MAT 127B-A

Winter 2021

Right Board

Overview:

Define (again) Derivative
Integral

Study spaces of functions

- Consider nice functions
  - polynomials
  - trig polynomials
- Consider limits of these to get more functions

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$$

useful as a soln to a diff eqn
common in phys.

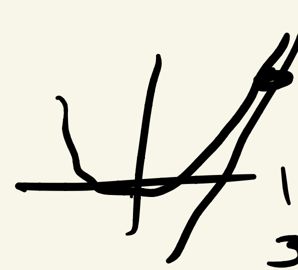
Weierstrass fn:

$$W(x) = \cos(x) + \frac{1}{2}\cos(3x) + \frac{1}{4}\cos(9x) + \cdots$$

extremely jagged.

Similar functions come up in physics
as paths of particles in a liquid.

Example:



① $f_1(x) = x^2$

$f_1'(3) = 2\cdot 3 = 6$ (rule) $(x^2)' = 2x$

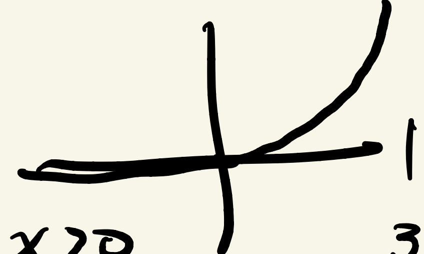
or using def:

$$f_1'(3) = \lim_{h\to 0} \frac{f_1(h+3) - f_1(3)}{h} = \lim_{h\to 0} \frac{(h+3)^2 - 3^2}{h}$$

$$= \lim_{h\to 0} \frac{h^2 + 6h + 9 - 9}{h}$$

$$= \lim_{h\to 0} (h+6) = 6$$

(2) $f_2(x) = \begin{cases} x^2 & x>0 \\ 0 & x=0 \\ 0 & x<0 \end{cases}$



3

Compute $f_2'(x) = \begin{cases} 2x & x>0 \\ 0 & x=0 \\ 0 & x<0 \end{cases}$

Since derivatives are local rules work in intervals

But at $x=0$ use the def

$$f_2'(0) = \lim_{h\to 0} \frac{f_2(h+0)-f_2(0)}{h} = \lim_{h\to 0} \frac{f_2(h)}{h}$$

Recall: $\lim_{x\to a} F(x) = L$ iff

$$\lim_{x\to a^+} F(x) = L = \lim_{x\to a^-} F(x).$$

here:

$$\lim_{h\to 0^+} \frac{f_2(h)}{h} = \lim_{h\to 0^+} \frac{h^2}{h} = \lim_{h\to 0^+} h = 0$$

$$\lim_{h\to 0^-} \frac{f_2(h)}{h} = \lim_{h\to 0^-} \frac{0}{h} = \lim_{h\to 0^-} 0 = 0$$

g_1

cts at 0

not diff at 0 since:

g_2:

$$g_2'(0) = \lim_{h \to 0} \frac{g_2(h)}{h}$$

use squeeze thm:

$$0 \leq g_2(h) \leq \cancel{x^3}\ h^2$$

so

$$0 \leq \frac{g_2(h)}{h} \leq h$$

$$\lim_{h\to 0} 0 = 0 \leq \lim_{h \to 0} \frac{g_2(h)}{h} \leq \lim_{h\to 0} h = 0$$

2 ✓

210106

Notation:

If f is cts in (a,b) write $f \in C(a,b)$
$= C^0(a,b)$

If f is diff in (a,b) write $f \in D^1(a,b)$

If f is ctsly diff in (a,b) write $f \in C^1(a,b)$

Picture:

$$C^0(a,b) \overset{\text{T8.17}}{\underset{\neq}{\supseteq}} D^1(a,b) \overset{\text{deff}}{\underset{\neq}{\supseteq}} C^1(a,b) \overset{\text{T8.17}}{\underset{\neq}{\supseteq}} D^2(a,b) \cdots$$

$f(x) = |x|$ $\qquad$ $x^2 \sin \frac{1}{x}$

Containment requires proof.

Proper content (not equal) requires examples.

Ex: ① $f(x)=|x|$ has $f(x) \in C^0\mathbb{R}$

but $f(x) \notin D^1\mathbb{R}$

Ex: ② $f(x)=\begin{cases} x^2\sin\frac{1}{x} & x\neq 0 \\ 0 & x=0 \end{cases}$ has $f(x) \in D^1\mathbb{R}$

but $f(x) \notin C^1\mathbb{R}$

In break out rms:

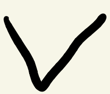
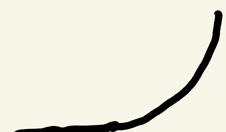
① More examples to see

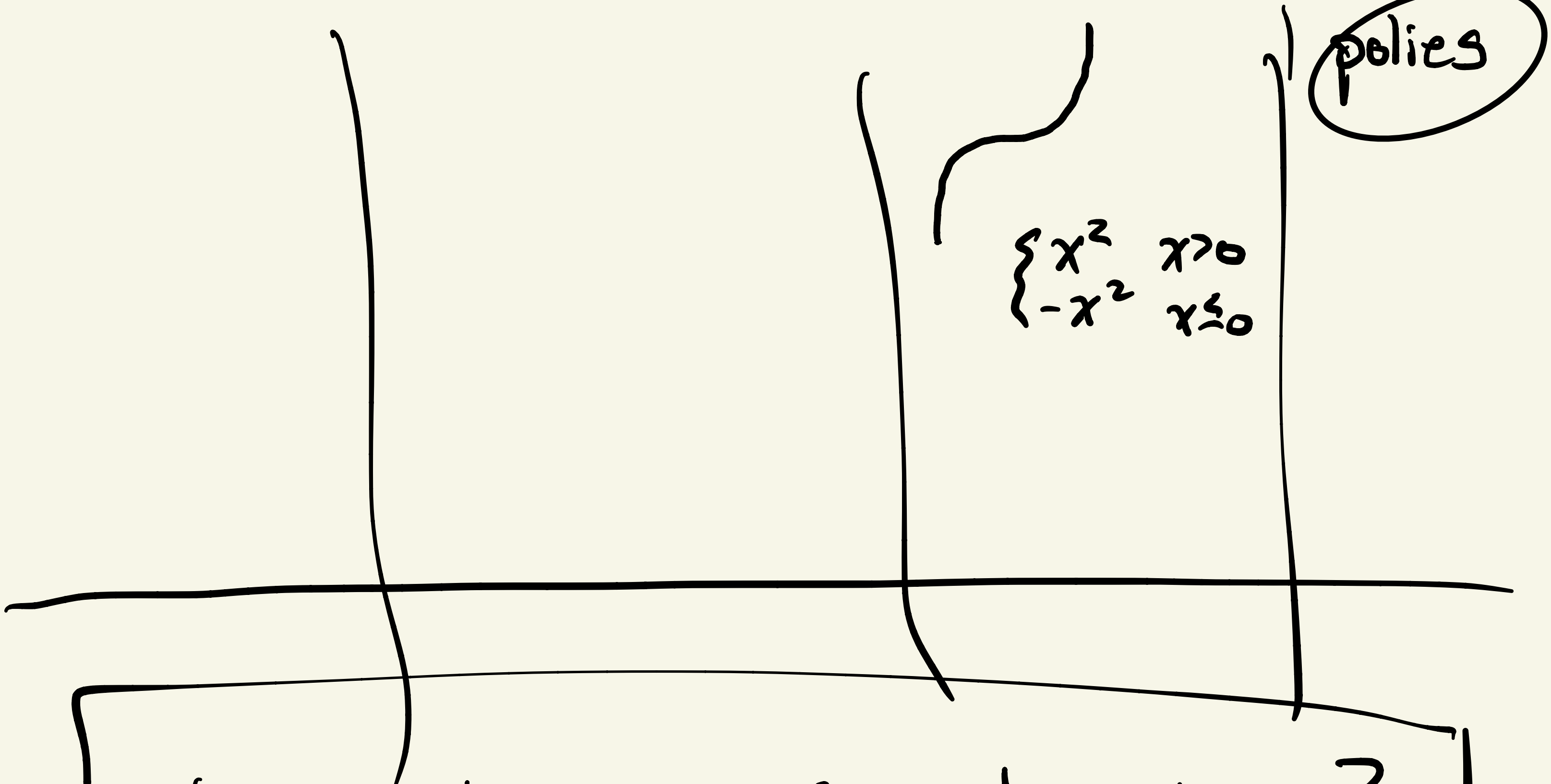
$C^0 \neq D^1$

$D^1 \neq C^1$

$C^1 \neq D^2$] New

What are C^∞ and D^∞?

C^0 not D^1	D^1 not C^1	C^1 not D^2	Same: $\underline{C^\infty}$ $\underline{D^\infty}$
 $\lvert x \rvert$	$x^2 \sin\frac{1}{x}$ 	 $\begin{cases} x^2 & x>0 \\ 0 & x \leq 0 \end{cases}$	e^x trig fns, $\frac{1}{x^n}$



$\boxed{\text{where does } x^3 \sin\frac{1}{x} \text{ live?}}$

210108

If $f, g : \mathbb{R} \to \mathbb{R}$

and $k \in \mathbb{R}$

Linearity: $k \cdot f \in \mathrm{Fun}(\mathbb{R})$

$f + g \in \mathrm{Fun}\,\mathbb{R}$

Product: $f \cdot g \in \mathrm{Fun}\,\mathbb{R}$

Composition: $g \circ f \in \mathrm{Fun}\,\mathbb{R}$

(or $f, g \in \mathrm{Fun}(\mathbb{R})$)

eg: $g = \sin(x)$

$f = \frac{1}{x}$

$k = 3$

$3 \cdot \frac{1}{x} = k \cdot f$

$\sin(x) + \frac{1}{x} = f + g$

$\frac{1}{x} \sin(x) = f \cdot g$

$g \circ f = \sin\left(\frac{1}{x}\right)$

$f \circ g = \dfrac{1}{\sin(x)}$

Continuity: If f, g cts (or $f, g \in C^0(\mathbb{R})$)

then $r \cdot f$, $f+g$, $f \cdot g$, $g \circ f$ are also

In particular if G is cts at $f(c)$

and f is cts at c, then

$$\lim_{x \to c} (G \circ f)(x) = (G \circ f)(c) = G(f(c)) = \lim_{y \to f(c)} G(y)$$

$G \circ f$ cts at c

eg: $\lim_{x \to \pi} \sin\left(\frac{1}{x}\right) = \sin \frac{1}{\pi} = \lim_{y \to \frac{1}{\pi}} \sin(y)$

$$\left[3\sin(x)\right]' = 3\sin'(x) = 3\cos(x)$$

$$\left(\sin(x) + \frac{1}{x}\right)' = \cos(x) - \frac{1}{x^2}$$

$$\left(\frac{1}{x}\sin(x)\right)' = -\frac{1}{x^2}\sin(x) + \frac{1}{x}\cos(x).$$

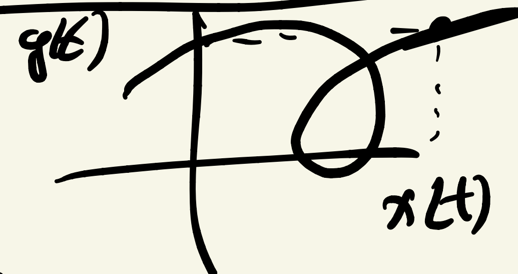
$$\left(\frac{\sin(x)}{x}\right)' = \frac{x\cos(x) - 1\cdot\sin(x)}{x^2}$$

$$\left[\sin\left(\frac{1}{x}\right)\right]' = \cos\left(\frac{1}{x}\right)\cdot\left(\frac{-1}{x^2}\right),$$

Proof of Chain Rule:

Use

Parametric Derivs!



Claim: $\lim_{s \to t} \frac{y(s) - y(t)}{x(s) - x(t)} = \frac{y'(t)}{x'(t)}$

and this is also the slope of the tangent line.

(useful for L'Hôpital rule.

if $x'(t) \neq 0$

What are possible tangent line slopes of $x'(t) = 0$?

Typically the line is vertical.

but any value is possible eg if

$y' = x' = 0$

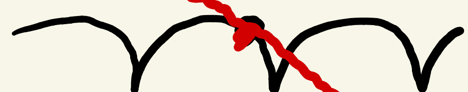
210111

Def: If $x(t), y(t) \in \mathrm{Fun}(\mathbb{R})$

then $\{(x(t), y(t)) \mid t \in \mathbb{R}\} \subseteq \mathbb{R}^2$

is the curve def. paramet. by x & y.

Ex: Cycloid:



If a wheel of radius 1 is rolling at $1 \frac{\text{rad}}{\text{sec}}$ then the posit. $P(t)$ of a point on the edge is

$$P(t) = (\overbrace{t + \cos(t)}^{x(t)}, \overbrace{1 + \sin(t)}^{y(t)})$$

center

so the posit. is given pair by x & y.

Lem: If $x(t), y(t) \in D'\mathbb{R}$
and $c \in \mathbb{R}$ has $x'(c) \neq 0$ then

$$\lim_{t \to c} \frac{y(t) - y(c)}{x(t) - x(c)} \overset{①}{=} \frac{y'(c)}{x'(c)} \overset{②}{=}$$

the slope of the line tang. to the parm. curve at $(x(c), y(c))$.

Ex: ② Find the slope of the motion of the point above when it is at the same height as the center

Derivatives of inverse functions:

Switch roles of x and y.

Def: If $f, g \in \mathrm{Fun}\,\mathbb{R}$

and $(f \circ g)(x) = x$

then call g the inverse to f or $g = f^{-1}$

Lem: If $g = f^{-1}$ then $(g \circ f)(x) = x$ also

so $f = g^{-1}$.

(Recall: $f \circ g$ is not usually $g \circ f$).

eg $\sin \frac{1}{x} \neq \frac{1}{\sin(x)}$

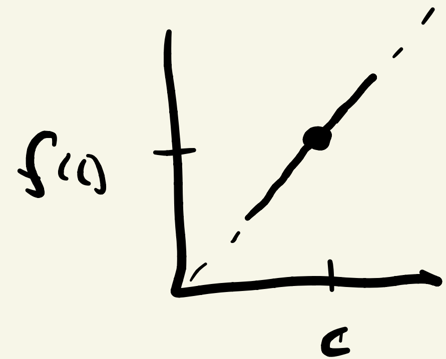
Lem: If $f = g^{-1}$ then the graphs of f and g differ by reflection over the line $x = y$.

Example:

If $f(x) = f^{-1}(x)$ and $c \in \mathbb{R}$ has $f(c) = c$ find the possible values for $f'(c)$.

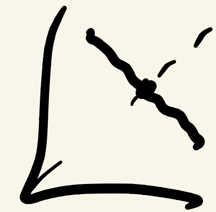
Hint: Can approach using graph or computationally.

Geometrically:



slope 1

or



slope -1

Computationally:

If $f = f^{-1}$

Lemma: If $g = f^{-1}$ so $(g \circ f)(x) = (x)$

Compute: $1 = D(x) = D(g \circ f)$

$$= [(Dg) \circ f] \cdot (Df)$$

so $(Df) = \dfrac{1}{(Dg) \circ f}$

or $(Dg^{-1})(c) = \dfrac{1}{(Dg) \circ g^{-1}(c)} = \dfrac{1}{g'(g^{-1}(c))}$

$\quad //$

$(g^{-1})'(c)$

So $\cancel{\ }$ $$(g^{-1})'(c) = \frac{1}{g'(g^{-1}(c))}$$

Now if $f = f^{-1}$ then

$$(f)'(c) = (f^{-1})'(c) = \frac{1}{f'(f^{-1}(c))}$$

and if $f^{-1}(c) = f(c) = c$

then $f'(c) = \dfrac{1}{f'(c)}$ so $[f'(c)]^2 = 1$

so $f'(c) = 1$ or -1

Spaces:

$$
\begin{array}{lll}
C^{\infty}\mathbb{R} = D^{\infty}\mathbb{R} & & \\
\cap^{+} & & \\
\vdots & & \\
C^{k}\mathbb{R} & f_{k+1}(x) & g_{2k+1}(x) \\
\cap^{+} & & \\
D^{k}\mathbb{R} & & g_{2k}(x) \\
\cap^{+} & & \\
\vdots & & \\
\cap^{+} & & \\
C^{2}\mathbb{R} & f_{3}(x) & g_{5}(x) \\
\cap^{+} & & \\
D^{2}\mathbb{R} & & g_{4}(x) \\
\cap^{+} & & \\
C^{1}\mathbb{R} & f_{2}(x) & g_{3}(x)
\end{array}
$$

$D^1\mathbb{R}$

$\cap$ $\neq$

Thm 12.6

$C^0\mathbb{R}$

$\cap$ $|$

New $D^0\mathbb{R}$

$\{g'(x) \mid g \in D^1\mathbb{R}\}$.

$\cap$ $|$

$\mathrm{Fun}\,\mathbb{R}$

$g_2(x)$

$f_1(x)$ $g_1(x)$ $W(x)$

$g_0(x)$

$f_0(x)$ $g_0^2(x)$

not a derivative. Pf on Wed.

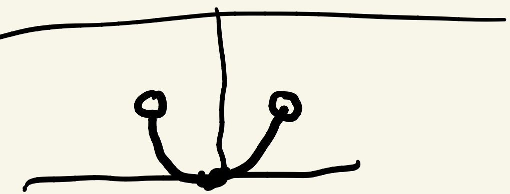
210113

Thm (8.27) (Fermat) (5.2.6 (A))
(Internal Extrema)

If $f \in D'(a,b)$

and $f(m) = \max \operatorname{Im}(f)$ (or $\min \operatorname{Im}(f)$)

then $f'(m) = 0$



e.g. $f(x) = x^2 \in D'(-1,1)$.
has min at 0
but no max

Proof:
$$f'(m) = \lim_{x \to m} \frac{f(x) - f(m)}{x - m}$$

$$f'(m) = \lim_{x\to m^+} \frac{f(x)-f(m)}{x-m} \le 0$$

$$= \lim_{x\to m^-} \frac{f(x)-f(m)}{x-m} \ge 0$$

so $f'(m)=0$

but If $x>m$ then $\frac{f(x)-f(m)}{x-m} \le 0$

since $f(m)$ is a max so $f(x)-f(m) \le 0$

similarly If $x<m$ then $\frac{f(x)-f(m)}{x-m} \ge 0$

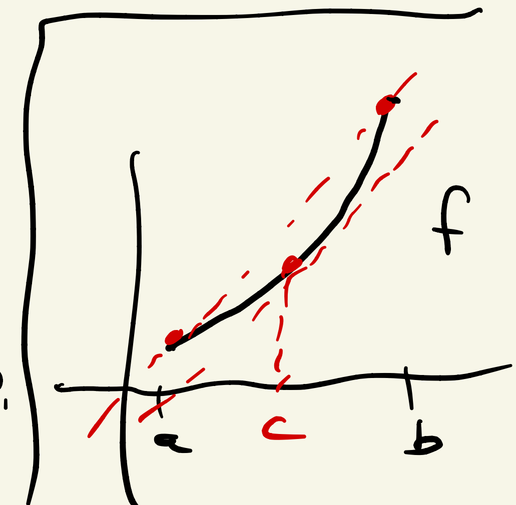
Thm (Mean Value Thm) (8.33 H)(5.3.2 A),

If $f \in C^0[a,b] \cap D'(a,b)$ then

$$\exists c \in (a,b) \text{ with } \underbrace{f'(c)}_{\substack{\text{slope of}\\ \text{tang line}\\ \text{at } c}} = \underbrace{\frac{f(b)-f(a)}{b-a}}_{\substack{\text{slope of secant}\\ \text{line at } a,b}}$$

Proof: Consider:

$$g(x) = f(x) - \underbrace{(x-a)\frac{f(b)-f(a)}{b-a}}_{\text{linear.}}$$

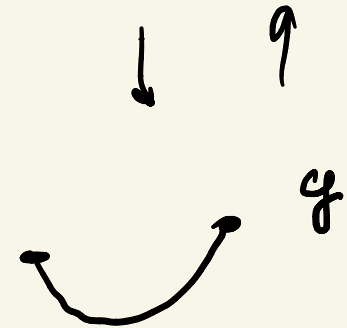


Check g satisfies the hyp of Rolle.

$g \in C^0[a,b] \cap D^1(a,b)$

since both are closed under addition and mult by const

and $f \in C^0 \cap D^1$ and $(x-a) \in C^0 \cap D^1$



Compute

$$g(a) = f(a) + 0 = f(a)$$

$$g(b) = f(b) - (\cancel{b-a})\,\frac{f(b)-f(a)}{\cancel{b-a}}$$

$$= \cancel{f(b)} - \cancel{f(b)} + f(a) = f(a)$$

Hence by Rolle's thm there is c

with $g'(c) = 0$ so $f'(c) = g'(c) + D\left[(x-a)\,\frac{f(b)-f(a)}{b-a}\right]$

$$= 0 + (1-0)\,\frac{f(b)-f(a)}{b-a}$$

6.2.10:

Need: $\forall \varepsilon > 0 \ \exists \ 0 < a < b < \varepsilon$

with $g(b) < g(a)$

Claim: $$g_0(x) = \begin{cases} \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases} \in D^0(\mathbb{R}),$$

This means: $\exists\ f \in D^1\mathbb{R}$ with $f' = g_0(x)$, Idea: ① $g_2'(x) = \begin{cases} 2x\sin\frac{1}{x} - \cos\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$

$$g_2(x) = \begin{cases} x^2 \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

or $\left\{\begin{matrix} \cancel{x^2\cos\frac{1}{x}} & x \neq 0 \\ 0 & x = 0 \end{matrix}\right\}' = \begin{cases} 2x\cos\frac{1}{x} + \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$

$$= \left\{\begin{matrix} 2x\cos\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{matrix}\right\} + g_0(x) = h(x) + g_0(x)$$

Thm: $\underline{C^0(\mathbb{R})} \subseteq D^0\underline{(\mathbb{R})}$ (12.6)

$h(x)$ is cts.

so by (12-6) $h(x) = k'(x)$

for some fn k

$$\left(\text{Hint: } k(x) = \int_0^x h(t)\,dt\right)$$

so $$g_0(x) = \{\qquad\}' - h(x) = \{x\}' - k'$$

$$= (\{x\} - k)'$$

210115

Note 9! From MAT 67 or MAT 22A Lin Alg

Fun$[a,b]$ or Fun(a,b) are $\mathbb{R}$ vector spaces,

(so can add functions or mult. by reals),

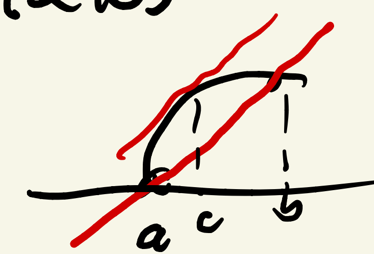
Also! every $D^n(a,b)$, $C^n(a,b)$, $C^n[a,b]$

and their intersections are vector subspaces.

Recall MVT:

$\forall f \in C^0[a,b] \cap D^1(a,b) \ \exists \ c \in (a,b)$

with $f'(c) = \dfrac{f(b) - f(a)}{b-a}$



$$\frac{-[f(b)-f(a)]g(a)-f(a)[g(b)-g(a)]}{} = \frac{0}{b-a} = 0.$$

Examples with MVT:

Ex: If $f \in D'(\mathbb{R})$

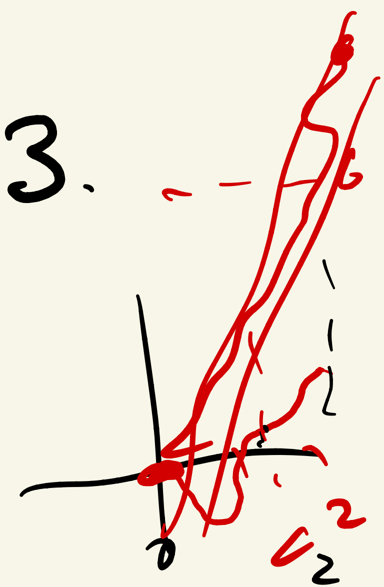
and $f(0)=0$

and $\forall x \in \mathbb{R}$, $f'(x) \le 3$.

and

Show that $f(2) \le 6$

Ans: Assume for contradiction that $f(2) > 6$.



Hence by MVT:

since $f \in D^1(\mathbb{R})$ have $f|_{[0,2]} \in C^0[0,2] \cap D^1(0,2)$

hence $\exists\, c \in (0,2)$ with

$$f'(2) = \frac{f(2)-f(0)}{2-0} = \frac{f(2)}{2} > \frac{6}{2} = 3$$

which is a contradiction

$\underline{210120}$ Consider $a < c < d < b$

Write if $f \in Fun(a,b)$ then

$$Im(f|_{[c,d]}) = \{f(x) \mid c \le x \le d\}.$$

(old)

Thms: (7.35) $\forall f(x) \in C^0(a,b)$ have $Im f_{[c,d]}$ closed and bounded

(IVP) (7.44) " " " " connected.

Hence $Im f_{[c,d]} = [f(m), f(M)]$ a closed interval or a point.

(new)

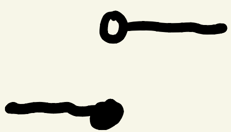
Thm: ⓐ $\exists f(x) \in D^0(a,b)$ with $Im f|_{[c,d]}$ not bounded.

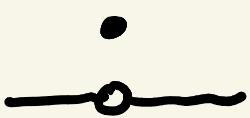
ⓑ " " " " not closed

Darboux ⓒ $\forall f(x) \in D^0(a,b)$ have $Im f|_{[c,d]}$ is connected.

First examples and then the proof of (c).
See HW for (a) & (b).

Using (c): If $\mathrm{Im}\, f|_{[a,b]}$ has exactly two values then it is not connected so $f \notin D^0(a,b)$.

Eg:  $= \begin{cases} 1 & x > 0 \\ 0 & x \leq 0 \end{cases} \notin D^0(\mathbb{R})$.

 $= \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases} \notin D^0(\mathbb{R})$.

In this case $M = c$. Compute:

$$0 < G'(c) = \lim_{x \to c^{+}} \frac{G(x) - G(c)}{x - c}$$

so $\forall \varepsilon > 0 \; \exists\, d - c > \delta > 0 \; \forall\, c < x < c + \delta$ we

have $$G'(c) - \frac{G(x) - G(c)}{x - c} < \varepsilon$$

Choose $\varepsilon = \frac{1}{2} G'(c) > 0$ so $\underset{\substack{\vee \\ 0}}{\left(1 - \frac{1}{2}\right) G'(c)\,(x - c)} < G(x) - G(c)$

hence $G(x) > G(c) = G(M)$ the contradiction desired.

210122

Thm (8.54) ($\frac{0}{0}$ L'Hospital Rule)

$\forall\, k, g \in D^1(a,b)$, $a < c < b$,

$k(c) = 0 = g(c)$ and $\lim_{x \to c} \frac{k'(x)}{g'(x)} = L$

then $\lim_{x \to c} \frac{k(x)}{g(x)} = L$

Proof: Note: $k(x) - k(c) = k(x)$

$g(x) - g(c) = g(x)$

$\forall \varepsilon > 0\ \exists \delta > 0\ \forall |x - c| < \delta$ have $\left| L - \frac{k'(x)}{g'(x)} \right| < \varepsilon$

hence $g'(x) \neq 0$

also $g(x) \neq 0$ [For contradiction if $g(x) = 0 = g(c)$ then by Rolle's then $\exists$ $|y-c| < |x-c| < \delta$ and $g'(y) = 0$ contradiction.]

and by GMVT $\exists$ $\xi \in [c, x]$

with $k(x) g'(\xi) = k'(\xi) g(x)$

so $\dfrac{k(x)}{g(x)} = \dfrac{k'(\xi)}{g'(\xi)}$ so $\left| L - \dfrac{k(x)}{g(x)} \right| < \varepsilon.$

Ex: $f(x) = e^x$ $\quad f^{(n)}(x) = e^x$

so $P_{3,c}(x) = e^c + e^c(x-c) + e^c \frac{(x-c)^2}{2} + e^c \frac{(x-c)^3}{6}$

eg: $P_{4,0}(-1) = .375$

$R_{4,0,-1} = \frac{1}{e} - .375$ is between $\frac{-1}{120}$ and $\frac{-1}{120e}$

actually $\frac{1}{e} \approx .368$

$P_{4,0}(2) = 7$ $\quad R_{4,0,2} = e^2 - 7$ between $\frac{4}{15}$ and $e^2 \frac{4}{15}$

$e^2 \approx 7.37$

Try for $f(x) = \frac{1}{x}$, $n=4$, $c=1$, $x = \frac{1}{2}$ and $x=3$

$x = \frac{1}{2}$ works well.

$x = 3$ terrible error at $\frac{1}{3} \approx 11$

Ex: $n = 2$ $\quad g'(t) = \left[f(x) - f(t) - f'(t)(x-t) - f''(t)\frac{(x-t)^2}{2}\right]'$

$$= 0 - f'(t) + f'(t) - f''(t)(x-t) + f''(t)(x-t) - f'''(t)\frac{(x-t)^2}{2}$$

cancel cance

210125

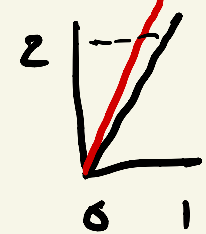
Examples: express e as a limit of rat. numbers:

$$e = 2.71828\ldots\ldots \quad \Big] \text{ limit.}$$

$$= 1 + 1 + \frac{1}{2} + \frac{1}{3!} + \frac{1}{4!} + \cdots$$

$$= 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}}$$

Ex: $F(x) = 2x \in \mathrm{Fun}[0,1]$



$f_n(x) = \left(2+\frac{1}{n}\right)x \in$ " "

$$\left.\begin{aligned} \|F(x)\|_{\sup} &= 2 \\ \|f_n(x)\|_{\sup} &= 2+\frac{1}{n} \end{aligned}\right] \quad \|\underbrace{f_n - F}_{\frac{1}{n}x}\|_{\sup} = \frac{1}{n}$$

$$\left.\begin{aligned} \|F(x)\|_{L^1} &= 1 \\ \|f_n(x)\|_{L^1} &= 1+\frac{1}{2n} \end{aligned}\right] \quad \|f_n - F\|_{L^1} = \frac{1}{2n}$$

$$\left.\begin{aligned} \|F(x)\|_{C^1} &= 2+2 \\ \|f_n(x)\|_{C^1} &= 4+\frac{2}{n} \end{aligned}\right] \quad \|f_n - F\|_{L^1} = \frac{3}{n}$$

Also for ptwise and unif!

$$|f_n(x) - F(x)| = \frac{1}{n}|x|$$

Q: $f_n \xrightarrow[n\to\infty]{\text{ptwise}} F$ Yes: $\forall\ |x|$ have $\frac{1}{n}|x| \to 0$

$f_n \xrightarrow[n\to\infty]{\text{unif}} F$ Yes:

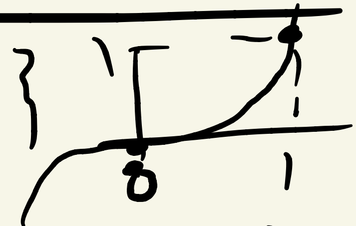
$f_n \xrightarrow[n\to\infty]{\|\cdot\|_{sup}\text{-norm}} F$ Yes. $\frac{1}{n} \to 0$

$f_n \xrightarrow[n\to\infty]{\|\cdot\|_{L^1}\text{-norm}} F$ Yes $\frac{1}{2n} \to 0$

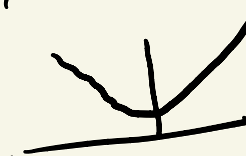
$$f_n \xrightarrow[n\to\infty]{\|\cdot\|_{C^1}\text{-norm}} F \quad \text{Yes} \quad \frac{2}{n} \to 0$$

---

① $f_n(x) = x^n$ $\qquad F(x) = \begin{cases} 1 & x \geq 1 \\ 0 & x < 1 \end{cases}$



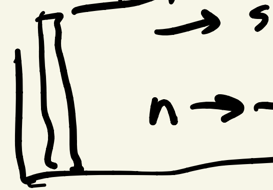
② $f_n(x) = \sqrt{x^2 + \frac{1}{n}}$ $\qquad F(x) = |x|$

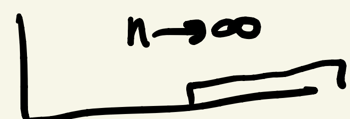


③ $f_n(x) = \dfrac{1}{x + \frac{1}{n}}$ $\qquad F(x) = \dfrac{1}{x}$

times.

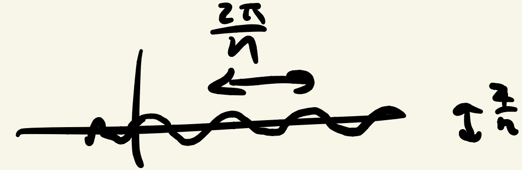
④ $f_n(x) = \begin{cases} 2^{-n} & 2^n \leq x \leq 2\cdot 2^n \\ 0 & \text{else} \end{cases}$



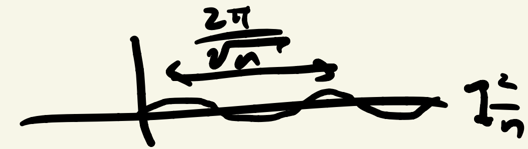


$n \to \infty$ or $n \to -\infty$

$F(x)=0$



(5) $\frac{1}{n}\sin(nx)=f_n(x)$ $\quad F(x)=0$



(6) $\frac{1}{n}\sin(\sqrt{n}x)=f_n(x)$ $\quad F(x)=0$

(7) $1+x+\frac{x^2}{2}+\cdots\frac{x^n}{n!}=f_n(x)$ $\quad F(x)=e^x$ $\quad R=e^{\xi}\frac{x^{n+1}}{(n+1)!}$

(8) $1+x+\cdots+x^n=f_n(x)$ $\quad F(x)=\frac{1}{1-x}$

210127

Thm 3.46: A sequence (a_n) in $\mathbb{R}$ is $|\cdot|$-Cauchy iff it is convergent.

Thm 9.13: A sequence $(f_n(x))$ in $\mathrm{Fun}(a,b)$ is unif-Cauchy iff it is unif-convergent.

Def: If every $(f_n(x))$ ~~is a~~ sequence of functions in a function space with a norm $\|\cdot\|$ converges iff it is $\|\cdot\|$-Cauchy then call the space

$\|\cdot\|$-norm-complete. ~~if~~

210129

Def: $\mathrm{Bdd}(a,b)$ = boundend functions on (a,b)

$= \{ f \in \mathrm{Fun}(a,b) \mid \exists B \in \mathbb{R}_{>0} \ \forall x \in (a,b)$ have $|f(x)| < B \}$.

$= \{$ " " $\mid \|f\|_{sup}$ exists (is finite) ($< \infty$) $\}$

Thm: If (f_n) is a sequence in $\mathrm{Bdd}(a,b)$ which converges uniformly to $f \in \mathrm{Fun}(a,b)$ then $f \in \mathrm{Bdd}(a,b)$.

Equivalently: $Bdd(a,b]$ is $\|\cdot\|_{sup}$-complete.

Proof: (see book for a proof not mentioning $\|\cdot\|_{sup}$)

Note: $\|f+g\|_{sup} \leq \|f\|_{sup} + \|g\|_{sup}$.

(always required of norms $\|\cdot\|$ and called the triangle inequality).

(check this also holds for $\|\cdot\|_{c'}$),

If (f_n) conv. unif. to f

then (f_n) conv. in $\|\cdot\|_{sup}$ to f.

so $\forall \varepsilon > 0 \quad \exists N_\varepsilon \quad \forall n \geq N_\varepsilon$ have $\|f - f_n\|_{sup} < \varepsilon$

If also each f_n is bdd.

then $\forall n$ have $\|f_n\|_{sup} < \infty$.

Combining: $\|f\|_{sup} \le \|f - f_{N_1}\|_{sup} + \|f_{N_1}\|_{sup}$

$\le 1 + \|f_{N_1}\| < \infty$

qed.

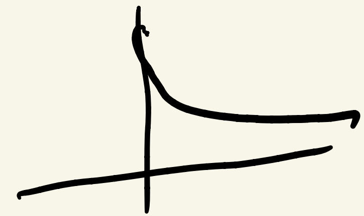
By: 9.14: $\|\omega(x)\|_{sup} < \infty$.

Ans: (1) $\|\omega\|_{sup} \le 1 + 1 = 2$.

since $\|f_n(x)\| \le \left\|\sum_{j=1}^{n} 2^{-j}\right\| < \sum_{j=1}^{\infty} 2^{-j} < 1$

(2) Consider $f_n(x) = \dfrac{1}{x + \frac{1}{n}} \in Fun(0, \infty)$

so $\quad \|f_n\|_{\sup} = \dfrac{1}{0+\frac{1}{n}} = n$



$$f_n \xrightarrow{\text{ptwise}} \frac{1}{x} \in \mathrm{Fun}(0,\infty)$$

but $\quad \|\tfrac{1}{x}\| = \infty \quad$ (does not exist).

210201

Thm: If (f_n) is a sequence in $D'(a,b)$, $c \in (a,b)$ with $(f_n(c))$ converging (in $\mathbb{R}$). and (f_n') converges uniformly to $g(x)$.

then: (HW) (f_n) converges uniformly to $f(x)$

and: (9.18) $g(x) = f'(x)$.

Proof:

Note:

1. Since $f_n' \xrightarrow{unif} g$ know

$\forall \varepsilon > 0 \quad \exists M_\varepsilon \quad \forall n, m \geq M_\varepsilon, \; p \in (a,b)$

have $\left| f_n'(p) - g(p) \right| < \dfrac{\varepsilon}{5}$

and $\left| f_n(p) - f_m(p) \right| < \dfrac{\varepsilon}{5}$

2. Since $f_n \longrightarrow f$ uniformly and hence pointwise

$\forall \varepsilon > 0, \; x, p \in (a,b) \quad \exists N_{\varepsilon, x, p} \quad \forall n \geq N_{\varepsilon, x, p}$

have $\left| f_n(p) - f(p) \right| < \dfrac{\varepsilon}{5} |p - x|$

and $|f_n(x) - f(x)| < \frac{\varepsilon}{5}|p-x|$

Claim: $\forall\, p \in (a,b),\ \varepsilon > 0\ \ \exists\, \delta = \underbrace{\delta}_{\text{from } ④}\ M_{\underbrace{\varepsilon,}_{\text{from } ①}}\ p, \varepsilon$

Check: $\forall\, |x-p| < \delta$ have $\boxed{\star}$

Compute: $\left| \frac{f(x)-f(p)}{x-p} - g(p) \right| \leq$

$\left| \frac{f(x)-f(p)}{x-p} - \frac{f_m(x)-f_m(p)}{x-p} \right| + \left| \frac{f_m(x)-f_m(p)}{x-p} - \frac{f_n(x)-f_n(p)}{x-p} \right|$

$$+\left|\frac{f_n(x)-f_n(p)}{x-p} - f_n'(p)\right| + \left| f_n'(p) - g(p)\right|$$

with $n = M_\varepsilon$ and $m \geq N_{\varepsilon, x, p}$.

so $$\left|\frac{f(x)-f(p)}{x-p} - g(p)\right| \leq$$

$$\left|\frac{f(x)-f_m(x)}{x-p}\right| + \left|\frac{f(p)-f_m(p)}{x-p}\right| + \underbrace{\left|(f_n-f_m)'(\xi_{n,m,x,p})\right|}_{\text{from MVT (3).}}$$

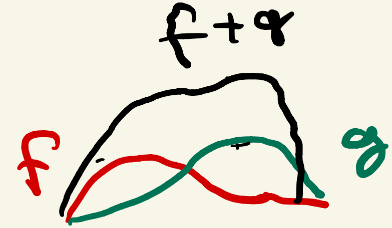
$$+\left|\frac{f_n(x)-f_n(p)}{x-p} - f_n'(p)\right| + \left|f_n'(p)-g(p)\right| < \frac{\varepsilon}{5}+\frac{\varepsilon}{5}+\frac{\varepsilon}{5}+\frac{\varepsilon}{5}+\frac{\varepsilon}{5}$$

$$=\varepsilon.$$

210203

Note: $\overbrace{\text{Triangle ineq.}}^{3}$ holds for $\|\cdot\|_{sup}$

since if $f, g \in Bdd[a,b]$ then,



$$\|f+g\|_{sup} = \sup_{x\in[a,b]} |f(x)+g(x)|$$

$$\le \sup_{x\in[a,b]} \left[|f(x)| + |g(x)|\right]$$

$$\le \sup_{x\in[a,b]} |f(x)| + \sup_{y\in[a,b]} |g(x)|$$

$$= \|f\|_{sup} + \|g\|_{sup}$$

✓.

Prop 13.21:

The metric ass to a norm $\|\cdot\|$ on a vectr space X is $d_{\|\cdot\|}(x,y) = d(x,y)$
$= \|x-y\| \in \mathbb{R}_{\geq 0}$.

if $x,y \in X$.

Ex: $d_{\|\cdot\|_{\sup}}(x, x^2) = \|x - x^2\|_{\sup} = \sup_{x\in[0,1]} |x - x^2|$

$x, x^2 \in Bdd[0,1]$

$= \left|\frac{1}{2} - \frac{1}{4}\right| = \frac{1}{4}.$

Examples:

Thm 3.46 $(\mathbb{R}, |\cdot|)$ is a Banach space.

Thm 9.13 shows: $(Bdd[a,b], \|\cdot\|_{sup})$ is a Banach space.

Thm (7.37, 9.13, 9.16)
$(\underline{C^0}[a,b], \|\cdot\|_{sup})$ is a Banach space.

Proof sketch:
First check $(C^0[a,b], \|\cdot\|_{sup})$ is a normed space,
Since $C^0[a,b] \subseteq Bdd[a,b]$ (by Thm 7.37)

it suffices to check $\|\cdot\|_{sup}$ is a norm on $Bdd[a,b]$.

properties ① easy.
② easy.
③ triangle ineq. done above.

Second check every $\|\cdot\|_{sup}$-Cauchy seq is $\|\cdot\|_{sup}$-conv (to another fn in $C^0[a,b]$.).

By 9.13 if (f_n) is $\|\cdot\|_{sup}$-Cauchy it is $\|\cdot\|_{sup}$-convergent to $f(x) \in Fun[a,b]$.

and by 9.16 $f(x)$ is cts so $f \in C^0[a,b]$.

qed.

Examples:

Recall: $\omega(x)$ is the ptwise limit of

$$f_n(x) = \sum_{j=0}^{n} 2^{-j} \cos(3^j x)$$

and figured out: $\omega(x) \in C^0$ and bdd.

Note: $\omega(x)$ is not differentiable even though $f_n(x)$ all are.

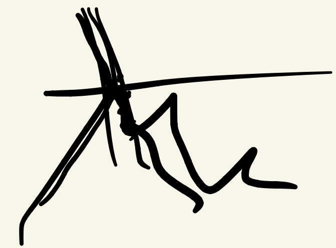
Check: $\omega'(0)$ does not exist.

Assume for contradiction $\omega'(0)$ exists.

Note: $\omega(x) = \lim_{n\to\infty} \sum_{j=0}^{n} 2^{-j} \cos(3^j x)$

$$\le \lim_{n\to\infty} \sum_{j=0}^{n} 2^{-j} = 2.$$

and $W(0) =$ " " $= 2 = \max_x W(x)$.



so $W'(0) = 0$

but: consider $x = \frac{\pi}{3^n}$

$$\text{so } W\left(\frac{\pi}{3^n}\right) = \sum_{j=0}^{n} 2^{-j} \cos(3^j x) \le \sum_{j=0}^{n} 2^{-j} = 2 - 2^{-n}$$

$$\text{but } \frac{W\left(\frac{\pi}{3^n}\right) - W(0)}{\frac{\pi}{3^n} - 0} = \frac{-2^{-n}}{\frac{\pi}{3^n}} = \frac{-3^n}{\pi 2^n} \longrightarrow 0$$

a contradiction

210205

Proof:

normed linear space

- $C[0,1]$ is a $\mathbb{R}$-vect. space from before (linearity of derivatives).

- $\|\cdot\|_{C^2}$ is a norm:
  - If $\|f\|_{C^2} = 0 = \|f\|_{sup} + \|f'\|_{sup} + \frac{1}{2}\|f''\|_{sup}$
    then $\|f\|_{sup} = (\|f'\|_{sup} = \|f''\|_{sup}) = 0$
    so $f(x) = 0$
  - If $k \in \mathbb{R}$ then $\|kf\|_{C^2} = |k|\|f\|_{sup} + |k|\|f'\|_{sup} + \frac{1}{2}|k|\|f''_{sup}\| = |k|\|f\|_{C^2}$.

since derivs are linear.

- If $f, g \in C^2[0,1]$ then

$$\|f+g\|_{C^2} = \|f+g\|_{sup} + \|f'+g'\|_{sup} + \frac{1}{2}\|f''+g''\|_{sup}$$

since $\|\cdot\|_{sup}$ is a norm,

$$\le \|f\|_{sup} + \|g\|_{sup} + \|f'\|_{sup} + \|g'\|_{sup} + \frac{1}{2}\|f''\|_{sup} + \frac{1}{2}\|g''\|_{sup}$$

$$= \|f\|_{C^2} + \|g\|_{C^2}$$

Cauchy $\Downarrow$ convergent

If (f_n) is $\|\cdot\|_{C^2}$-Cauchy in $C^2[0,1]$

then $\forall \varepsilon > 0 \; \exists M \; \forall n,m \geq M$ have $\|f_n - f_m\|_{C^2} < \varepsilon$

$$\|f_n - f_m\|_{sup} + \|f_n' - f_m'\|_{sup} + \frac{1}{2}\|f_n'' - f_m''\|_{sup}$$

so each term is $< \varepsilon$.

hence (f_n) is $\|\cdot\|_{sup}$ - Cauchy hence $\|\cdot\|_{sup}$-convergent by 9.16

also (f_n') is " "

and (f_n'') is " "

so $f_n \xrightarrow{\text{unif}} f$, $f_n' \xrightarrow{\text{unif}} g$, $f_n'' \xrightarrow{\text{unif}} h$

so by 9.18 $g = f'$

and by 9.18 $h = g' = f''$

Hence: $\forall \varepsilon > 0$ $\exists N, M, L$ $\forall n \geq N, m \geq M, l \geq L$ have

$$\|f_n - f\|_{\sup} < \frac{\varepsilon}{3},\ \|f_m' - f'\|_{\sup} < \frac{\varepsilon}{3},\ \|f_l'' - f''\|_{\sup} < \frac{\varepsilon}{3}$$

so $\forall \varepsilon >$ $\exists R = \max(N, M, L)$ $\forall r \geq R$ have $\|f_r - f\|_{C^2} < \varepsilon$

and (f_n) is $\|\cdot\|_{C^2}$-convergent.

q.e.d.

Power Series:

Thm 9.21/9.22: Cauchy for series:

If $(X, \|\cdot\|)$ is a normed vector space.

and (f_n) is a sequence in X

and $F_N = \sum_{n=0}^{N} f_n$ then

(a) (F_N) is $\|\cdot\|$-Cauchy iff

$\forall \varepsilon > 0 \ \exists M \ \forall n, m \geq M$ have $\left\| \sum_{j>n}^{m} f_j \right\| < \varepsilon$

(b) (F_N) is $\|\cdot\|$-Cauchy if (not only if)

$\lim_{N\to\infty} \sum_{n=0}^{N} \|f_n\|$ exists (or $< \infty$).

Question: Find x for which

Ⓐ conv. / div.

Ⓑ conv. / div.

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = 2 \quad \text{exists} \qquad x = \frac{1}{2}$$

$$\sum 1^n \quad \text{does not exist} \qquad x = 1$$

Ⓐ

$$\sum \frac{1}{n!} \cdot 1^n = e$$

$x=1$

B

always converges

210208 Power Series & $\mathbb{R}$-analytic fns.

Review $\mathbb{R}$-series in Ch 4.

Thm (10.3): If $S(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$ both possible. in $[0,\infty]$

pointwise limit.

then there is $R = R_S$ the radius of conv (Def 10.4)

with (a) If $|x-c| > R$ then $S(x)$ diverges.

(b) If $|x-c| < R$ then $S(x)$ converges absolutely (hence $S(x)$ converges ptwise in $(c-R, c+R)$)

(c) If $r < R$ then $S(x)$ converges uniformly in $[c-r, c+r]$

(d) $S(c+R), S(c-R)$ are mysterious and usually avoid.

Proof: (Recall Ex 4.2 if $0<b<1$ then $\sum_{n=0}^{\infty} b^n = \frac{1}{1-b}$)

For simple notation assume $c=0$.

Set $R_s = \sup\{|x| \mid S(x) \text{ converges}\}$.

(a) is immediate from the definition.

(b) If $S(x)$ converges any $|y|<|x|$ then by 4.6 $\sup\{|a_n x^n|\} = M < \infty$

and $\left|\sum_{n=0}^{\infty} a_n y^n\right| \leq \sum_{n=0}^{\infty} |a_n y^n| = \sum_{n=0}^{\infty} |a_n x^n| \left|\frac{y}{x}\right|^n$

$\leq M \frac{1}{1-\frac{y}{x}}$ which is finite.

ⓒ $\left\| \sum_{n=N}^{\infty} a_n y^n \right\|_{\sup} \leq \sum_{n=N}^{\infty} \| a_n y^n \|_{\sup} \leq \sum_{n=N}^{\infty} M \frac{y^n}{x^n}$

$= M \frac{y^N}{x^N} \frac{1}{1-\frac{y}{x}}$

By Thm 9.21. S_n conv. unif. in $[-r, r]$ $\to 0$.

Ans: ① Use Ratio test:

If $\lim_{n\to\infty} \left| \frac{x^{n+1}}{x^n} \right| < 1$

$\parallel$

$\lim_{n\to\infty} x = x$

so $R \geq 1$

and $\sum 1$ diverges so $R = 1$.

② Use Ratio test:

If $\lim_{n\to\infty} \left| \dfrac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} \right| < 1$

$$\parallel$$

$\lim_{n\to\infty} \left| \dfrac{x}{n+1} \right| = 0$ for any value of x.

so $R = \infty$.

③ $R =$

④ $\lim_{n\to\infty} \left|x \frac{n}{n+1}\right| = |x| < 1$ ✓

⑥ $R = 2$

Algebraic properties of power series:

If $S(x)$ and $T(x)$ are power series both about c with radii of convergence R_S and R_T, with $R_S \le R_T$

Then: • If $k \in \mathbb{R} - \{0\}$ then $k\,S(x)$ has $R_{kS} = R_S$.

- $S(x)+T(x)$ has $R_{S+T} \geq R_S$

Prop 10.15 • $S(x) \cdot T(x)$ has $R_{ST} \geq R_S$

Prop 10.20 • $S'(x)$ has $R_{S'} = R_S$

$\int S(x)\,dx$ has $R_{\int S} = R_S$

Cor: If $S(x)$ is a power series with radius of conv. R_S.

then for any $r < R_S$

have $S(x) \in C^\infty[c-r, c+r]$

210210

Cor: If $S(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$ is a limit of a power series with $R_S > 0$

then $\sum a_n (x-c)^n$ is the Taylor series of $S(x)$ about c.

(that is $\sum_{n=0}^{\infty} \frac{S^{(n)}(c)}{n!} (x-c)^n$).

Pf: Need to show: $S^{(n)}(c) \overset{?}{=} a_n n!$

but, by 10.22 $S^{(k)}(x) = \sum_{n=k}^{\infty} a_n\, n(n-1)\cdots(n-k+1)\,(x-c)^{n-k}$

so $S^{(k)}(c) = a_k\, k!$ ← 0 to ...

Cor: If $S(x) = \sum a_n (x-c)^n$, $R_S > 0$

$T(x) = \sum b_n (x-c)^n$, $R_T > 0$

and $S(x) = T(x)$ in any neighborhood of c then every $a_n = b_n$.

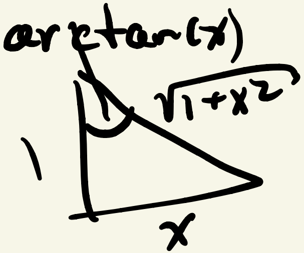
Pf: $a_n = \frac{S^{(n)}(c)}{n!} = \frac{T^{(n)}(c)}{n!} = b_n$.

Example: Find a rational approximation to $\frac{\pi}{\sqrt{3}}$.

Find a power series for $\arctan(x)$.

Recall: (8.23):

$$\left(\arctan(x)\right)' = \frac{1}{\tan'(\arctan(x))} = \frac{1}{\sec^2(\arctan(x))}$$



$$= \frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-x^2)^n$$

$R=1$

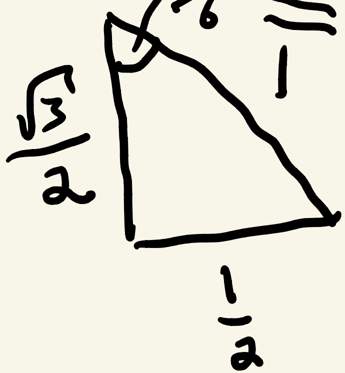
$R=1$ $\quad \sum \frac{1}{1-x} = 1 + x + \cdots = \sum x^n$

hence! $\quad (\arctan(x))' = \sum (-1)^n x^{2n}$

$$\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

~~conv~~ with $R=1$

Use



$$\frac{\pi}{6} = \arctan\left(\frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) = \frac{1}{2\sqrt{3}} < 1$$

so

$$\frac{\pi}{6} = \sum (-1)^n \frac{\left(\frac{1}{2\sqrt{3}}\right)^{2n+1}}{2n+1}$$

Estimate $\dfrac{\pi}{\sqrt{3}} = 2\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)\,3^n}$

so $\dfrac{\pi}{6} = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)\,3^n\sqrt{3}}$

so $\dfrac{\sqrt{3}\,\pi}{6} = \displaystyle\sum \frac{(-1)^n}{(2n+1)\,3^n}$

$\dfrac{\sqrt{3}\,\pi}{6} = \dfrac{\pi}{2\sqrt{3}}$

so $\dfrac{\pi}{\sqrt{3}} = 2\left[1 - \dfrac{1}{3\cdot 3} + \dfrac{1}{5\cdot 9} - \dfrac{1}{7\cdot 27}\right.$

$$+ \frac{1}{9\cdot 81} - \frac{1}{11\cdot 243}\Big]$$

$$\approx 1.8136$$

Prop 10.29: $\varphi(x) = \begin{cases} e^{-\frac{1}{x}} & x > 0 \\ 0 & x \leq 0 \end{cases}$.

has $\varphi(x) \in C^\infty \mathbb{R}$.

and $\varphi^{(k)}(0) = 0$

so Taylor series for $\varphi(x)$ about 0 is $0 \neq \varphi(x)$.

so $\varphi(x)$ is not the limit of a power series about 0.

210217

Notation for Riemann Integrals (Areas).

§11.2:

Def: If $[a,b]$ is an interval and $P \subseteq [a,b]$ which is finite and contains a and b, then call P a <u>partition</u> of $[a,b]$.

write: $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$

with associated intervals $I_k = [x_{k-1}, x_k]$

with lengths $l_k = x_k - x_{k-1}$

Write $\Pi[a,b]$ for the set of all partitions of $[a,b]$ (so $P \in \Pi[a,b]$).

Call Q a <u>refinement</u> of P if $P \subseteq Q$ and both are partitions of $[a,b]$.

Def: If $f(x) \in Bdd[a,b]$ and $P \in \Pi[a,b]$ then

(a) $$U(f;P) = \sum_{i=1}^{n} l_i \left[\sup_{a \in I_i} f(a)\right]$$

(b) $$L(f;P) = \sum_{i=1}^{n} l_i \left[\inf_{a \in I_i} f(a)\right]$$

Ideas: Compare $RInt[a,b]$ to $\underbrace{D^0[a,b]}_{\text{derivatives,}}$

$$\begin{tikzcd} D^0 \arrow[r, shift right, "\text{Integral}"'] & D^1 \arrow[l, shift right, "D"'] \end{tikzcd}$$

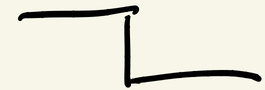
Compare: $\int_0^x f'(y)\,dy$ to $f(x)$ (FTC).

Compare $\left(RInt[a,b],\ \int_a^b 1\cdot 1\,dx\right)$. to a Banach space.

Examples:

① $\int_0^1 x\,dx = \frac{1}{2}$ as above

② $\int_0^1 f(x)\,dx = \frac{1}{2}$ $\quad f \notin D^0[0,1]$



$$f(x) = \begin{cases} 1 & x \le \frac{1}{2} \\ 0 & x > \frac{1}{2} \end{cases}$$

③ $\int_0^1 g(x)\,dx = 0$

$\int |g(x)|\,dx = 0$ but $g \neq 0$ Not Banach.

$$g(x) = \begin{cases} 1 & x = 0 \\ 0 & x > 0 \end{cases}$$

④ $\int_0^1 h(x)\,dx =$ Does not exist

$$h(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

$\overline{\int_0^1} h(x)\,dx = 1$

$\underline{\int_0^1} h(x)\,dx = 0$

so $h \notin RInt[0,1]$

210219

Cauchy criteria examples:

consider $R_m \in \Pi[0,1]$

with $R_m = \{0, \frac{1}{m}, \frac{2}{m}, \dots, \frac{m-1}{m}, 1\}$.

Find: $U(f; R_m) - L(f; R_m)$

① $f(x) = x$ (last time):

found $U(f; R_m) - L(f; R_m) = \left(\frac{1}{2} + \frac{1}{2m}\right) - \left(\frac{1}{2} - \frac{1}{2m}\right)$

$= \frac{1}{m} \xrightarrow{m \to \infty} 0$

so by Cauchy critrion $f(x) = x$ is R. Integrable, on $[0,1]$.

Try:

② $f(x) = \chi_{\{1\}}(x) = \begin{cases} 1 & x = 1 \\ 0 & x \neq 1 \end{cases}$.

③ $f(x) = \chi_{[\frac{1}{2}, 1]}(x) = \begin{cases} 1 & x \geq \frac{1}{2} \\ 0 & x < \frac{1}{2} \end{cases}$

④ $f(x) = \chi_{\mathbb{Q}}(x) = \left\{ \begin{matrix} 1 & x = \frac{a}{b} \\ 0 & \text{otherwise} \end{matrix} \right\}.$

Ans:

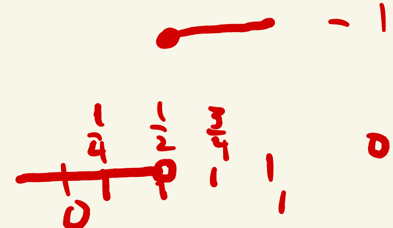
② $$U(f; R_m) = \overbrace{0+0+\cdots+\frac{1}{m}}^{m\text{-terms}}$$
$$L(f; R_m) = 0+0+\cdots+0$$

so $U(f;R_m) - L(f;R_m) = \frac{1}{m}$

③ $$U(f; R_m) = 0+0+\cdots+0+\frac{1}{m}+\frac{1}{m}+\cdots+\frac{1}{m}$$
$$L(f; R_m) = \underbrace{0+\cdots\cdots+0}+\underbrace{\frac{1}{m}+\cdots+\frac{1}{m}}$$

so $U(f;R_m) - L(f;R_m) = \frac{1}{m}$

$$U(f; R_4) = 0 + \frac{1}{4}+\frac{1}{4}+\frac{1}{4}$$
$$L(f; R_4) = 0+0+\frac{1}{4}+\frac{1}{4}$$
$$U(f; R_5) = 0+0+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}$$
$$L(f; R_5) = 0+0+0+\frac{1}{5}+\frac{1}{5}.$$



So both 2 and 3 are R. Int. fns.

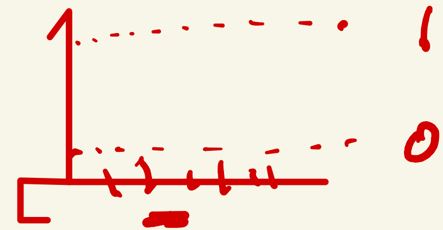
since $\lim_{m\to 0} \frac{1}{m} = 0$.

4 $U(f;R_m) = \frac{1}{m} + \frac{1}{m} + \cdots + \frac{1}{m} = 1$

$L(f;R_m) = 0 + 0 + \cdots + 0 = 0$

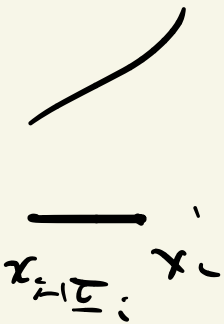
$U(f;R_m) - L(f;R_m) = 1$

$\lim_{m\to\infty} 1 = 1 \neq 0$



Thm 11.30 increasing functions are Riemann integrable:

Proof: If $f \in Bdd[a,b]$ and increasing fcn:



$$U(f;R_m) - L(f;R_m)$$

$$= \sum_{i=1}^{m} l_i \left[f(x_i) - f(x_{i-1})\right]$$

$$= \frac{b-a}{m}\left[f(b) - f(x_{m-1}) + f(x_{m-1}) - f(x_{m-2}) + \cdots - \cdots f(a)\right]$$

$$= \frac{b-a}{m}\left[f(b)-f(a)\right]$$

which approaches 0 as $m \to \infty$.

210222

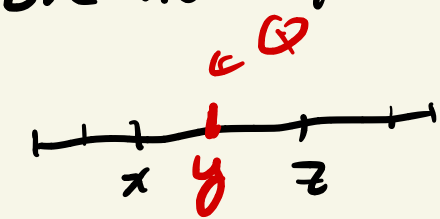
Thm 11.20: If $P \subseteq Q \in \Pi[a,b]$ and $f \in Bdd[a,b]$ then

(a) $U(f;P) \geq U(f;Q)$

(b) $L(f;P) \leq L(f;Q)$

Proof sketch of (a)

Since P and Q are finite it suffices to assume Q has only one more point then P.



Since only one interval in P is subdiv. in Q it suffices to take

$$P = \{x, z\} \subseteq Q = \{x, y, z\}$$

Compute:

$$U(f;Q) = (y-x)\sup_{\alpha\in[x,y]} f(\alpha) + (z-y)\sup_{\beta\in[y,z]} f(\beta)$$

$$\leq (y-x)\sup_{\gamma\in[x,z]} f(\gamma) + (z-y)\sup_{\gamma\in[x,z]} f(\gamma)$$

$$= U(f;P).$$

RInt$[0,1]$ is a $\mathbb{R}$. vector space.

$\int : \text{RInt}[0,1] \longrightarrow \mathbb{R}$ is linear.

Need: Recall! $\int_0^1 f(x)\,dx = \overline{\int_0^1} f(x)\,dx = \underline{\int_0^1} f(x)\,dx$

if $f \in \text{RInt}[0,1]$.

Prop 11.32:

If $f \in \text{RInt}[0,1]$ and $c \in \mathbb{R}$

then $cf \in \text{RInt}[0,1]$, and $\int_0^1 cf(x)\,dx = c\int_0^1 f(x)\,dx$

Proof: If $c > 0$

Compute: $\overline{\int_0^1} cf(x)\,dx = \inf_P U(cf;P) = \inf_P \sum_{i=1}^n l_i \sup_{x \in I_i} cf(x)$

$= c \cdot \inf_P \sum l_i \sup_{x \in I} f(x) = c \overline{\int_0^1} f(x)\,dx.$

since $f \in \text{RInt}[0,1]$

similarly $\underline{\int_0^1} cf(x)\,dx = c\underline{\int_0^1} f(x)\,dx = c\int_0^1 f(x)\,dx$

Hence also $cf \in RInt[0,1]$.

Check what happens in $c<0$.

get $\overline{\int_0^1} cf(x)\,dx = c\underline{\int_0^1} f(x)\,dx$

$\int$ is monotone:

Prop 11.39: If $f \leq g$ pointwise both in $Bdd[a,b]$ then

(a) $\overline{\int_a^b} f(x)\,dx \leq \overline{\int_a^b} g(x)\,dx$

(b) $\underline{\int_a^b} f(x)\,dx \leq \underline{\int_a^b} g(x)\,dx$

Proof of (a) ((b) similar).

Compute:

$$\overline{\int_a^b} f(x)\,dx = \inf_P \sum_{i=1}^n l_i \sup_{x\in I_i} f(x)$$

$$\leq \inf_P \sum_{i=1}^n l_i \sup_{x\in I_i} g(x) = \overline{\int_a^b} g(x)\,dx$$

Cor: If $f, g \in RInt[a,b]$ and $f \leq g$ pointwise then $\int_a^b f(x)\,d(x) \leq \int_a^b g(x)\,dx$.

210224

Claim (scaling): If $c \in \mathbb{R}$ and $f \in RInt[a,b]$ then $\|cf\|_{L^1} = |c| \, \|f\|_{L^1}$.

Proof: Use 11.32: $\int c g(x)\,dx = c \int g(x)\,dx$.

Compute: $\|cf\|_{L^1} = \int |c||f(x)|\,dx \overset{11.32}{=} |c| \int |f(x)|\,dx$

$= |c| \, \|f\|_{L^1}$.

q.e.d.

Claim (triangle): If $f, g \in RInt[a,b]$ then $\|f+g\|_{L^1} \leq \|f\|_{L^1} + \|g\|_{L^1}$

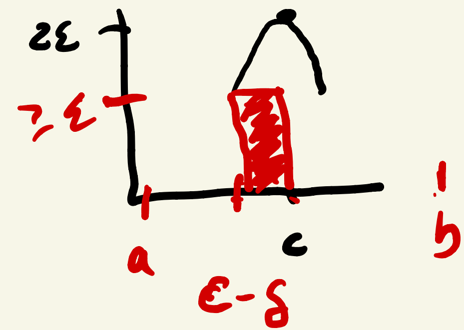
Proof: Use 11.33 $\int (f+g) = \int f + \int g$

and 11.36 if $f \leq g$ then $\int f \leq \int g$

$$\|f+g\|_{L^1} = \int_a^b |f+g|\,dx \underset{11.36}{\leq} \int_a^b (|f|+|g|)\,dx$$

$$\underset{11.83}{=} \int_a^b |f|\,dx + \int_a^b |g|\,dx = \|f\|_{L^1} + \|g\|_{L^1}$$

Claim (11.42): If $f \in C^0[a,b]$ and $\|f\|_{L^1} = 0$ then $f = 0$.



Proof: Assume for contradiction $f \in C^0[a,b]$, $f(c) \neq 0$ and $\|f\|_{L^1} = 0$.

Write $|f(c)| = 2\varepsilon$

Since f is cts: $\exists\ 0<\delta$ $\forall\ |x-c|<\delta$

have $\big||f(c)|-|f(x)|\big|<\varepsilon$ so $|f(x)| \geq \varepsilon$

So if $P=\{a, c-\delta, c, b\}$ or $\{a, c, c+\delta, b\}$.

then $L(|f|; P) \geq pos + \delta\cdot\varepsilon + pos \geq \delta\cdot\varepsilon>0$

$$L(|f|) \geq L(|f|;P)$$

$$\|f\|_{L^1} = L(|f|)$$

contradicting $\|f\|_{L^1}=0$.

q.ed.

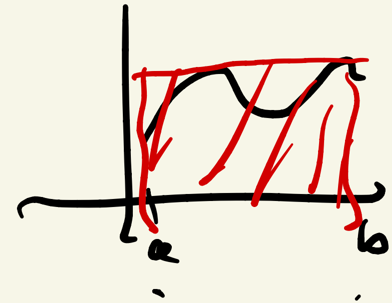
Cor: $(C^0[a,b], \|\cdot\|_{L^1})$ is a normed linear space.

Recall: $(C^0[a,b], \|\cdot\|_{\sup})$ is also a " "

Claim: If $f\in RInt[a,b]$ then

$$\|f\|_{L^1} \leq (b-a)\|f\|_{\sup}.$$

Proof: Consider the partition $P = \{a, b\}$,



$$\|f\|_{L^1} = \int_a^b |f(x)|\,dx \leq U(|f|; \{a,b\})$$

$$= (b-a) \sup_{x \in [a,b]} |f(x)|$$

$$= (b-a)\|f\|_{\sup}$$

Use this later.

210226

Thm 11.50: If $f \in \mathrm{Bdd}[0,1]$ and $\forall \varepsilon > 0$ have $f|_{[\varepsilon,1]} \in \mathrm{RInt}[\varepsilon,1]$

then $f \in \mathrm{RInt}[0,1]$.

Hence: $f(x) = \begin{cases} \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases} \in \mathrm{RInt}[0,1]$.

Proof: Write $M = \|f\|_{\sup}$

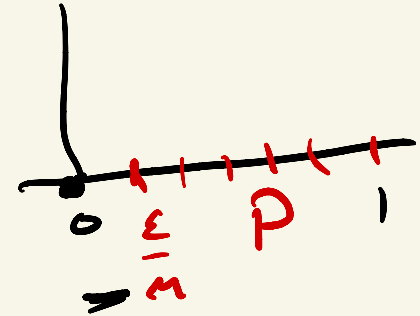
$\forall \varepsilon > 0 \quad \exists P \in \Pi[\frac{\varepsilon}{M}, 1]$

with $U(f|_{[\frac{\varepsilon}{M},1]}; P) - L(f|_{[\frac{\varepsilon}{M},1]}; P) < \varepsilon$

Hence $P \cup \{0\} = \hat{P} \in \Pi[0,1]$

and $U(f) - L(f) \le U(f;\hat{P}) - L(f;\hat{P})$

$$\le \underbrace{\left(\frac{\varepsilon}{M} - 0\right)\left[\sup_{x\in[0,\frac{\varepsilon}{M}]} f(x) - \inf_{x\in[0,\frac{\varepsilon}{M}]} f(x)\right]}_{\text{first interval}} + \underbrace{\varepsilon}_{\text{all others,}}$$



$$\le \frac{\varepsilon}{M} 2M + \varepsilon = 3\varepsilon$$

Hence $f \in RInt[0,1]$.

Ch: 12: Point of integrals: Fund Thm of Calc.

Example: $f(x) = x^2$

Notation: If $f \in RInt[a,b]$ write

$$(Jf)(x) = \int_a^x f(y)\,dy$$

Note: $(Jf)(x)$ exists since by 11.44 $f|_{[a,x]} \in RInt[a,x]$

Note: $(Jf)(a) = 0$

Note: $Jf \in Bdd[a,b]$

since $|Jf(x)| \leq (x-a)\,\|f\|_{sup}$

which is finite since $f \in Bdd[a,b]$

Examp $f(x) = x^2$

$$(Jf)(x) = \frac{1}{3}x^3$$

so $(DJf)(x) = \left(\frac{1}{3}x^3\right)' = x^2 = f(x).$

and $(Df)(x) = 2x$

and $(JDf)(x) = x^2 = f(x)$

Question: For which functions f do we get:

$$DJf = f$$
$$JDf = f.$$

210301

Thm 12.4 If $f \in RInt[a,b]$ and f is cts at $c \in [a,b]$ then $D\bar{J}f(c) = f(c)$.

Proof: Since f is cts at c:

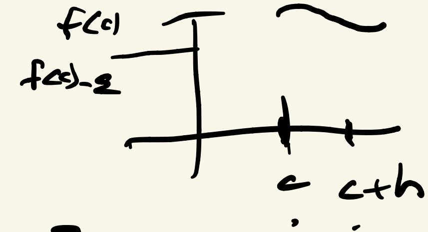
$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall |h| \leq \delta$ have

$f(c+h) \in [f(c)-\varepsilon, f(c)+\varepsilon]$

so $\sup\limits_{|h| \leq \delta} f(c+h) \in$ " "

and $\inf\limits_{|h| \leq \delta} f(c+h) \in$ " "

so $$\int_c^{c+h} f(y)\,dy \le U(f;\{c,c+h\}) = h\cdot\sup$$
$$\ge L(f;\{c,c+h\}) = h\cdot\inf.$$



and $$\int_c^{c+h} f(y)\,dy \in [h(f(c)-\varepsilon),\ h(f(c)+\varepsilon)]$$

Compute:

$$DJf(c) = \lim_{h\to 0} \frac{Jf(c+h) - Jf(c)}{h}$$

$$= \lim_{h\to 0} \frac{\int_a^{c+h} f(y)\,dy - \int_a^{c} f(y)\,dy}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\int_c^{c+h} f(y)\,dy\right] \in [f(c)-\varepsilon,\ f(c)+\varepsilon]$$

So $DJf(c) = f(c)$. qed.

Examples: (12.14)

Using integrals to define functions.

Def: $\ln(x) = \int_1^x \frac{dx}{x}$

Claim: $\ln(xy) = \ln(x) + \ln(y)$

Proof: $\ln(x) + \ln(y)$

$$= \int_1^x \frac{1}{t}\,dt + \int_1^y \frac{1}{t}\,dt$$

$$= \int_1^x \frac{1}{t}\,dt + \int_x^{xy} \frac{x}{s}\,\frac{1}{x}\,ds$$

$$= \int_1^{xy} \frac{1}{t}\,dt$$

$s = t\,x$

$\frac{1}{x} s = t$

$\frac{1}{x} ds = dt$

$t=1$
$s=x$

$t=y$
$s=xy$

$= \ln(xy)$

(12.15) $\mathcal{E}rf(x) = \frac{2}{\sqrt{\pi}} \int e^{-t^2} dt$

Ex: Show that there is a unique function $F(x)$ with

$$F'''(x) = e^{-x^2}$$

and $F(0) = F'(0) = F(1) = 0.$

Proof: Since e^{-x^2} is cts so a derivative, and $g = \int(e^{-x^2}) = \int_0^x e^{-t^2} dt$ has $g' = e^{-t^2}$ and only $g + c$ have this deriv.

so $h(x) = J(g)(x)$ only $h(x) + cx + d$ have e^{-x^2} as second deriv.

$k(x) = J(h)(x)$

so $k'''(x) = e^{-x^2}$

and only $k(x) + \frac{1}{2}cx^2 + dx + b = F(x)$

have e^{-x^2} as third deriv. $= F'''(x)$

Use initial conditions:

$F(0) = k(0) + \frac{1}{2}c\cdot 0 + d\cdot 0 + b = 0$

$F'(0) = k'(0) + c\cdot 0 + d + 0 = 0$

$F(1) = k(1) + \frac{1}{2}c\cdot 1 + d\cdot 1 + b = 0$

so $b = -k(0)$

$d = -k'(0)$

$\frac{1}{2}c = -k(1) - d - b$

$= -k(1) + k'(0) + k(0)$

and there is only one choice for $F(x)$.

Consider:

$$f(x) = \begin{cases} x^2 \cos \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$f(x) \in D'[-1,1]$ but $f(x) \notin C'[-1,1]$

and $$Df(x) = \begin{cases} 2x\cos\frac{1}{x} - \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}.$$

which is not continuous.

Question: is $\displaystyle\int_0^x Df(y)\,dy \overset{?}{=} f(x) - f(0).$

$= f(x).$

Yes:

Thm 12.1: If $f(x) \in D'(a,b)$ and $f(x) \in C^0[a,b]$ and $f'(x) \in \mathrm{RInt}[a,b]$.

then $\int Df(b) = f(b) - f(a)$

Proof: $\forall \varepsilon \ \exists P \in \Pi[a,b]$

with $\quad Lf' \leq L(f';P) + \varepsilon$

and $\quad \mathcal{U} f' \geq \mathcal{U}(f'; P) - \varepsilon$

Compute:

$$\mathcal{J} D f(x) = \int_a^x f'(t)\,dt$$

$$= \mathcal{L} f' \leq L(f'; P) + \varepsilon = \sum_{i=1}^{n} l_i \inf_{t \in I_i} f'(t) + \varepsilon$$

$$\underset{\text{MVT}}{\leq} \sum_{i=1}^{n} l_i \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} + \varepsilon$$

$$\underset{\text{telescopes}}{=} f(x) - f(a) + \varepsilon$$

similarly $U f' \geq f(x) - f(a) - \varepsilon$.

and done.

210303

Cor: If $F = f'$

then $\forall g \in C^1[a,b]$ with $g(a) = g(b) = 0$

have $\int_a^b f(t) g'(t)\,dt = -\int_a^b F(t) g(t)\,dt + 0$

Def: If $F, f \in RInt[a,b]$

and $\forall g \in C^1[a,b]$ with $g(a) = g(b) = 0$

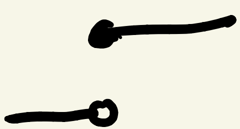
have $\int_a^b f(t) g'(t)\,dt = -\int_a^b F(t) g(t)\,dt$

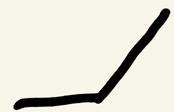
call F a <u>weak</u> <u>derivative of</u> f.

Example: $F(t) = \left\{\begin{matrix} 1 & x \geq 0 \\ 0 & x < 0 \end{matrix}\right\} \in RInt[-1,1]$

is a weak deriv. of!

$$f(t) = \left\{\begin{matrix} x & x \geq 0 \\ 0 & x < 0 \end{matrix}\right\} \in \text{" "}$$

F: 

f: 

Note: $f'(t) = \left\{\begin{matrix} 1 & x > 0 \\ \text{undef} & x = 0 \\ 0 & x < 0 \end{matrix}\right\}$.

Check: Compute: for any g as above

$$\int_{-1}^{1} f(t)\,g'(t)\,dt = \int_0^1 t\,g'(t)\,dt$$

$$\underset{\text{parts}}{=} -\int_0^1 1\cdot g(t)\,dt = -\int_{-1}^{1} F(t)\,g(t)\,dt.$$

Proof:

$$\left| \int_a^b f_n(t)\,dt - \int_a^b f(t)\,dt \right|$$

$$= \left| \int_a^b \left(f_n(t) - f(t) \right) dt \right|$$

$$\leq \int_a^b \left| f_n(t) - f(t) \right| dt$$

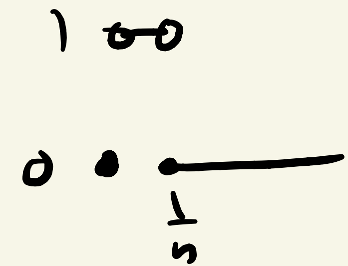
$$= \| f_n - f \|_{L^1} \longrightarrow 0$$

since $(f_n) \xrightarrow{L^1} f$,

210365

Ex: 12.22:

(a) $f_n(x) = \left\{\begin{array}{ll} 1 & 0 < x < \frac{1}{n} \\ 0 & \text{else} \end{array}\right\} \in R\text{Int}[0,1]$



(f_n) conv. ptwise to 0

does not. conv. unif.

conv. in L^1 to 0.

[conv. in

$\|f_n - 0\|_{L^1} = \int |f_n(x) - 0|\,dx = \frac{1}{n} \xrightarrow[n\to\infty]{} 0$

(b) $f_n(x) = \begin{cases} 1 & 0 \leq n < \frac{1}{n} \\ 0 & \text{else} \end{cases} \in RInt[0,1]$

(f_n) con. ptwise to $\begin{cases} 1 & x=0 \\ 0 & \text{else} \end{cases}$ 

does not con. unif. 

conv. in L^1 to 0

conv. in L^1 to $\begin{cases} 1 & x>0 \\ 0 & \text{else} \end{cases}$

C^1 conv $\Longrightarrow$ unif. conv $\Longrightarrow$ pointwise conv

unif. conv $\Longrightarrow$ L^1 conv.

12-17

Example 12.19:

If $S(x) = \sum_{n=0}^{\infty} a_n x^n$

is a power series about 0

with radius of conv. R.

and $|a|, |b| < R$

then $S(x)$ conv. unif. in $[a,b]$

hence " " in L^1 in $[a,b]$

hence $\int_a^b S(x)\,dx = \sum_{n=0}^{\infty} a_n \int_a^b x^n\,dx$

$$= \sum_{n=0}^{\infty} a_n \frac{x^{n+1}}{n+1} \Big|_a^b$$

$n+1 \to n$

$$= \left(\sum_{n=1}^{\infty} a_{n-1} \frac{x^n}{n} \right) \Big|_a^b$$

Ex: $\ln(2) = \sum_{n=1}^{\infty} \frac{1}{n 2^n} = \frac{1}{2} + \frac{1}{8} + \frac{1}{24} + \cdots$

converges fast.

Since: $\ln(x) = \int_1^x \frac{dy}{y}$

So $\ln(1-t) = \int_1^{1-t} \frac{dy}{y}$

take $x = 1-t$

take $y = 1-s$

$$= \int_0^t \frac{-ds}{1-s}$$

$$= -\int_0^t \sum_{n=0}^{\infty} s^n \, ds$$

$$= -\sum_{n=1}^{\infty} \frac{s^n}{n} \Big|_0^t$$

so $-\ln(1-t) = \ln\left(\frac{1}{1-t}\right) =$ " "

so if $t = \frac{1}{2}$ $\quad \ln(2) = \sum_{n=1}^{\infty} \frac{1}{n2^n}$

Add more functions to integrate:

Using L^1-Cauchy sequences which only converge pointwise to unbounded fns.

From MAT21: Improper integrals.

210308)

Example: 12.25

$$\int_0^1 \frac{dx}{x^{1-\alpha}} \overset{\text{def}}{=} \lim_{\varepsilon \to 0^+} \int_\varepsilon^1 \frac{dx}{x^{1-\alpha}}$$

$$= \lim_{\varepsilon \to 0^+} \left[\frac{x^\alpha}{\alpha} \right]_\varepsilon^1$$

$$= \begin{cases} \frac{1}{\alpha} - \lim_{\varepsilon \to 0^+} \frac{\varepsilon^\alpha}{\alpha} = \frac{1}{\alpha} & \text{if } \alpha > 0 \\ \text{d.n.e.} & \text{if } \alpha \leq 0. \end{cases}$$

Example 12.27

$$\int_0^1 \frac{dx}{x^{1-\alpha}} = + \int_1^\infty y^{1-\alpha}\,\frac{dy}{y^2} = \int_1^\infty \frac{dy}{y^{1+\alpha}}$$

$$y = \frac{1}{x} \qquad x=0 \quad y=\infty$$

$$x=1 \quad y=1$$

$$x = \frac{1}{y}$$

$$dx = \frac{-dy}{y^2}$$

$$= \begin{cases} \dfrac{1}{\alpha} & \alpha > 0 \\ \text{d.n.e.} & \alpha \le 0 \end{cases}$$

Thm: 12.33:

If $|f| \le |g|$ with $f, g \in \text{Fun}[a,b]$, (Thanks Rahal).

$\forall \varepsilon > 0,\ f \in \text{RInt}[a+\varepsilon, b]$,

and g is abs. imp. int. on $[a,b]$

then (a) f " "

and (b) $\int_a^b f(x)\,dx$ exists.

Proof:

(a) $|f| \le |g|$ so $\lim\limits_{\varepsilon \to 0^+} \int_{a+\varepsilon}^{b} |f|\,dx \le \lim\limits_{\varepsilon \to 0^+} \int_{a+\varepsilon}^{b} |g|\,dx$

$= L$

L exists by hypothesis.

and as $\varepsilon \to 0^+$ $\int_{a+\varepsilon}^{b} |f|\,dx$ is increasing

so by monotone convergence (3.29)

$\int_{a}^{b} |f|\,dx$ exists.

(b) Def: $f_+(x) = \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ 0 & \text{else} \end{cases}$.

and $f_-(x) = \begin{cases} -f(x) & \text{if } f(x) \leq 0 \\ 0 & \text{else} \end{cases}$, so $\quad f(x) = f_+(x) - f_-(x)$

and $\quad f_+(x) = |f_+(x)|$

$\qquad f_-(x) = |f_-(x)|.$

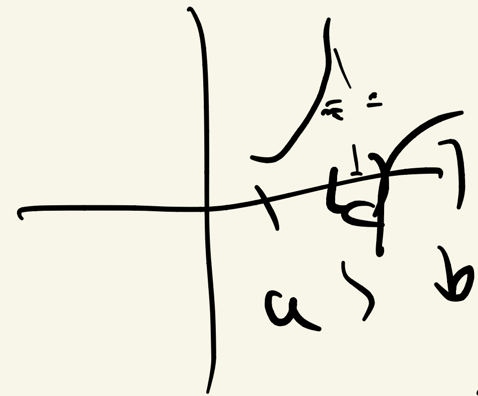
hence: $\quad \int_{a+\varepsilon}^{b} f(x)\,dx = \underbrace{\int_{a+\varepsilon}^{b} |f_+(x)|\,dx}_{\downarrow} - \underbrace{\int_{a+\varepsilon}^{b} |f_-(x)|\,dx}_{\downarrow}$

and by ⓐ $\qquad$ and $\qquad$ ~~exist.~~ limits exist.

so $\quad \int_a^b f(x)\,dx = \int_a^b |f_+(x)|\,dx - \int_a^b |f_-(x)|\,dx$

which exists, $\qquad$ q.e.d.

Note: f has an abs. conv. impr. int. on $[a,b]$ iff $\left(f|_{[a+\frac{1}{n},b]}\right)$ is L^1-Cauchy.



Def 12.39:
If $a<c<b$
and $f\in \mathrm{Fun}([a,b]-\{c\})$
and $\forall\ \varepsilon>0$ have $f\in \mathrm{RInt}([a,b]-(c-\varepsilon,c+\varepsilon))$

then write

$$\underbrace{\text{P.V.}}_{\text{principal value}} \int_a^b f(x)\,dx = \lim_{\varepsilon \to 0} \left[\int_a^{c-\varepsilon} f(x)\,dx + \int_{c+\varepsilon}^b f(x)\,dx \right]$$

if the limit exists.

Example: $\frac{1}{x}$ is not abs. imp. int. on $[0,1]$ or $[-1,0]$

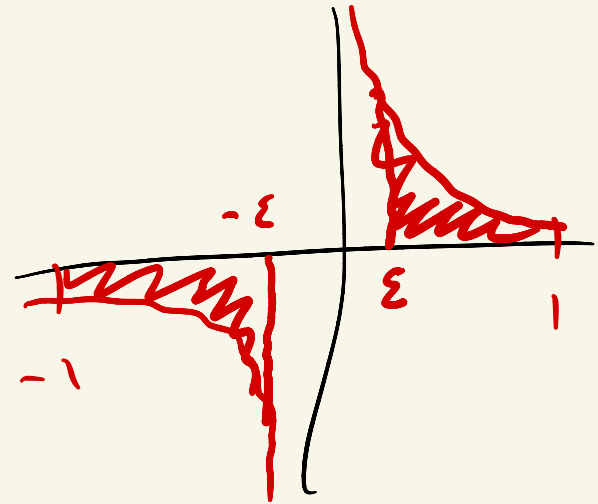
12.38

and also not imp. int. on $[0,1]$ or $[-1,0]$

but $\text{P.V.} \int_{-1}^{1} \frac{dx}{x} = 0$

Example: $\frac{1}{x^2}$

$\text{P.V.} \int_{-1}^{1} \frac{dx}{x^2}$ does not exist.



210310

Asside: 1 $\int_0^1 \frac{e^t}{t}\,dt$

$$= \lim_{\varepsilon \to 0^+} \int_\varepsilon^1 \frac{e^t}{t}\,dt$$

$$\geq \lim_{\varepsilon \to 0^+} \int_\varepsilon^1 \frac{1}{t}\,dt$$

$$= \lim_{\varepsilon \to 0^+} \ln(t)\Big|_\varepsilon^1 \quad \text{diverges.}$$

Asside ② $\int_{-\infty}^{-1} \tilde{f}(t)\,dt$

$$= \lim_{n\to\infty} \int_{-n}^{-1} \frac{e^t - 1}{t}\,dt$$

$$= \lim_{n\to\infty} \int_{-n}^{-1} \frac{e^t}{t}\,dt - \lim_{n\to\infty} \int_{-n}^{-1} \frac{dt}{t}$$

$$= \text{convergent} - \lim_{n\to\infty} \left[-\ln(n) + \ln(1)\right]$$

diverges.

---

§12.7:

Recall: Thm 8.4.6 Taylor with Lagrange Error

If f is cts on $[a,b]$ and $f^{(n+1)} \in \mathrm{Fun}[a,b]$

and $a < c < b$ then write

$$P_n(x) = \sum_{r=0}^{n} \frac{f^{(r)}(c)}{r!}(x-c)^r$$

and $P_n(x) + R_n(x) = f(x)$.

and $\forall x \ \exists \xi$ with $R_n(x) = \dfrac{f^{(n+1)}(\xi)}{(n+1)!}(x-c)^{n+1}$

New: Thm 12.48 Taylor with integral Error.

If also $f^{(n+1)} \in \mathrm{RInt}[a,b]$

then $$R_n(x) = \frac{1}{n!}\int_c^x f^{(n+1)}(t)\,(x-t)^n\,dt$$

Proof of 12.48: Induct on n:

Base case $n=0$ use FTC 12.1

(If f cts on $[a,b]$ with $f' \in RInt[a,b]$

then $\int_a^b f'(t)\,dt = f\Big|_a^b$)

Induction step use parts 12.10

(If f, g cts on $[a,b]$ with $f', g' \in RInt[a,b]$

then $\int_a^b f(t)g'(t)\,dt = -\int_a^b f'(t)g(t)\,dt + \Big[f(t)g(t)\Big]_a^b$)

210312

Properties of fuctions:

Continuity at a point.

Differentiability at a point.

Integrability on an interval (finite)

(abs) improperly integrable on an interval (possibly infinite).

Properties of sequences of functions:

Convergence to a function: $\|\cdot\|_{C^1}$ $\Downarrow$ or Cauchy in any of the three $\|\cdot\|$'s.

$$\|\cdot\|_{unif} \Rightarrow \text{pointwise}, \qquad \|\cdot\|_{unif} \Rightarrow \|\cdot\|_{L^1}$$

Banach spaces:

$(C^0[0,1], \|\cdot\|_{unif})$

$(Bdd[0,1] \cap D^0[0,1], \|\cdot\|_{unif})$

$(Bdd[0,1], \|\cdot\|_{unif})$

$(C^1[0,1], \|\cdot\|_{C^1})$

Normed Lin. space:

$(C^0[0,1], \|\cdot\|_{L^1})$

Asside

$L^1[0,1] = \{ f \in Fun[0,1] \mid$
$\exists (f_n) \xrightarrow{ptwse} f$
and (f_n) is $\|\cdot\|_{L^1}$-Cauchy
and $f_n \in RInt[0,1] \} \big/ \sim$

where $f \sim g$ if f_n, g_n above
have $\|f_n - g_n\|_{L^1} \longrightarrow 0$.

$\big($ $L^1[0,1]$, $\|\cdot\|_{L^1}$ $\big)$ is Banach

Taylor example:

Find a bound on $g(1)$ if:

$$g(x) = J^{100} f(x). = \int_{-1}^{x_{100}} \int \int \cdots \left(\int_{-1}^{x_2} \left(\int_{-1}^{x_1} f(t)\,dt \right) dx_1 \right) dx_2 \cdots dx_{100}$$

$$f(x) = \begin{cases} e^{\frac{1}{x^2-1}} & |x| \le 1 \\ 0 & \end{cases}$$

and $(Jf)(x) = \int_{-1}^{x} f(t)\,dt$

Answer: Why is this a Taylor series?

Recall: $f^{(n)}(-1) = 0$ (HW).

so $P_n(x) = \sum_{r=0}^{n} \frac{f^{(r)}(-1)}{r!} x^r = 0$

$c=-1$ so $R_n(x) = f(x) - P_n(x) = f(x)$

so bounds on R (eg Lag bound) is a bound on f.

Similarly $g^{(n)}(-1) = 0$ if $c=-1$ the $P_n(x) = 0$

so for g also $R_n(x) = g(x)$.

and

and $R_{100}(x) = g(x) \underset{\text{Lag.}}{=} \frac{g^{(100)}(\xi)}{(100)!}(x-c)^{100} = \frac{f(\xi)(x+1)^{100}}{(100)!}$

$$\leq \frac{1 \cdot (x+1)^{100}}{(100)!}$$

so $g(1) \leq \dfrac{2^{100}}{(100)!}$

Problem:

(a) Show that

$$\sum_{n=0}^{\infty} 4^{-n} a_n \cos(3^n x) = f(x)$$

converges pointwise to $f(x) \in C^1\mathbb{R}$

if $a_n =$ nth digit of π.

$\in \{0,1,2,\dots,9\}$.

(b) Find other a_n's for which $f \notin C^1\mathbb{R}$.

Ans: (a) Need unif. convergence of the partial sums and the derivatives of the partial sums ($\|\cdot\|_{C^1}$ - norm convergence),

Derivative: $f'(x) = -\sum \left(\frac{3}{4}\right)^n a_n \sin(3^n x)$

converges uniformly since $a_n \leq 9$ bdd.

so $|f'(x)| \leq \sum \left(\frac{3}{4}\right)^n \cdot 9 \cdot 1$

which converges since $\frac{3}{4} < 1$

(b) Use $a_n = 8^n$