W(x)\|_{sup}$ with $W(x)$ given below as a unif. limit.

② Find a sequence g_n of functions converging ptwise to g. with $\|g_n\|_{sup} < \infty$ but $\|g\|_{sup} = \infty$.

(can not conv. uniformly by 9.14).

Thm 9.16 If (f_n) is a seq. in $C^0(a,b)$ conv. unif. to $f \in Fun(a,b)$ then $f \in C^0(a,b)$

Equivalently: $C^0(a,b)$ is $\|\cdot\|_{sup}$-complete.

Proof: Note: $f \in C^0(a,b)$ iff.

$\forall c \in (a,b),\ \varepsilon > 0\ \exists\, \delta_{c,\varepsilon}$ $\forall\ |x-c| < \delta_{c,\varepsilon}$

want

have $|f(x) - f(c)| < \varepsilon$

Assume

$\forall n, c, \varepsilon > 0\ \exists\, \delta_{n,c,\varepsilon/4}$ $\forall\ |x-c| < \delta_{n,c,\varepsilon/4}$

have $|f_n(x) - f_n(c)| < \frac{\varepsilon}{4}$

and $f_n \xrightarrow{\|\cdot\|_{\sup}} f$ or

$\forall\ \varepsilon > 0\ \exists\ N_{\varepsilon/4}$ $\forall\ n \geq N_{\varepsilon/4}$ have

$\|f - f_n\|_{sup} < \varepsilon/4$ (A*)

Combining

choose

$\forall c \in (a,b), \ \varepsilon > 0 \quad \exists \, \delta_{c,\varepsilon} = \delta_{N_{\frac{\varepsilon}{4}}, c, \frac{\varepsilon}{4}} \quad \forall |x - c| < \delta$

have $|f(c) - f(x)| \leq |f(c) - f_{N_{\frac{\varepsilon}{4}}}(c)| + |f_{N_{\frac{\varepsilon}{4}}}(c) - f_{N_{\frac{\varepsilon}{4}}}(x)|$

$+ |f_{N_{\frac{\varepsilon}{4}}}(x) - f(x)|$

$< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} < \varepsilon$

(*A) (*) (AB)

210201 Properties	pointwise	uniform	$\|\cdot\|_{C^1}$-norm
Bounded	x $f_n(x)$	9.13	✓
Continuous	x $g_n(x)$	9.16	✓
Differentiable	x	x $h_n(x)$	9.18 (today).

where

$$\underbrace{f_n(x) = \frac{1}{x+\frac{1}{n}}}_{\text{bounded in } Fun(0,1)} \xrightarrow[\substack{\text{not uniform} \\ 9.13}]{\text{ptwise}} \underbrace{f(x) = \frac{1}{x}}_{\text{not bounded in } Fun(0,1)}$$

$$\underbrace{g_n(x) = x^n} \xrightarrow[\substack{\text{not unif.} \\ 9.16}]{\text{ptwise}} \underbrace{g(x) = \begin{cases} 1 & x=1 \\ 0 & x \neq 1 \end{cases}}$$

cts in Fun $[0,1]$ $\qquad$ not cts in Fun $[0,1]$

$$\underbrace{h_n(x) = \sqrt{x^2 + \tfrac{1}{n}}}_{\text{differentiable in } [-1,1]} \xrightarrow[\text{not in } \|\cdot\|_{C^1}]{\text{unif.}} \underbrace{h(x) = |x|}_{\text{not diff. in } [-1,1]}$$

9.18

Useful facts:

① Since $f_n' \xrightarrow{\text{unif.}} g$ have

$\forall \varepsilon > 0 \quad \exists N_\varepsilon \quad \forall n, m \geq N_\varepsilon, \; p \in (a,b)$

have $\left|f_n'(p)-g(p)\right| < \frac{\varepsilon}{5}$

and $\left|f_n'(p)-f_m'(p)\right| < \frac{\varepsilon}{5}$

② Since $f_n \xrightarrow{\text{unif}} f$ have

$f_n \xrightarrow{\text{ptwise}} f$ so

$\forall \varepsilon > 0,\ p, x \in (a,b)\ \exists M_{\varepsilon, p, x}\ \forall m \geq M_{\varepsilon, p, x}$

have $\left|f_m(p) - f(p)\right| < \frac{\varepsilon}{5}\left|p-x\right|$

(3) Since f_n are differentiable so
are $f_n - f_m$ so by MVT get.

$\forall n, m \in \mathbb{N}, \; x, p \in (a,b) \; \exists \underbrace{\xi_{n,m,x,p}}_{\text{between } x \text{ and } p}$

with $\dfrac{(f_n - f_m)(x) - (f_n - f_m)(p)}{x - p} = (f_n - f_m)'(\xi_{n,m,x,p})$

(4) Since f_n' exists have
$\forall n \in \mathbb{N}, \; p \in (a,b), \; \varepsilon > 0 \; \exists \delta_{n,p,\varepsilon} \; \forall |x-p| < \delta_{n,p,\varepsilon}$
have $\left| \dfrac{f_n(x) - f_n(p)}{x - p} - f_n'(p) \right| < \dfrac{\varepsilon}{5}$

Recall: $W(x)$ is the uniform limit of $k_n(x) = \sum_{j=0}^{n} 2^{-j} \cos(3^j x)$

and from Fri! $W(x)$ is cts. (9.16)

Thm later! every cts fn. is a derivative

Instead use 9.18 to find a fn. $V(x)$ with $V'(x) = W(x)$.

Choosing $f_n(x) = \sum_{j=0}^{n} 6^{-j} \sin(3^j x)$

so $\quad f_n'(x) = k_n(x)$

and $\quad f_n'(x)$ conv. unif to $W(x)$]

so $\quad f_n(x)$ conv. unif to $V(x)$]

with $V'(x) = W(x)$.

210203 Ch 13 sections 2, 4 & 6.

Def 13.20: A **normed vector space** $(X, \|\cdot\|)$ is a $\mathbb{R}$-vector space X and $\|\cdot\| : X \longrightarrow \mathbb{R}_{\geq 0}$ with

1. $\|x\| = 0$ iff $x = 0$
2. $\|kx\| = |k|\,\|x\|$ if $k \in \mathbb{R}$.
3. $\|x+y\| \leq \|x\| + \|y\|$.

Ex: $(\mathbb{R}, |\cdot|)$ is a normed vect sp.

Ex: $(Bdd[a,b], \|\cdot\|_{sup})$ is a normed vect. sp.

Note: $(Fun[a,b], \|\cdot\|_{sup})$ is not a normed vect. sp since: otherwise $\|f\|_{sup}$ does not exist. eg $\|\frac{1}{x}\|_{sup}$ does not exist.

Ex: $(D^0[a,b] \cap Bdd[a,b], \|\cdot\|_{sup})$ is a normed vect. sp.

Ex: $(C^1[a,b], \|\cdot\|_{C^1})$ is a normed vect. sp where $\|f\|_{C^1} = \underbrace{\|f\|_{sup}} + \underbrace{\|f'\|_{sup}}$

Note: If $f \in C^1[a,b]$ then f' is cts so by 7.37 f' has a max & min.
so $\|f'\|_{sup}$ exists.
and if $f \in C^1$ then f is cts, and has max & min so $\|f\|_{sup}$ exist.

Prop 13.21: The <u>metric</u> associated to a norm $\|\cdot\|$ is $d_{\|\cdot\|}(x,y) = d(x,y) = \|x-y\|$.

Ex: If $x, x^2 \in Bdd[0,1]$ then

$$d_{\|\cdot\|_{sup}}(x, x^2) = \|x - x^2\|_{sup} = \sup_{x \in [0,1]} |x - x^2|$$

$$= \left|\frac{1}{2} - \frac{1}{4}\right| = \frac{1}{4}.$$

Def: 13.45
Prop 13.21
Def 3.10

A sequence (x_n) in a normed vect. sp $(X, \|\cdot\|)$ is $\|\cdot\|$-convergent to $x \in X$ if $\forall \varepsilon > 0 \ \exists N \ \forall n \geq N$ have $\|x_n - x\| < \varepsilon$

and $\|\cdot\|$-Cauchy if $\forall \varepsilon > 0 \ \exists M \ \forall n, m \geq M$ have $\|x_n - x_m\| < \varepsilon$

Def 13.54: A normed vector space $(X, \|\cdot\|)$ is a <u>Banach space</u> if every $\|\cdot\|$-Cauchy sequence (x_n) in X is $\|\cdot\|$-convergent.

Ex ② Next Example:

$$g_n(x) = \sum_{j=0}^{n} 4^{-j}\left[(x-a_j)^2 \cos\left(\frac{1}{x-a_j}\right)\right]$$

Note: $g_n(x) \in D^0[0,1] \cap \text{Bdd}[0,1]$

~~so~~ and (g_n) is $\|\cdot\|_{sup}$-Cauchy

Hence: Using Thm ② above there is a limit. $u(x) \in D^0[0,1] \cap \text{Bdd}[0,1]$.

In particular:

Check! $U(x)$ is not continuous at any a_j.

and can choose $\{a_j\}$ to be all rational numbers in $(0,1)$.

eg $a_0 = \frac{1}{2}$, $a_1 = \frac{1}{3}$, $a_2 = \frac{2}{3}$,

$a_3 = \frac{1}{4}$, ...

Result! $U(x)$ is a derivative but not continuous at any rational value in $(0,1)$.

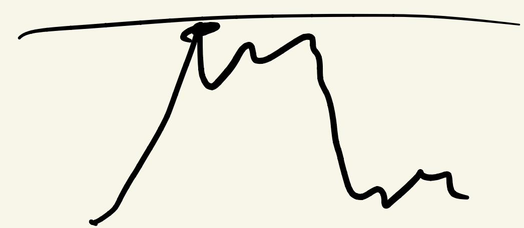
Check: $W(x)$ is not differentiable at 0.

Assume for contradiction $W'(0)$ exists.

Note: $\forall x$ have $|W(x)| \leq \sum_{j=0}^{\infty} 2^{-j} = 2$.

but also $W(0) =$ " ".

So $W'(0) = 0$



but choose $x_n = \frac{\pi}{3^n}$

so $$\lim_{n\to\infty} \frac{w(x_n) - w(0)}{x_n - 0} = w(0) = 0$$

$$\|$$

$$\frac{w\left(\frac{\pi}{3^n}\right) - 2}{\frac{\pi}{3^n}} = \frac{(2-2^n) - 2}{\frac{\pi}{3^n}} = \frac{-3^n}{\pi 2^n}$$

which does not approach 0 a contradiction

210205 Midterm next week:

See old exams on web site.

Recall: $(C^0[0,1], \|\cdot\|_{sup})$ is a Banach space

Thm: $(C^2[0,1], \|\cdot\|_{C^2})$ is a Banach space

if $\|f\|_{C^2} = \|f\|_{sup} + \|f'\|_{sup} + \frac{1}{2}\|f''\|_{sup}$

Eg: $\|x\|_{C^2} = \|x\|_{sup} + \|1\|_{sup} + \frac{1}{2}\|0\|_{sup} = 2$

in $C^2[0,1]$

$\|x^2\|_{C^2} = \|x^2\|_{sup} + \|2x\|_{sup} + \frac{1}{2}\|2\|_{sup} = 4$

Last: check $\|\cdot\|_{C^2}$-Cauchy $\overset{?}{\Longrightarrow}$ $\|\cdot\|_{C^2}$-convergent

If (f_n) is $\|\cdot\|_{C^2}$-Cauchy then

$\forall \varepsilon > 0 \ \exists M \ \forall n,m \geq M$ have $\|f_n - f_m\|_{C^2} < \varepsilon$

$$= \|f_n - f_m\|_{\sup} + \|f_n' - f_m'\|_{\sup} + \frac{1}{2}\|f_n'' - f_m''\|_{\sup}$$

so (f_n) is $\|\cdot\|_{\sup}$-Cauchy hence $\|\cdot\|_{\sup}$-conv to f

(f_n') " " " " to $g = f'$

(f_n'') " " " " to $h = f''$

hence by thm 9.18: $g = f'$

and " " ; $h = g' = f''$

Hence:

$\forall \varepsilon > 0 \ \exists L, M, N \ \forall l \geq L, m \geq M, n \geq N$ have

$$\| f_n - f \|_{sup} < \frac{\varepsilon}{3}, \quad \| f_n' - f' \|_{sup} < \frac{\varepsilon}{3},$$

$$\| f_n'' - f'' \|_{sup} < \frac{\varepsilon}{3}$$

so

$\forall \varepsilon > 0 \ \exists R = \max\{L, M, N\} \ \forall r \geq R$ have

$$\| f_n - f \|_{C^2} < \varepsilon$$

hence (f_n) converges in $\| \cdot \|_{C^2}$-norm to f.

qed

Try to show:

Ⓐ If $f, g \in C^2[0,1]$ then
$\|fg\|_{C^2} \leq \|f\|_{C^2} \|g\|_{C^2}$.

If instead $\|f\|_{\tilde{C}^2} = \|f\|_{sup} + \|f'\|_{sup} + \|f''\|_{sup}$.

Ⓑ then $\|fg\|_{\tilde{C}^2} \nleq \|f\|_{\tilde{C}^2} \|g\|_{\tilde{C}^2}$ for
some choice of $f, g \in C^2$,

$$\left(\|f\| + \|f'\| + \tfrac{1}{2}\|f''\|\right)\left(\|g\| + \|g'\| + \tfrac{1}{2}\|g''\|\right)$$

$$\|fg\|_{C^2} = \|fg\| + \|f'g + fg'\| + \tfrac{1}{2}\|f''g + f'g' + f'g' + fg''\|$$

B Take: $f(x) = g(x) = x$

$\|x\|_{C^2} = 2$ (above) ✓.

$\|x^2\|_{C^2} = 4$ " "

$\|x^2\|_{\tilde{C}^2} = 5$ $\quad \|x\|_{\tilde{C}^2} = 2$ ✗

Notation:

* The power series about c with coefficients a_n is

$$\underbrace{\sum_{n=0}^{\infty}}_{\lim_{N\to\infty} \sum_{n=0}^{N}} a_n (x-c)^n$$

Example:

(A) $\sum_{n=0}^{\infty} x^n$ $(c=0, a_n=1)$ $\sim \frac{1}{1-x}$

(B) $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ $(c=0, a_n=\frac{1}{n!})$ $\sim e^x$

Note: (A) $\sum (\frac{1}{2})^n = 2$ converges any $|x|<1$

$\sum (2)^n = 1+2+4\ldots$ diverges $|x|\geq 1$

(B) $\sum \frac{(1)^n}{n!}$ converges to e

$$\sum \frac{(\ )^n}{n!} \quad \text{never diverge}\ ,$$

210208) Midterm Friday;

Web page: Covers through today (ch 10),
(from ch: 8, 9, 10, 13)
2 old exams up, (Hunter Notes).
Held in regular zoom session.
1 sheet notes allowed.

Use cameras.
- Try out setup in Quiz this week.
Instructions posted tomorrow.

Choose $R_S = \sup\{|x| \mid S(x) \text{ converges}\}$.

(a) Holds by definition of R_S.

(b) Assume $S(x)$ converges and show that if $|y| < |x|$ then $S(y)$ conv. absolutely ~~unif.~~

Since $S(x) = \sum a_n x^n$ converges by 4.6 there is $\max_n |a_n x^n| = M < \infty$.

Hence $|S(y)| = |\sum a_n y^n| \leq \sum |a_n y^n| = \sum |a_n x^n| \left|\frac{y}{x}\right|^n$

$$\leq M \frac{1}{1 - \left|\frac{y}{x}\right|} < \infty .$$

(c) As in 9.21. For unif. conv it suffices to check: $\left\| \sum_{n=N}^{\infty} a_n y^n \right\|_{sup} \leq \sum_{n=N}^{\infty} \| a_n y^n \|_{sup}$

$$\leq M \frac{1}{1-\left|\frac{y}{x}\right|} \left|\frac{y}{x}\right|^N \xrightarrow[N\to\infty]{} 0$$

Examples to find radius of conv (R).

① $\sum_{n=0}^{\infty} x^n \underset{\text{Tay-Lag}}{=\!=} \frac{1}{1-x}$ if $x \in (-1,1)$ $\quad R=1$

② $\sum \frac{x^n}{n!} \underset{\text{Tay-Lag}}{=\!=} e^x$ if $x \in \mathbb{R}$ $\quad R=\infty$

③ $\sum n!\, x^n = 1$ if $x=0$ and div. otherwise. $\quad R=0$

④ $\sum (n+1)x^n = \frac{1}{(1-x)^2}$ if $x \in (-1,1)$ $\quad R=1$

⑤ $\sum \frac{1}{n} x^n = -\ln(1-x)$ if $x \in (-1,1)$ $\quad R=1$

⑥ $\sum 2^n n^2 x^n$ $\quad x \in \left(-\frac{1}{2}, \frac{1}{2}\right)$ $\quad R=\frac{1}{2}$

⑦ $\sum \left(\frac{x}{2}\right)^n = \frac{1}{1-\frac{x}{2}} = \frac{2}{2-x}$ if $x \in (-2,2)$, $R=2$

⑧ $\sum \frac{1}{2}\left(\frac{x+1}{2}\right)^n = \frac{1}{2\left(1-\frac{x+1}{2}\right)} = \frac{1}{1-x}$ if $x \in (\underset{\substack{\parallel \\ -1-2}}{-3}, \underset{\substack{\parallel \\ -1+2}}{1})$, $R=2$

⑨ $\sum 2^n \sin(n) x^n$ $\quad R \geq \frac{1}{2}$

using comparison and ratio test

210210 Today: More power series.

Thm 10.22: If $S(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$

is a limit of a power series with radius of convergence $R_s > 0$

then $S'(x) = \sum_{n=0}^{\infty} a_n n (x-c)^{n-1} = \sum_{n=0}^{\infty} a_{n+1}(n+1)(x-c)^n$

is also a limit of a power series with $R_{s'} = R_s$.

Proof: Use 9.18: If $(f_n') \xrightarrow{\text{unif}} g$, $f_n \xrightarrow[\text{pt}]{\text{at a}} f$

then $f' = g$.

Recall 10.3 $\left(\sum_{n=0}^{N} a_n (x-c)^n\right) \xrightarrow{\text{unif}} S(x)$

in $[c-r, c+r]$ if $r < R_S$.

(not unif. in all of $(c-R_S, c+R_S)$ where it converges ptwise).

Check (similar to last time):

If $S(x)$ converges and $|y-c| < |x-c|$

then $\sum a_{n+1}(n+1)(y-c)^n$ converges.

Hence by 10.3 again $\sum a_{n+1}(n+1)(x-c)^n = T(x)$,

conv. unif. in $[c-r, c+r]$.

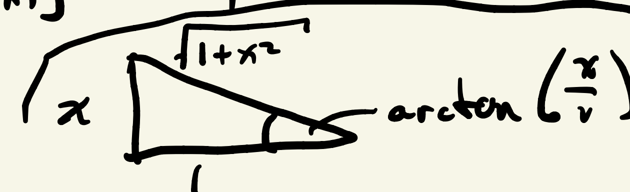
Therefore by 9.18 have $S'(x) = T(x)$.

Approximate $\frac{\pi}{2\sqrt{3}}$ by a sequence of rational numbers.

Recall: (8.23):

$$(\arctan(x))' = \frac{1}{\tan'(\arctan(x))}$$

$$= \frac{1}{\sec^2(\arctan(x))} = \frac{1}{1+x^2}$$



$$= \sum_{n=0}^{\infty} (-x^2)^n \quad \text{with } R = 1$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\text{so} \quad \arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

$$\text{if} \quad |x| < 1 = R.$$

$$\frac{\pi}{6} = \arctan\Big(\underbrace{\tfrac{1}{\sqrt{3}}}_{<1}\Big) = \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{1}{\sqrt{3}}\right)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)\, 3^n \sqrt{3}}$$

$$\frac{\sqrt{3}\,\pi}{2\cdot 3} = \frac{\pi}{2\sqrt{3}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)\,3^n}$$

Prop 10.25: If $f(0)=1$ and $f'(x)=f(x)$,

then $f(x) = Ce^x$.

Proof: $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

and $(e^x)' = \sum_{n=1}^{\infty} n\,\frac{x^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$= e^x$

Also $e^x \neq 0$.

Compute: $$\left(\frac{f(x)}{e^x} \right)' = \frac{f'(x)e^x - f(x)(e^x)'}{(e^x)^2}$$

$$= 0$$

So by 8.34: $$\frac{f(x)}{e^x} = C$$ a constant.

so $$f(x) = Ce^x$$ end.

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Derivative: ~ limit of diffence quotient

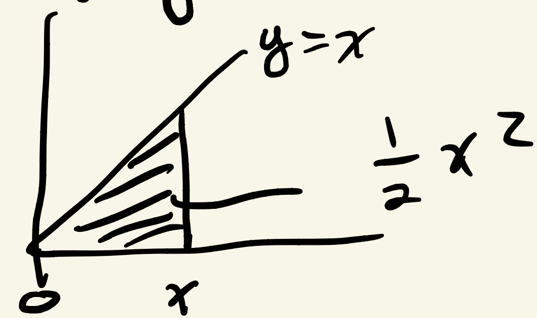
~ slope.

$$
\begin{tikzcd}
D^0 & D^1 \arrow[l, "D"'] \arrow[l, dashed, bend left]
\end{tikzcd}
$$

Next: Invert the derivative.

Integrals: FTC $\quad \int_0^x f'(y)\,dy \overset{\text{often}}{=} f(x)$

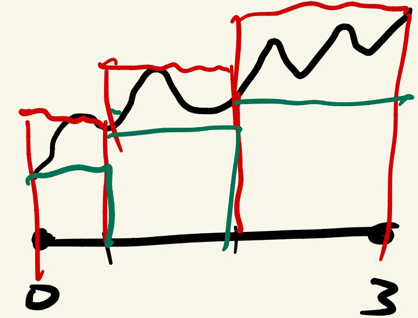
defined using area



Explore which functions can be integrated.

Example: In $[0, 3]$

$$a = 0, \quad b = 3$$



$P \in \Pi[0,3]$

$$P = \{0, \tfrac{1}{2}, 2, 3\}$$

$$x_0 = 0, \quad x_1 = \tfrac{1}{2}, \quad x_2 = 2, \quad x_3 = 3 \qquad 3 = n$$

$I_1 = [0, \tfrac{1}{2}]$ $\quad \ell_1 = \tfrac{1}{2}$

$I_2 = [\tfrac{1}{2}, 2]$ $\quad \ell_2 = \tfrac{3}{2}$

$I_3 = [2, 3]$ $\quad \ell_3 = 1$

$Q = \{0, \tfrac{1}{2}, 1, 2, 3\}$ is a refinement of P

Def: If $f \in \mathrm{Bdd}[a,b]$ and $P \in \Pi[a,b]$

write (a) $U(f;P) = \sum_{i=1}^{n} l_i \left[\sup_{a \in I_i} f(a)\right]$

(b) $L(f;P) = \sum_{i=1}^{n} l_i \left[\inf_{a \in I_i} f(a)\right]$

(c) $U(f) = \inf_{P \in \Pi[a,b]} U(f;P) = \overline{\int}_{x=a}^{b} f(x)\,dx$

(d) $L(f) = \sup_{P \in \Pi[a,b]} L(f;P) = \underline{\int}_{x=a}^{b} f(x)\,dx$

(e) If $U(f) = L(f)$ write

$$\int_{x=a}^{b} f(x)\,dx = \mathcal{U}(f) = L(f)$$

and call f (Riemann) integrable

and write $f \in R\,\mathrm{Int}[a,b]$

Breakout rooms: $[a,b] = [0,1]$ and $f(x) = x$

$$P_n = \left\{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1\right\} \in \Pi[0,1]$$

Find: (a) I_i (b) ℓ_i (c) $\mathcal{U}(f;P_n)$

(d) $L(f;P_n)$ (e) $\mathcal{U}(f)$

(f) For which n and m is P_n a refinement of P_m?

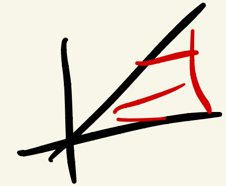
Goal: Invert derivative: $D^0 \xleftarrow{\;?\;} D^1$

Compare: $\left[\underbrace{RInt[a,b]}_{\text{integrable}} \right.$ to $\underbrace{D^0[a,b]}_{\text{derivatives.}}$

(FTC) $\left[\int_0^x f'(y)\,dy \right.$ to $f(x)$

$\left[\left(RInt[a,b], \int_a^b |\cdot|\,dx\right) \right.$ to a Banach space.

Examples:



① $\int_0^1 x\,dx = \frac{1}{2}$

② $\int_0^1 f(x)\,dx = \frac{1}{2}$

$f \in RInt[0,1]$

$f \notin D^0[0,1]$

$f(x) = \begin{cases} 1 & x \le \frac{1}{2} \\ 0 & x > \frac{1}{2} \end{cases}$

③ $\int_0^1 g(x)\,dx = 0$

$\left.\begin{array}{l} g \text{ is nonzero} \\ \int |g(x)|\,dx = 0 \end{array}\right]$ Not Banach,

$g(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases}$

④ $\int_0^1 h(x)\,dx =$ Does not exist

$h(x) \doteq \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$

$\left.\overline{\int_0^1} h(x)\,dx = U(h) = 1\right]$ $h \notin RInt[0,1]$,

$\int_0^1 h(x)\,dx = L(h) = 0$)

2/02/19

Thm 11.28: $C^0[0,1] \subseteq R\,Int[0,1]$

(step toward showing $C^0[0,1] \subseteq D^0[0,1]$)

Thm: If $\|f\|_{L^1} = \int_0^1 |f(x)|\,dx$

then $(C^0[0,1], \|\cdot\|_{L^1})$ is a normed linear space but not Banach.

(Next week).

Cauchy Criteria (for proof of 11.28).

Def: $\mathrm{RInt}[0,1] = \{ f \in \mathrm{Bdd}[0,1] \mid \underset{\overline{\int_0^1} f(x)\,dx}{\overset{\shortparallel}{U(f)}} = \underset{\underline{\int_0^1} f(x)\,dx}{\overset{\shortparallel}{L(f)}} \}$

$$\overset{11.23}{=} \{ f \in \mathrm{Bdd}[0,1] \mid \forall \varepsilon > 0 \ \ \exists P_\varepsilon \in \Pi[0,1] \text{ with } U(f;P_\varepsilon) - L(f;P_\varepsilon) < \varepsilon \}$$

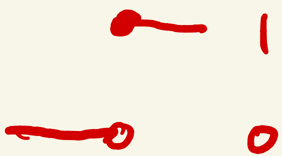
$$\overset{11.26}{=} \{ f \in \mathrm{Bdd}[0,1] \mid \exists (Q_n)_{n \in \mathbb{N}},\ Q_n \in \Pi[0,1] \text{ with } \lim_{n \to \infty} (U(f;Q_n) - L(f;Q_n)) = 0 \}$$

② 

$U(f;R_m) - L(f;R_m) = \frac{1}{m}$

so since $\lim_{m\to\infty} \frac{1}{m} = 0$

get: $f(x) \in RInt[0,1]$.

③ 

$U(f;R_m) - L(f;R_m) = \frac{1}{m}$

$0+0+0 \cdots +0+\frac{1}{m}+\frac{1}{m}+\cdots+\frac{1}{m}$

$0+0+0 \cdots +0+0+\frac{1}{m}+\cdots+\frac{1}{m}$

so since $\lim_{m\to\infty} \frac{1}{m} = 0$

get $f(x) \in RInt[0,1]$

④

$f(x)$ — 1

R_m — 0

$$\underbrace{U(f;R_m)}_{\frac{1}{m}+\cdots+\frac{1}{m}} - \underbrace{L(f;R_m)}_{0+0+\cdots+0} = 1$$

but $\lim_{m\to\infty} 1 = 1 \neq 0$

so can not conclude ~~that~~ $f(x)$ is in $RInt[0,1]$.

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$$\underbrace{C^0[0,1] \subseteq \mathrm{RInt}[0,1]}_{\mathbb{R}\text{- vector space}} \xrightarrow[\text{linear map}]{\int \cdot \, dx} \mathbb{R}$$

monotone.

Prop 11.21: If P, Q are partitions of $[a,b]$ and $f \in \mathrm{Bdd}[a,b]$ then $L(f;P) \leq U(f;Q)$

Proof: $L(f;P) \underset{\substack{11.20\\ (b)}}{\leq} L(f;P \cup Q) \underset{\text{def}}{\leq} U(f;P\cup Q) \underset{\substack{11.20\\ (a)}}{\leq} U(f;Q)$

Prop 11.22: If $f \in \mathrm{Bdd}[a,b]$ then

$$L(f) \leq U(f).$$

Proof: Assume for contradiction that $L(f) = U(f) + 2\varepsilon$ $\quad \varepsilon > 0$

Hence $\exists\, P, Q$ with:

$$L(f) < L(f;P) + \varepsilon$$

$$U(f) > U(f;Q) - \varepsilon$$

and $L(f) < L(f;P) + \varepsilon \underset{11.21}{\leq} U(f;Q) + \varepsilon < U(f) + 2\varepsilon$

$= L(f)$

contradiction,

11.33: If $f, g \in \mathrm{RInt}[a,b]$

then $f+g \in \mathrm{RInt}[a,b]$

and $$\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

Proof: Compute:

$$\overline{\int_a^b} [f(x)+g(x)]\,dx = \inf_P \sum_{i=1}^n l_i \sup_{x \in I_i} (f(x)+g(x))$$

$$\leq \inf_P \sum_{i=1}^n l_i \left[\sup_{x \in I_i} f(x) + \sup_{y \in I_i} g(y)\right]$$

$U(f+g)$

$$= \inf_P \sum_{i=1}^n l_i \sup_{x\in I_i} f(x) + \inf_P \sum_{i=1}^n l_i \sup_{y\in I_i} g(y)$$

$$= \overline{\int_a^b} f(x)\,dx + \overline{\int_a^b} g(y)\,dy.$$

similarly. $L(f+g) = \underline{\int_a^b} [f(x)+g(x)]\,dx \geq \underline{\int_a^b} f(x)\,dx + \underline{\int_a^b} g(y)\,dy$

But $f, g \in RInt[a,b]$

so $U(f+g) \leq \int_a^b f(x)\,dx + \int_a^b g(y)\,dy \leq L(f+g) \underset{11.22}{\leq} U(f+g)$

Hence the $\leq$ are all $=$.

and $U(f+g) = L(f+g)$ and $f+g \in RInt[a,b]$

with $\int (f+g) = \int f + \int g$.

$\boxed{210224}$ More study of $RInt[a,b]$

Know: $RInt[a,b]$ is a vect. space.

11.32 / 11.33

$C^0[a,b] \subseteq RInt[a,b]$ 11.28

$Inc[a,b] \subseteq$ " " 11.30

but not above

$f(x) = \begin{cases} 1 & x = \frac{1}{2} \\ 0 & \text{else} \end{cases} \in RInt[0,1]$

$\left(\int_0^1 f(x)\,dx = 0 \right)$

Def: If $f \in RInt[a,b]$ then

$$\|f\|_{L^1} = \int_a^b |f(x)|\,dx$$

Recall (13.20): If X is a $\mathbb{R}$-vector space then $(X, \|\cdot\|)$ is a normed linear space if

zero: If $\|f\| = 0$ then $f = 0$. ✓ for C^0 ✗

scaling: If $c \in \mathbb{R}$ then $\|cf\| = |c|\,\|f\|$ ✓

triangle: If $f, g \in X$ then
$$\|f+g\| \le \|f\| + \|g\|.$$ ✓

fails.

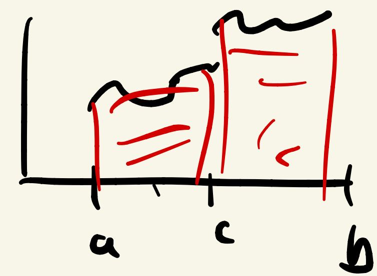
Note: $(\mathrm{RInt}[a,b], \|\cdot\|_{L^1})$ is not a normed linear space since if $f = \begin{cases} 1 & x = \frac{1}{2} \\ 0 & \text{else} \end{cases}$

then as above $\|f\|_{L^1} = \int_0^1 |f|\,dx = \int_0^1 f(x)\,dx = 0$ but $f \neq 0$.

Other Riemann. Integrable functions.

Claim (11.44) If $a \le c \le b$ then $f \in Bdd[a,b]$ is R. Int. iff $f|_{[a,c]}$ and $f|_{[c,b]}$ are both R. Int.

and $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$.



Proof: Sketch:

$\Leftarrow$ Assume $f|_{[a,c]}$ and $f|_{[c,b]}$ are

12. Integ. and show:

$$L(f) = U(f) = \int f|_{[a,c]} + \int f|_{[c,b]}$$

Compute: If $\varepsilon > 0$ then

$$\int_a^c f|_{[a,c]}\,dx = L(f|_{[a,c]}) \leq L(f|_{[a,c]}; Q_1) + \varepsilon$$

$$\int_c^b f|_{[c,b]}\,dx = L(f|_{[c,b]}) \leq L(f|_{[c,b]}; Q_2) + \varepsilon$$

so $L(f) \geq L(f; Q_1 \cup Q_2) = L(f_{[a,c]}; Q_1)$

$+ L(f_{[c,b]}; Q_2) \geq L(f|_{[a,c]}) - \varepsilon + L(f|_{[c,b]}) - \varepsilon$

so $L(f) = \int_a^c f|_{[a,c]}\,dx + \int_c^b f|_{[c,b]}\,dx$

$U(f) =$ " "

is similar.

$\Longrightarrow$ is also similar.

210226) Riem. Int. fns:

11.28 $C^0[a,b] \subseteq RInt[a,b]$

11.30 $Inc[a,b] \subseteq RInt[a,b]$

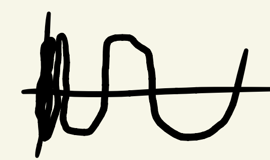
11.44 $f \in RInt[a,b]$ iff $f|_{[a,c]}$ $\{ f|_{[c,b]}$ are

Ex: Thomae: (HW):

$$T(x) = \begin{cases} \frac{1}{n} & \text{if } x = \frac{a}{n} \\ 0 & \text{else} \end{cases} \in RInt[0,1]$$

$$\chi_{\mathbb{Q}}(x) = \begin{cases} 1 & \text{if } x = \frac{a}{n} \\ 0 & \text{else} \end{cases} \notin \text{ " \quad " }$$

Question: $f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$



is $f(x) \in R\,Int[0,1]$?

$$g(x) = \begin{cases} \sin \dfrac{1}{\sin(\frac{1}{x})} & x \neq \frac{1}{n\pi} \text{ or } 0 \\ 0 & \text{else} \end{cases}$$

is $g(x) \in R\,Int[0,1]$?

Note: Write $\operatorname{disc}(f) = \{x \mid f \text{ is not cts at } x\}$.

If $f \in C^0[a,b]$ $\quad \operatorname{disc}(f) = \emptyset$

- If f as above then $\operatorname{disc}(f) = \{0\}$
- If g as above then $\operatorname{disc}(g) = \left\{\frac{1}{n\pi}\right\} \cup \{0\}$
- $\operatorname{disc}(\chi_{\mathbb{Q}}) = [0,1]$
- $\operatorname{disc}(T) = \mathbb{Q} \cap [0,1]$

Cor: 11.53: If $f \in \text{Bdd}[a,b]$ and $\text{disc}(f)$ is a finite set then $f \in \text{R Int}[a,b]$

Proof: Skip: Use: Prop 11.50 and Prop 11.44.

Show: $g(x) = \begin{cases} \sin\left(\frac{1}{\sin \frac{1}{x}}\right) & x \neq \frac{1}{n\pi} \text{ or } 0 \\ 0 & \text{else} \end{cases}$.

is R. Int.

using both 11.50 and 11.53

Proof: $\text{dis}(g) = \{\frac{1}{n\pi}\} \cup \{0\}$.

$\forall\, \varepsilon > 0$ $\text{disc}(g|_{[\varepsilon,1]})$ is finite.

hence by 11.53 $g|_{[\varepsilon,1]} \in RInt[\varepsilon,1]$.

hence by 11.60 $g \in RInt[0,1]$.

Thm 11.61: There is a collection Z of subsets of $[a,b]$ (called zero measure) so that if $f \in Bdd[a,b]$ then $f \in RInt[a,b]$ iff $\text{disc}(f) \in Z$.

Note: - every countable subset of $[a,b]$ is in Z.
- there are uncountable subsets of $[a,b]$ also in Z.

eg a ~~the~~ 0-measure Cantor set.

(see MAT 205 Meas Thy).

Eg: $f(x) = x^2$ in $Bdd[0,1]$

$$(Df)(x) = (x^2)' = 2x$$

$$(Jf)(x) = \int_0^x y^2\,dy = \frac{1}{3}x^3$$

$$(DJf)(x) = x^2 = f(x)$$

$$(JDf)(x) = x^2 = f(x)$$

Question for which functions f

do we get

$$(DJf) = f$$
$$(JDf) = f$$

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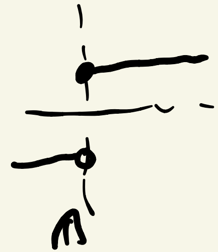
$$\{f \in C^1[a,b] \mid f(a) = 0\}$$



$$|x|$$

$$\begin{tikzcd}
C^0[a,b] \arrow[d, phantom, "\cap"] & & |x| \arrow[d, phantom, "\in"] & & \{f \in C^1[a,b] \mid f(a)=0\} \arrow[d, phantom, "\parallel"] \\
RInt[a,b] \arrow[rr, "12.6 \quad J"] \arrow[dr, phantom, "\text{cor to } 12.4"] & & C^0[a,b] \arrow[rr, shift right, "J"', "\text{bijective}"] \arrow[d, phantom, "\parallel"] & & C_0^1[a,b] \arrow[ll, shift right, "D"', "8.35 \text{ inclusion}"] \arrow[d, phantom, "\cap"] \\
& D^0[a,b] \arrow[rrr, leftrightarrow, "D"] & & & D_b^1[a,b]
\end{tikzcd}$$

$$\text{before} \longleftarrow\!\!+ \; x^2 \sin\frac{1}{x}$$

Cor to 12.4:

If $f \in C^0[a,b]$ then $f \in D^0[a,b]$.

Proof: f is the derivative of $Jf(x) = \int_a^x f(t)\,dt$.

by 12.4.

Cor to 12.4 and 8.35

$D: C^1_0[a,b] \longrightarrow C^0[a,b]$ is a bijection.

Proof: By 12.4 D is onto since

$D(Jf) = f$.

By 8.35 D is injective since

if $Df = Dg$ or $f' = g'$

then $f = g + c$

but sine $\underset{\displaystyle g(a)+C}{\overset{\shortparallel}{f(a)}} = g(a) = 0$ have $C = 0$ so $f = g$.

210303 § 12.2: Algebraic prop. of integrals:

Product Rule: 8.19 for derivs.

Becomes: Integration by Parts 12.10.

Recall: Thm 12.1 (FTC):

If f is differentiable in (a,b)

and continuous in $[a,b]$

and f' is R. Integrable in $[a,b]$

then $\int_a^b f'(t)\,dt = f(b) - f(a) = f\Big|_a^b$

Thm 12.10: If $f, g \in D'(a,b) \cap C^\circ[a,b]$
and $f', g' \in R\,Int[a,b]$ hyp for 12.1

then: $$\int_a^b f(t)\, g'(t)\,dt = -\int_a^b f'(t)\, g(t)\,dt + f(t)g(t)\Big|_a^b$$

Proof: Recall: 8.19: $(fg)' = f'g + fg'$

want ~~this fn.~~ $f(t)g(t)$ to satisfy the hyp of 12.1.

- $f(t)g(t) \in D'(a,b)$ by 8.19
- $f(t)g(t) \in C^\circ[a,b]$ from Ch. 7.
- by hyp. $f' \in R\,Int[a,b]$

since g is cts by 11.28 have $g \in RInt[a,b]$

by HW 7.4.6 have $f'g \in RInt[a,b]$

and by 11.33 have seems so $f'g + fg' \in RInt[a,b]$

Appy 12.1 to fg:

$$\int_a^b [f(t)g(t)]' dt = f(t)g(t)\Big|_a^b$$

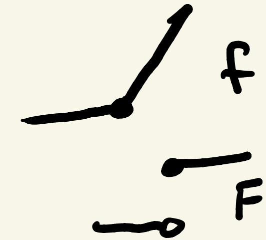
$$\int_a^b [f'(t)g(t) + f(t)g'(t)]\,dt$$

$$= \int_a^b f'(t)g(t)\,dt + \int_a^b f(t)g'(t)\,dt$$

hence $\int_a^b f'(t)g(t)\,dt = -\int_a^b f(t)g'(t)\,dt + f(t)g(t)\Big|_a^b$

Example: If $f(x) = \begin{cases} x & x \geq 0 \\ 0 & x < 0 \end{cases} \in RInt[-1,1]$

and $F(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases} \in RInt[-1,1]$



then F is a weak der. of f.

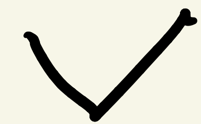
Proof: Compute: Choose g as above.

LHS: $\displaystyle\int_{-1}^{1} f(t)\, g'(t)\, dt = \int_0^1 t\, g'(t)\, dt$

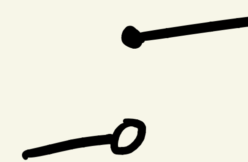
$$\overset{\text{parts}}{=} -\int_0^1 1 \cdot g(t)\, dt + \cancel{t g(t)}\Big|_0^1 \to 0$$

$$= -\int_{-1}^{1} F(t)\, g(t)\, dt. \qquad : \text{RHS.}$$

Example: Which has a weak derivative and what is one?



ⓐ $f(x) = |x|$ $\quad \in RInt[-1,1]$



ⓑ $k(x) = \left\{\begin{array}{ll} 1 & x > 0 \\ 0 & x < 0 \end{array}\right\} \in$ " "

Ans: ⓐ $\displaystyle\int_{-1}^{1} |t|\, g'(t)\, dt = \int_{-1}^{0} (-t)\, g'(t)\, dt + \int_{0}^{1} t\, g'(t)\, dt$

$\underline{\underline{\text{parts}}}$ $+\int_{-1}^{0} 1\cdot g(t)\,dt - \int_{0}^{1} 1\cdot g(t)\,dt = -\int_{-1}^{1} K(t)g(t)\,dt$

if $K(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$

ⓑ $\int_{-1}^{1} k(t)g'(t)\,dt = \int_{0}^{1} g'(t)\,dt = g(1) - g(0) = -g(0)$

Note: For ⓐ $f'(x) = \begin{cases} 1 & x > 0 \\ \text{undet} & x = 0 \\ -1 & x < 0 \end{cases}$ sim. to $K(x)$.

for ⓑ There is no bounded function $R(t)$

with $\int_{-1}^{1} R(t)g(t)\,dt = g(0)$.

Sequences:

Use L^1-Cauchy and L^1-convergent:

Recall:

Def: If (f_n) is a sequence in $RInt[a,b]$ then it is $\underline{L^1\text{-Cauchy}}$ if

$\forall \varepsilon > 0 \; \exists M \; \forall n,m \geq M$ have

$$\|f_n - f_m\|_{L^1} = \int_a^b |f_n(t) - f_m(t)|\,dt < \varepsilon$$

and it converges in L^1 to $f \in RInt$.

if $\forall \varepsilon > 0 \ \exists N \ \forall n \geq N$ have

$$\|f_n - f\|_{L^1} = \int_a^b |f_n(t) - f(t)| dt < \varepsilon$$

Note: If (f_n) conv. in L^1 to f

$$\left((f_n) \xrightarrow{L^1} f \right)$$

then $\int_a^b f(t) dt = \lim_{n \to \infty} \int_a^b f_n(t) dt$

210305

$$C^1\text{-conv.} \implies \text{unif conv.} \implies \text{pointwise}$$

$$\text{unif conv.} \overset{\text{new}}{\implies} L^1\text{-conv}$$

Ex 12.19: If $S(x) = \sum_{n=0}^{\infty} a_n x^n$ is a power series centered at 0 with radius of conv. R.

and $\quad |a|, |b| < R \quad$ then

$S(x)$ conv. unif in $[a,b]$

" " in L^1 in $[a,b]$

so

So $$\int_a^b S(x)\,dx = \sum_{n=0}^{\infty} a_n \int_a^b x^n\,dx$$

$$= \sum_{n=0}^{\infty} a_n \left.\frac{x^{n+1}}{n+1}\right|_a^b$$

$n+1 \to n \longrightarrow$ $$= \sum_{n=1}^{\infty} a_{n-1} \left.\frac{x^n}{n}\right|_a^b$$

Ex: 12.20!

$$\ln(2) = \sum_{n=1}^{\infty} \frac{1}{n2^n} = \frac{1}{2} + \frac{1}{8} + \frac{1}{24} + \frac{1}{64} + \cdots$$

quickly conv.

Recall! $\ln(x) = \int_1^x \frac{dt}{t}$

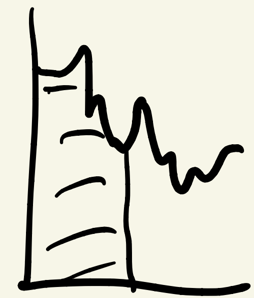
$$x = 1-y \quad \ln(1-y) = \int_1^{1-y} \frac{dt}{t}$$

$$t = 1-s, \quad dt = -ds \quad = \int_0^y \frac{-ds}{1-s} = \int_0^y \left[-\sum_{n=0}^{\infty} s^n\right] ds$$

$$= -\sum_{n=1}^{\infty} \left[\frac{y^n}{n} - \frac{0^n}{n}\right]$$

so $y = \frac{1}{2}$ $\quad -\ln(1-\frac{1}{2}) = \ln \frac{1}{1-\frac{1}{2}} = \ln(2)$

$$= \sum_{n=1}^{\infty} \frac{1}{n2^n}$$



Ex 9.24 Weierstrass:

$$\int_0^{\frac{\pi}{3}} \underbrace{\sum_{n=0}^{\infty} 2^{-n}\cos(3^n x)}_{\text{unif. conv. to a cts (but not differentiable) function}} dx$$

hence L^1 conv

$$= \sum_{n=0}^{\infty} 2^{-n} \int_0^{\frac{\pi}{3}} \cos(3^n x)\, dx$$

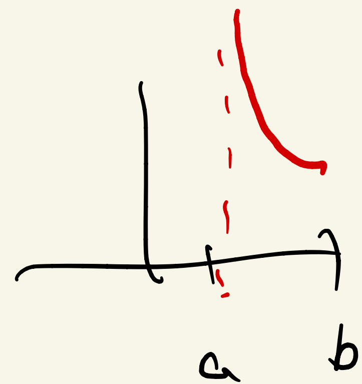
$$= \sum_{n=0}^{\infty} 2^{-n} 3^{-n} \sin(3^n x) \Big|_0^{\frac{\pi}{3}}$$

$$= 2^0 \cdot 3^0 \left[\sin\left(\frac{\pi}{3}\right) - \sin(0)\right] + \sum_{n=1}^{\infty} 0$$

$$= \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

210308

Def: If $f \in \mathrm{Fun}(a, b]$
12.24
and $\forall \varepsilon > 0$ have $f \in \mathrm{RInt}[a+\varepsilon, b]$
write $\int_a^b f(x)\,dx = \lim_{\varepsilon \to 0^+} \int_{a+\varepsilon}^b f(x)\,dx$



and call this the improper integral of f on $[a, b]$.

Note: this limit may not exist.

Def 12.26 If $f \in \mathrm{Fun}[a, \infty)$ and $\forall\, n$ have $f \in \mathrm{RInt}[a, n]$

write $\displaystyle\int_a^\infty f(x)\,dx = \lim_{n\to\infty} \int_a^n f(x)\,dx$

Def: 12.32: If $f \in \mathrm{Fun}(a, b]$

and $\forall \varepsilon > 0$ have $f \in \mathrm{RInt}[a+\varepsilon, b]$

and $\displaystyle\lim_{\varepsilon\to 0^+} \underbrace{\int_{a+\varepsilon}^b |f(x)|\,dx}_{\| f|_{[a+\varepsilon, b]} \|_{L^1}}$ exists then f is absolutely improperly integrable on $[a,b]$.

Similarly for $[a,\infty)$ analagous to 12.26.

Note: being abs. imp. int. is equiv. to $\left(f|_{[a+\frac{1}{n},\, b]}\right)$ being L^1-Cauchy.

$\underbrace{\forall \varepsilon>0,\ f\in RInt[a+\varepsilon, b]}$

(Thanks Rahul)

Thm 12.33: If $|f|\le|g|$, and

g is abs. impr. int. on $[a,b]$

then (a) f is also.

and (b) $\int_a^b f(x)\,dx$ exisits,

Proof: (a) $\int_a^b |f(x)|dx$

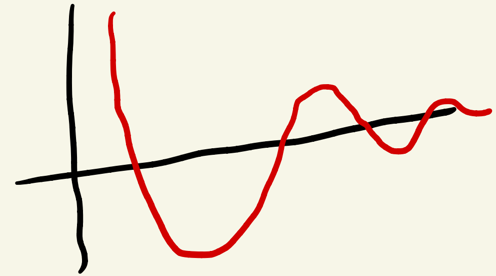
$$= \lim_{\varepsilon\to 0^+} \int_{a+\varepsilon}^b |f(x)|dx \leq \lim_{\varepsilon\to 0^+} \int_{a+\varepsilon}^b |g(x)|dx = L$$

and as $\varepsilon \to 0^+$ $\int_{a+\varepsilon}^b |f(x)|\,dx$ is monotone increasing.

Hence by 3.29 (monotone convergence thm)

$\int_a^b |f(x)|\,dx$ exists.

(b) Write $f_+(x) = \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ 0 & \text{else} \end{cases}$



and $f_-(x) = \begin{cases} -f(x) & \text{if } f(x) \leq 0 \\ 0 & \text{else} \end{cases}$

hence $f(x) = f_+(x) - f_-(x)$

and $f_+(x) = |f_+(x)|$, $f_-(x) = |f_-(x)|$

but $|f_\pm(x)| \leq |g(x)|$

so by (a) f_+ and f_- are improperly integrable.

so $\displaystyle\int_a^b f(x)\,dx = \lim_{\varepsilon\to 0^+} \int_{a+\varepsilon}^b f(x)\,dx$

$$= \lim_{\varepsilon\to 0^+} \left[\int_{a+\varepsilon}^b f_+(x)\,dx - \int_{a+\varepsilon}^b f_-(x)\,dx \right]$$

$$= \lim_{\varepsilon\to 0^+} \int_{a+\varepsilon}^b f_+(x)\,dx - \lim_{\varepsilon\to 0^+} \int_{a+\varepsilon}^b f_-(x)\,dx$$

which both exist. q.e.d.

210310

Claim: p.v. $\int_{-\infty}^{1} \frac{e^t}{t}\,dt$ exists.

First part: (improper)

$$\int_{-\infty}^{-1} \frac{e^t}{t}\,dt = \lim_{n\to\infty} \int_{-n}^{-1} \frac{e^t}{t}\,dt \le \lim_{n\to\infty} \int_{-n}^{-1} e^t\,dt$$

$$= \lim_{n\to\infty} \left[\frac{1}{e} - \frac{1}{e^n}\right]$$

$$= \frac{1}{e} \text{ exists.}$$

Note: if $t \le -1$

then $\left|\frac{e^t}{t}\right| \le e^t$

so $\int_{-\infty}^{-1} \frac{e^t}{t}\,dt$ exists.

Second part: (principal value)

Consider subtracting the singularity:

$$\tilde{f}(t) = \begin{cases} \dfrac{e^t}{t} - \dfrac{1}{t} & t \neq 0 \\ 1 & t = 0 \end{cases}$$

Check $\tilde{f}(t)$ is cts: Compute:

$$\lim_{t\to 0}\left[\frac{e^t}{t} - \frac{1}{t}\right] = \lim_{t\to 0} \frac{e^t - 1}{t} \underset{\text{L'Hô}}{=} \lim_{t\to 0} \frac{e^t}{1} = 1$$

Why is $\frac{1}{t}$ the right thing to subtract;

Recall $e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!} = 1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \cdots$

so

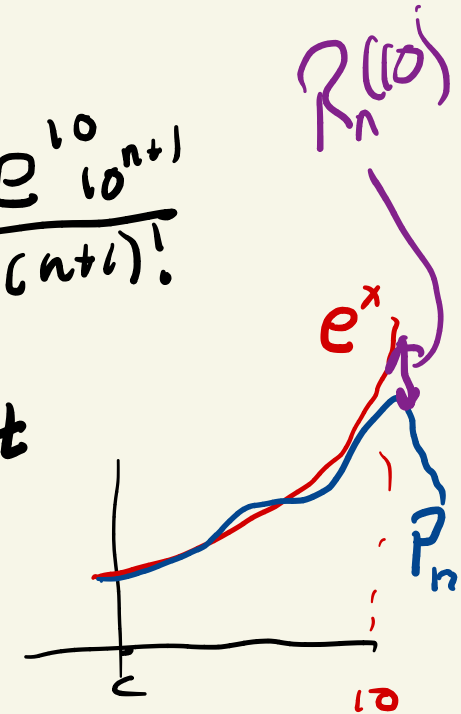
$$\frac{e^t}{t} = \frac{1}{t} + 1 + \frac{t}{2} + \frac{t^2}{6} + \cdots$$

$$= \frac{1}{t} + \underbrace{\sum_{n=0}^{\infty} \frac{t^n}{(n+1)!}}_{\text{power series}}$$

power series with $R = \infty$

so cts.

Example: Consider both formulae for $R_n(10)$ if $f(x) = e^x$ and $c=0$



Lagrange: $R_n(10) = \frac{e^{\xi}}{(n+1)!} 10^{n+1} \leq \frac{e^{10}\, 10^{n+1}}{(n+1)!}$

integral: $R_n(10) = \frac{1}{n!} \int_0^{10} e^t (10-t)^n \, dt$

so $R_n(10) \leq \frac{1}{n!} \int_0^{10} e^{10} (10-t)^n \, dt$

$$= \frac{e^{10}}{n!} \int_0^{10} s^n \, ds \qquad s = 10 - t$$

$$= \frac{e^{10}}{n!} \frac{10^{n+1}}{n+1} \qquad \text{Same as above.}$$

and $$R_n(10) \leq \frac{1}{n!} \int_0^{10} e^t \, 10^n \, dt$$

$$= \frac{10^n}{n!} \left(e^{10} - 1 \right) < \frac{10^n e^{10}}{n!}$$

$\frac{n+1}{10}$ times the above bound.

Proof of Thm 10.48:

Use induction on n

For the base case $n=0$ use FTC 12.1

For the induction step use int by parts 12.10.

Recall: FTC 12.1:

If f is cts in $[a,b]$, $f' \in RInt[a,b]$

then $\int_a^b f'(t)dt = f(t)\Big|_a^b$

Base case: $R_0(x) = f(x) - P_0(x) = f(x) - f(c)$

$$= f(t)\Big|_c^x \underset{FTC}{=\!=} \int_c^x f'(t)\,dt$$

$$= \frac{1}{0!}\int_c^x f^{(1)}(t)\,(x-t)^0\,dt$$

Induction step:

Recall 12.10: If u, v are cts in $[a,b]$ and $u', v' \in RInt[a,b]$ then

$$\int_a^b u(t)\,v'(t)\,dt = -\int_a^b u'(t)\,v(t)\,dt + \big[u(t)v(t)\big]\Big|_a^b$$

210312) Office hrs Monday at usual class time, email otherwise.

Summary of 127B:

Spaces:

MVT / Darboux give restrictions

$$
\begin{tikzcd}
 & & & \overbrace{\phantom{C^{\omega}}}^{\substack{\text{power ser.} \\ \mathbb{R}\text{-analytic}}} \\
 & & & C^{\omega}[0,1] \arrow[d, phantom, "\cap" description] \\
D^{0}[0,1] \arrow[d, phantom, "\cup" description] & D^{1}[0,1] \arrow[l, "D"'] \arrow[d, phantom, "\cup" description] & & C^{\infty}[0,1] \\
C^{0}[0,1] & C^{1}[0,1] \arrow[l, "D"'] \arrow[r, phantom, "\cap \quad \cdots"] & &
\end{tikzcd}
$$

$$\underset{\cong}{\mathrm{RInt}[0,1]} \xrightarrow[J]{\text{bijectn.}} \underset{\cong}{\underbrace{C^0_0[0,1]}_{f(0)=0}}$$

Good examples:

9.6: $f_n(x) = \dfrac{x^2}{\sqrt{x^2+\frac{1}{n}}}$ $\quad V \longrightarrow V \quad |x|$

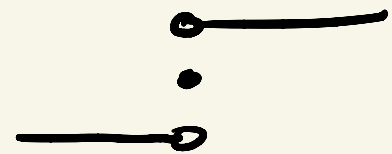
9.24: $\displaystyle\sum_{n=0}^{N} 2^{-n}\cos(3^n x) \longrightarrow W(x)$

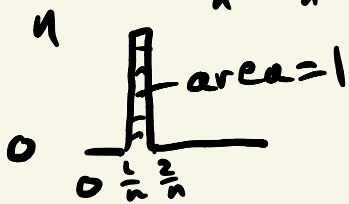
8.9/10 $\qquad x^a \sin\frac{1}{x}$

10.1 $\sum a_n x^n$ eg: e^x, $\sin(x)$, $\frac{1}{1-x}$

11.15 $f_n = \begin{cases} 1 & \text{fin many} \\ 0 & \text{else} \end{cases}$ $\longrightarrow$ Dir $\begin{cases} 1 & x \in \mathbb{Q} \\ 0 & \text{else} \end{cases}$

11.16 " " $\longrightarrow$ Thomae $\begin{cases} \frac{1}{n} & x = \frac{a}{n} \\ 0 & \text{else} \end{cases}$

13.89  $\longrightarrow$ 

12.22  $\longrightarrow$ 0

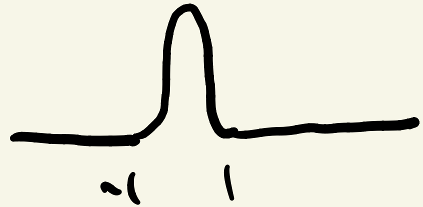
$\left.\frac{1}{x^p}\right]$ improp int? depends on p.

12.25

10.33 $\left\{ e^{\frac{1}{x^2-1}} \right.$ $|x| \leq 1$

1

0



$\{\, 0 \quad$ else

smooth. not $\mathbb{R}$-analytic.
with compact support.

Problem:

Show if $f \in \text{Bdd}[-1,1]$

then (a) $xf(x)$ is cts at 0

(b) $x^2 f(x)$ is differentiable at 0.

(c) Find $f \in \text{Fun}[-1,1]$ with

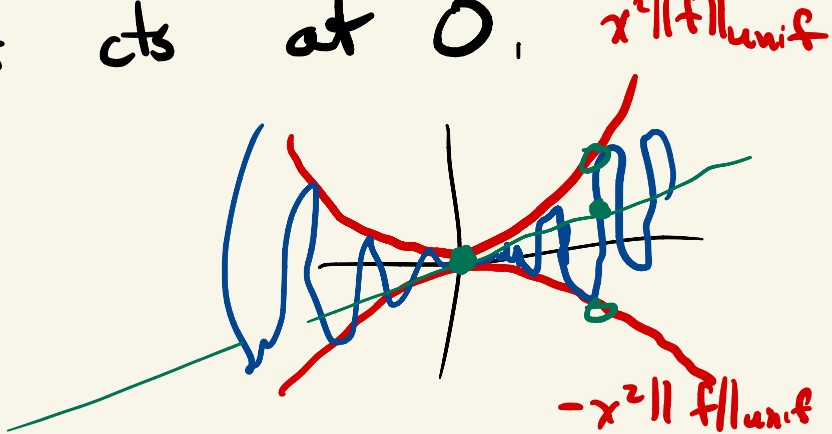
$x f(x)$ not cts at 0.

Ans: (c) $f(x) = \begin{cases} \frac{1}{x^2} & x \neq 0 \\ 5 & x = 0 \end{cases}$

so $x f(x) = \begin{cases} \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$

which is not cts at 0.

(b) Sketch:



Since $f \in \mathrm{Bdd}[-1,1]$ have

$\sup_{x \in [-1,1]} |f(x)| = \|f\|_{\sup} = M$ is finite.

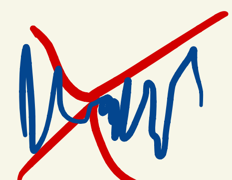
so $|x^2 f(x)| \leq x^2 M$

so $$[x^2 f(x)]'(0) = \lim_{h \to 0} \frac{h^2 f(h) - 0}{h} = \lim_{h \to 0} h f(h)$$

so by sandwich:

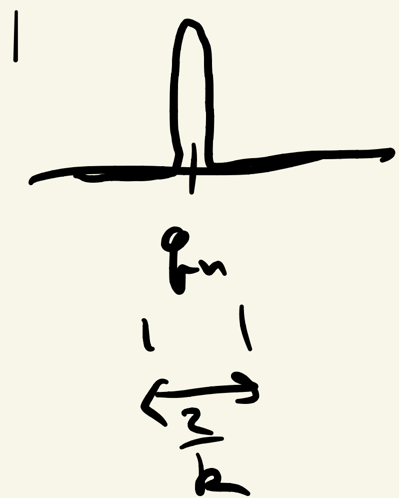
$$0 = -\lim_{h \to 0} hM \leq (x^2 f(x))'(0) \leq \lim_{h \to 0} hM = 0$$

(a) similar



Now consider $g_{n,k}$ with support in $[q_n - \frac{1}{k}, q_n + \frac{1}{k}]$

and $g_{n,k}(q_n) = 1 = \|g_{n,k}\|_{\sup}$



then take

$$G_N = \sum_{n=1}^{N} g_{n,n^3}$$

Check: L^1-Cauchy