MAT 127B-B

Winter 2021

Right Board

Outline:

Define: Derivatives

and Integrals.

Fo.

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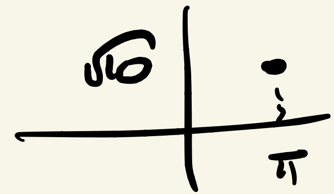
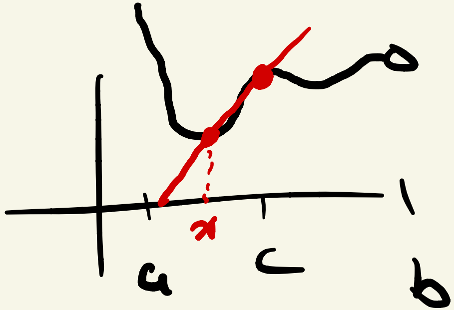
analogous to the plane $\mathbb{R}^2$



approx. points using

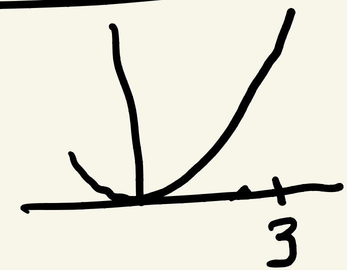
rationals: $(\pi, \sqrt{10}) = (3.14159\ldots, 3.166\ldots)$

Both are examples of
Banach algebras.
see Functional analysis course
for more examples.



Ex: 1 $f_1(x) = x^2$

$f_1'(3) = 2 \cdot 3 = 6$ by rules



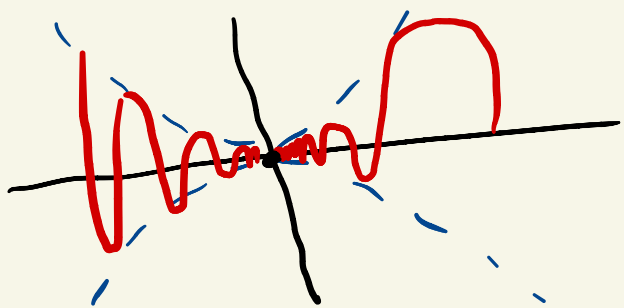
also using the def:

$$f_1'(3) = \lim_{h \to 0} \frac{f_1(h+3) - f_1(3)}{h}$$

$$= \lim_{h \to 0} \frac{(h+3)^2 - (3)^2}{h} = \lim_{h\to 0} \frac{h^2 + 6h + 9 - 9}{h}$$

$$= \lim_{h \to 0} (h+6) = 6$$

Ex 3: $f_3(x) = \begin{cases} x^2 \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$



(A) $f_3(x)$ is continuous at any $x \neq 0$

for $x=0$ compute:

$$\lim_{x \to 0} x^2 \sin\frac{1}{x}$$

$$-x^2 \leq x^2 \sin\frac{1}{x} \leq x^2$$

$$\lim_{x\to 0}(-x^2) = 0 \leq \lim_{x \to 0} x^2 \sin\frac{1}{x} \leq \lim_{x\to 0} x^2 = 0$$

$$\underbrace{\lim_{x \to 0} x^2 \sin\frac{1}{x} = 0}_{\text{by squeeze thm.}} = f(0)$$

so cts at 0 as well.

Next try in Breakout rooms:

$$g_1(x) = \begin{cases} x \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

(A) cts? (B) diff?

$$g_2(x) = \begin{cases} x^2 & x = \frac{a}{b} \text{ rational,} \\ 0 & x \text{ otherwise.} \end{cases}$$

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Spaces of functions:

If f is cts in (a,b) write

$$f \in C(a,b) = C^0(a,b)$$

If f is diff in (a,b) write

$$f \in D^1(a,b)$$

If f is ctsly diff in (a,b) write

$$f \in C^1(a,b)$$

Relationships:

$$C^0(0,1) \underset{\neq}{\overset{\text{T8.17}}{\supsetneq}} D^1(0,1) \underset{\neq}{\overset{\text{def}}{\supsetneq}} C^1(0,1) \underset{\neq}{\overset{\text{T8.17}}{\supsetneq}} D^2(0,1) ? \ldots$$

ω $\qquad |x|$ $\cancel{\&}$

For containments need proof.

For not equal need example functions.

Eg: (1) $f(x)=|x|$ has $f(x) \in C^0$ but $f(x) \notin D^1$

(2) $f(x)=\begin{cases} x^2 \sin\frac{1}{x} & x\neq 0 \\ 0 & x=0 \end{cases}$ has $f(x) \in D^1$ but $f(x) \notin C^1$

$(f' \notin C^0)$.

from last time

In rooms: Find more examples distinguishing: C^0 from D^1

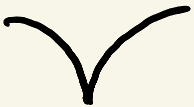
D^1 from C^1

C^1 from D^2

Figure out what C^∞ and D^∞ are.

C^0	D^1	C^1	D^2	$C^\infty = D^\infty$ smooth

$\sqrt{|x|}$



$$\begin{cases} x\ln|x| & x\neq 0 \\ 0 & x=0 \end{cases}$$

$x^2\sin\frac{1}{x}$

polynomial
trig.

$$\begin{cases} \boxed{x\ln(x)} & x\neq 0 \\ 0 & x=0 \end{cases}$$

$$\lim_{x\to 0}(x\ln|x|)=\lim_{x\to 0}\frac{\ln|x|}{\frac{1}{x}} \overset{L'H}{=} \lim_{x\to 0}\frac{\frac{1}{x}}{\frac{-1}{x^2}} = \lim_{x\to 0}(-x)=0$$

cts? ✓

Compute

$$f(c) = f(c) + \underbrace{\lim_{x\to c} \frac{f(x)-f(c)}{x-c}}_{\text{exists by hypothesis}} \cdot \underbrace{\lim_{x\to c} (x-c)}_{0}$$

$$= \lim_{x\to c} f(c) + \lim_{x\to c} \left[\frac{f(x)-f(c)}{x-c} \cdot (x-c) \right]$$

$$= \lim_{x\to c} \left[f(c) + f(x) - f(c) \right]$$

$$= \lim_{x\to c} f(x)$$

✓

210108

eg: $g(x) = \sin(x)$, $f(x) = \frac{1}{x}$, $k = 3$

$$3 \cdot \sin(x) = 3 \cdot g$$

$$\sin(x) + \frac{1}{x} = g + f$$

$$\frac{1}{x} \sin(x) = g \cdot f$$

$$\sin\left(\frac{1}{x}\right) = g \circ f$$

$$\frac{1}{\sin(x)} = f \circ g$$

eg: $\lim_{x \to \pi} \sin\left(\frac{1}{x}\right) = \sin\frac{1}{\pi} = \lim_{y \to \frac{1}{\pi}} \sin(y)$

$$\left(3\sin(x)\right)' = 3\sin'(x) = 3\cos(x)$$

$$\left[\frac{1}{x} + \sin(x)\right]' = \frac{-1}{x^2} + \cos(x)$$

$$\left(\frac{1}{x}\sin(x)\right)' = \frac{-1}{x^2}\sin(x) + \frac{1}{x}\cos(x).$$

$$\left(\frac{\sin(x)}{x}\right)' = \frac{x\cos(x) - 1\cdot\sin(x)}{x^2}$$

$$\left[\sin\left(\frac{1}{x}\right)\right]' = \cos\left(\frac{1}{x}\right)\cdot\left(\frac{-1}{x^2}\right).$$

Question: Are $\cancel{C^k(\mathbb{R})}$, $\cancel{D^k(\mathbb{R})}$

$C^1(\mathbb{R})$, $D^2(\mathbb{R})$, $C^2(\mathbb{R})$ ...

and $C^\infty(\mathbb{R}) = D^\infty(\mathbb{R})$

also preserved by these operations?

$$\mathrm{Fun}(\mathbb{R}) \supsetneq C^0(\mathbb{R}) \supsetneq D^1(\mathbb{R}) \supsetneq C^1(\mathbb{R}) \supsetneq D^2(\mathbb{R}) \supsetneq \cdots$$

$\mathrm{Fun}(\mathbb{R})$	$C^0(\mathbb{R})$	$D^1(\mathbb{R})$	$C^1(\mathbb{R})$	$D^2(\mathbb{R})$
	127A	8.19/8.21	$k\cdot$, $+$	$k\cdot$, $+$,
$k\cdot$, $+$,	$k\cdot$, $+$,	$k\cdot$, $+$,	$\bullet$, 0 $\supseteq$	$\bullet$, 0
$\bullet$, 0	$\bullet$, 0	$\bullet$, 0		

Note: If $f \in C^1(\mathbb{R})$ then f' is cts.
so $k\cdot(f')$ is cts so $(k\cdot f)'$ is cts.
so $k\cdot f \in C^1\mathbb{R}$

Note: If $f \in D^2(\mathbb{R})$ then f'' exists
so $k\cdot f''$ exists so $(k\cdot f)''$ exists
so $k\cdot f \in D^2(\mathbb{R})$

Note: If $f, g \in C^2 \mathbb{R}$ then

$$(f \cdot g)'' = [f' \cdot g + f \cdot g']'$$

$$= \underline{f}'' \cdot g + f' \cdot g' + f' \cdot g' + f \cdot \underline{g}''$$

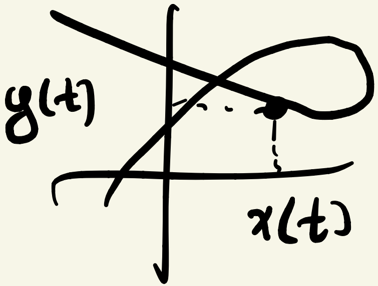
which exists since f'' and g'' exist.

Thm: If $f, g \in C^k(\mathbb{R})$ or $D^k(\mathbb{R})$, $k \in \mathbb{R}$

then kf, $f+g$, $f \cdot g$, $g \circ f$ $\in C^k(\mathbb{R})$ or $D^k(\mathbb{R})$

also.

Parameterized curves and slopes:



Claim: $$\lim_{s \to t} \frac{y(s) - y(t)}{x(s) - x(t)} = \frac{y'(t)}{x'(t)}$$

as long as $x'(t) \neq 0$

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in this case this is the slope of the tangent line to the curve at $(x(t), y(t))$

Question: What can the slope of the tangent line be if

$$x'(t) = 0 \quad ?$$

vertical is most common.
could be not defined

or any other slope if $\frac{y'}{x'} = \frac{0}{0}$.

210111

If $x(t), y(t) \in \mathrm{Fun}\,\mathbb{R}$

then $\{(x(t), y(t)) \mid t \in \mathbb{R}\} \subseteq \mathbb{R}^2$

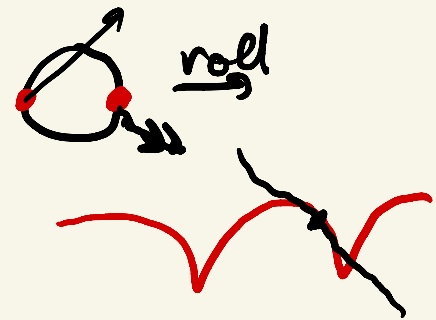
is the curve param by x and y.

Ex: Cycloid

If a radius 1 wheel rolls at $1\,\frac{\text{rad}}{\text{sec}}$,

a point on the edge has position

$$P(t) = (\underbrace{t + \cos(t)}_{x(t)}, \underbrace{1 + \sin(t)}_{y(t)})$$



Lem: If $x(t), y(t) \in D'(\mathbb{R})$ and $c \in \mathbb{R}$ with $x'(c) \neq 0$ then

$$\lim_{t \to c} \frac{y(t)-y(c)}{x(t)-x(c)} \underset{①}{=} \frac{y'(c)}{x'(c)} \underset{②}{=}$$ the slope of the tang line to the param curve at $(x(c), y(c))$

Ex: Find the slope of the motion of the point when it is at the same height (1) as the center

Ans: ② $\text{slope} = \frac{y'(t)}{x'(t)} = \frac{\cos(t)}{1-\sin(t)}$

Consider t with $y(t) = 1 = 1 + \sin(t)$

so $\sin(t) = 0$ or $t = n\pi$
for n an integer.

so slope $= \dfrac{\cos(n\pi)}{1 - \sin(n\pi)} = \cos(n\pi) = \pm 1$.

Inverse Functions:

switching roles of x and y:

Def: If $(f \circ g)(x) = x$ then
call f the inv. fn. to g
and write $f^{-1} = g$.

Lem: If $f^{-1} = g$ then

$$(g \circ f)(x) = x \quad \text{so} \quad g^{-1} = f \text{ also}$$

(unlike $\sin \frac{1}{x} = \sin x \circ \frac{1}{x} \neq \frac{1}{\sin(x)} = \frac{1}{x} \circ \sin x$),

Compute: If $f(x), g(x) \in D^1(\mathbb{R})$

and $g = f^{-1}$ so $(f \circ g)(x) = x$ then:

$$1 = D(x) = D(f \circ g) \underset{\text{ch. rl.}}{=} [(Df) \circ g] \cdot (Dg)$$

$$\text{so } (f^{-1})'(c) = (Df^{-1})(c) = (Dg)(c) = \frac{1}{(Df) \circ g(c)}$$

$$= \frac{1}{f'(f^{-1}(c))}$$

Note: If $g^{-1} = f$ then the graphs of f and g differ by: reflect through the line $x = y$.

Bestiary:

$C^\infty \mathbb{R} = D^\infty \mathbb{R}$ $\quad F(x),\ x^k,\ \sin(x),\ e^x$

$\subsetneq$

$\vdots$

$\subsetneq$

$C^k \mathbb{R} \quad f_{k+1} \quad g_{2k+1}$

$\subsetneq$

$D^k \mathbb{R} \quad\quad g_{2k}$

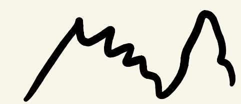
$\subsetneq$

$\vdots \quad\quad \vdots$

$\subsetneq$

$$f_k(x) = \begin{cases} x^k & x > 0 \\ 0 & x \le 0 \end{cases}$$

$$g_k(x) = \begin{cases} x^k \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$W(x) =$ 

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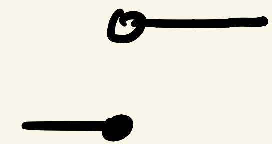
$f_0(x) =$ 

$C^2\mathbb{R}$ $\quad f_3 \quad g_5$

$\cap\neq$

$D^2\mathbb{R}$ $\quad\quad\quad g_4$

$\cap\neq$

$C^1\mathbb{R}$ $\quad f_2 \quad g_3$

$\cap\neq$

$D^1\mathbb{R}$ $\quad\quad\quad g_2$

$\cap\neq$

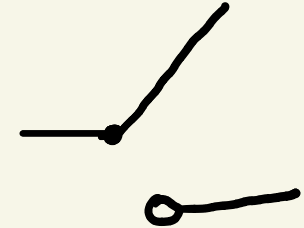
$C^0\mathbb{R}$ $\quad f_1 \quad g_1$

$\cap\neq$

$D^0\mathbb{R}$ $\quad\quad\quad g_0$

$\cap\neq$

$\mathrm{Fun}\,\mathbb{R}$ $\quad f_0 \quad g_0^2$

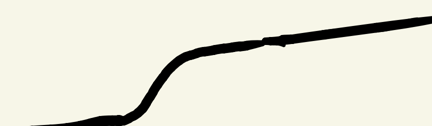
$f_1(x) =$ 

$Df_1(x) =$ 

$\neq$

f_0

$$F(x) = \begin{cases} e^{-\frac{1}{x}} & x > 0 \\ 0 & x \leq 0 \end{cases}$$



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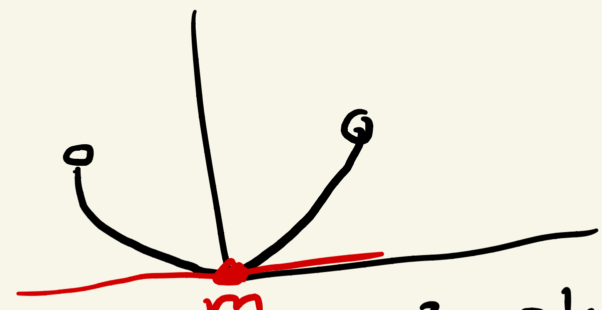
Thm (Fermet) (8.27 H) (5.2.6 A):

(Internal Extrema)

If $f \in D^1(a,b)$

and $f(M) = \mathrm{Max}(\mathrm{Im} f)$ (or $\min(\mathrm{Im} f)$).

then $f'(M) = 0$



$f(x) = x^2 \in D^1(-1,1)$

No M

Proof: $f'(M) = \lim\limits_{x \to M} \dfrac{f(x)-f(M)}{x-M}$

so $f'(M) = \lim\limits_{x \to M^+} \dfrac{f(x)-f(M)}{x-M} \leq 0$

$\quad = \lim\limits_{x \to M^-}$ " " ≥ 0

so $f'(M) = 0$.

If $x > M$ then $\dfrac{f(x)-f(M)}{x-M} \leq 0$

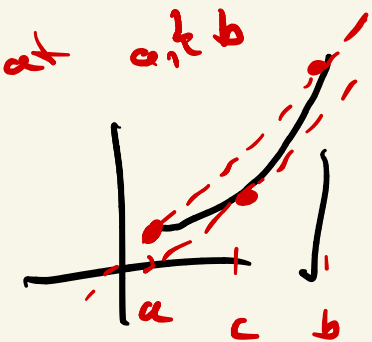
$f(x) - f(M) \leq 0$ and $x - M > 0$.

If $x < M$ then $\dfrac{f(x)-f(M)}{x-M} \geq 0$ similarly

Thm (Mean Value) (8.3.3 H) (5.3.2 A)

If $f \in C^0[a,b] \cap D^1(a,b)$ then

$\exists\, c \in (a,b)$ with $\underbrace{f'(c)}_{\text{slope of tang. at } c} = \underbrace{\dfrac{f(b)-f(a)}{b-a}}_{\text{slope of secant line at } a,\, b}$



Proof: To use Rolle's thm.

take $g(x) = f(x) - \underbrace{(x-a)\underbrace{\dfrac{f(b)-f(a)}{b-a}}_{\text{number}}}_{\text{lin.}}$

$g \in C^0[a,b] \cap D^1(a,b)$ since both function spaces are preserved by adding fns and mult by const. and $f \in C^0 \cap D^1$ by hyp and $x-a \in C^0 \cap D^1$ by computation.

Compute $g(a) = f(a) - 0 = f(a)$

$$g(b) = f(b) - \cancel{(b-a)}\frac{f(b)-f(a)}{\cancel{b-a}}$$

$$= \cancel{f(b)} - \cancel{f(b)} + f(a) = f(a)$$

Same

Hence the hyp. of Rolle's Thm hold and

get $c \in (a,b)$ with $g'(c) = 0$

hence $f'(c) = g'(c) + (x-a)' \dfrac{f(b)-f(a)}{b-a}$

so $f'(c) = 0 + 1 \cdot \dfrac{f(b)-f(a)}{b-a}$ ✓.

2/10/15

Asside: From Lin Alg: MAT 22A or MAT 67.

$\mathrm{Fun}(a,b)$ or $\mathrm{Fun}[a,b]$ are $\mathbb{R}$ vect. spaces

and every $D^k(a,b)$, $C^k[a,b]$, $C^k(a,b)$

and their intersections are

vector subspaces.

Hence: if $f, g \in D^1[a,b] \cap C^0[a,b]$

then $f+g \in$ " ".

Recall: MVT!

If $h \in C^0[a,b] \cap D^1(a,b)$ then $\exists c \in (a,b)$

with $h'(c) = \dfrac{h(b)-h(a)}{b-a}$

Computing: RHS

$$\frac{h(b)-h(a)}{b-a} = \frac{[f(b)-f(a)]g(b) - f(b)[g(b)-g(a)] - \{[f(b)-f(a)]g(a) - f(a)[g(b)-g(a)]\}}{b-a}$$

$$= 0.$$

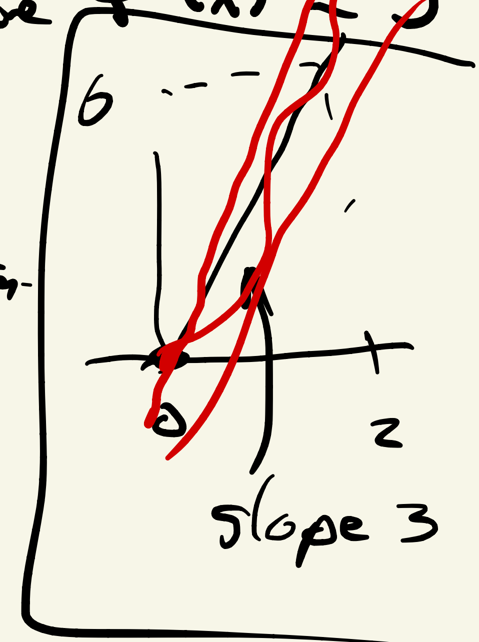
Or use Rolle's Thm since $h(b)=h(a)$]

Example: If $f \in D'(\mathbb{R})$ and $f(0)=0$ and $\forall x$ have $f'(x) \leq 3$ show that $f(2) \leq 6$.

Ans: Use MVT: and contradiction.

Assume for contradiction that $f(2) > 6$ and by MVT get $c \in (0,2)$ with $f'(c) = \dfrac{f(2)-f(0)}{2-0} > 3$

a contradiction.



Thm (Darboux) (Proof involves MVT):

$\forall f \in D^0(\mathbb{R})$ have $\underline{Im(f|_{[x,y]})}$ is connected.

equivalently:

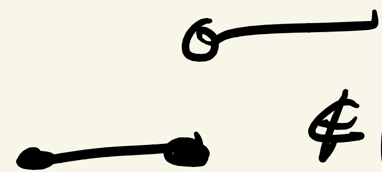
$\forall F \in D^1(\mathbb{R})$ and α is between

$F'(x)$ and $F'(y)$ then $\exists\ c \in (x,y)$

with $F'(c) = \alpha$

$\boxed{210120}$

Ex: If f is a function taking exactly two values then f is not a derivative.

eg: $f(x) = \left\{\begin{matrix} 1 & x>0 \\ 0 & x \le 0 \end{matrix}\right\} =$  $\notin D^0(\mathbb{R})$

$f(x) = \left\{\begin{matrix} 1 & x=0 \\ 0 & x \neq 0 \end{matrix}\right\} =$  $\notin D^0(\mathbb{R})$.

Cor: If $f, g \in D^0(a,b)$ and $e \in (a,b)$ and $\forall x \neq e$ have $f(x) = g(x)$ then $f(e) = g(e)$.

Proof: Since $f, g \in D^0(a,b)$ have $f-g \in D^0(a,b)$

but $(f-g)(x) = \begin{cases} 0 & x \neq e \\ \underbrace{f(e)-g(e)}_{\neq 0} & x = e \end{cases}$

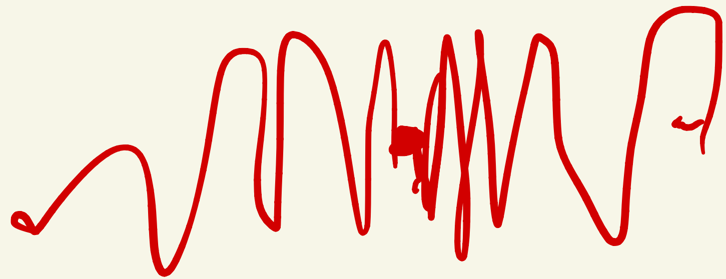
If $f(e) \neq g(e)$ then takes exactly two values and hence is not a deriv. fn.

so $f-g \notin D^0(a,b)$ a contradiction

Ex: $f(x) = \begin{cases} x^2 \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases} \in D^1(\mathbb{R})$.

computed $f'(x) = \begin{cases} 2x\sin\frac{1}{x} - \cos\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases} \in D^0(\mathbb{R})$

so $g(x) = \begin{cases} & x \neq 0 \\ \frac{1}{2} & x = 0 \end{cases} \notin D^0(\mathbb{R})$



either $F'(c) < \alpha < F'(d)$] similar.

or [" " $> \alpha >$ " "] consider below

Consider $G(x) = F(x) - \alpha x \in D^1(a,b)$

so $G'(x) = F'(x) - \alpha \in D^0(a,b)$

so [$G'(c) > 0 > G'(d)$

and the goal is to find e with $G'(e) \stackrel{?}{=} 0$

Since $G \in D^1(a,b) \subseteq C^0(a,b)$

by 7.37 $\exists\, M \in [c,d]$ with $G(M) = \text{Max Im } G|_{[c,d]}$

ⓐ If $M \in (c,d)$ then by Int. Ext Thm 8.27

have $G'(M) = 0$ so take $e = M$.

ⓒ If $M = c$ then get x with $G(x) > G(c) = G(M)$ contradicting the choice of M.

ⓓ If $M = d$ ...] similar to ⓒ

Compute $0 < G'(c) = \lim\limits_{x \to c^+} \dfrac{G(x) - G(c)}{x - c}$

so $\forall\, \varepsilon > 0\ \exists\, d - c > \delta > 0\ \forall\, c < x < c + \delta$ have $\left| G'(c) - \dfrac{G(x) - G(c)}{x - c} \right| < \varepsilon$

Choose $\dfrac{G'(c)}{2} > 0$ and any $c < x < c + \delta$

so $0 < \frac{1}{2} G'(c) \cdot (x-c) < G(x) - G(c)$

getting $G(x) > G(c) = G(m)$

a contradiction.

210122

Thm: (8.54) ($\frac{0}{0}$ version of L'Hôspital's limit rule)

$\forall k, g \in D'(a,b)$, $a<c<b$, $k(c)=0=g(c)$ and $\lim_{x\to c} \frac{k'(x)}{g'(x)} = L$ then $\lim_{x\to c} \frac{k(x)}{g(x)} = L$

Proof: Using GMVT and Rolle's Thm.

Note: $k(x)-k(c)=k(x)$ and $g(x)-g(c)=g(x)$.

$\forall \varepsilon>0 \; \exists \delta>0 \; \forall \; |x-c|<\delta$ have $\left|\frac{k'(x)}{g'(x)} - L\right| < \varepsilon$

Hence: $g'(x) \neq 0$

Also $g(x) \neq 0$ [Pf: Ass. for contrad. that $g(x) = 0 = g(c)$
so by Rolle's thm $\exists\ |y-c| \leq |x-c| < \delta$
with $g'(y) = 0$ a contradiction.]

Check:

$\forall \varepsilon > 0\ \exists\ \delta > 0\ \forall\ |y-c| < \delta$

by GMVT $\exists\ x$ between c and y with

$$k(y)\, g'(x) = k'(x)\, g(y)$$

but $g(y) \neq 0$ and $g'(x) \neq 0$

so $\dfrac{k(y)}{g(y)} = \dfrac{k'(x)}{g'(x)}$ so $\left| \dfrac{k(y)}{g(y)} - L \right| < \varepsilon.$

qed.

Examples:

$f(x) = e^x$ so $f^{(n)}(x) = e^x$

so $P_{n,c}(x) = e^c + e^c(x-c) + \cdots + e^c \frac{(x-c)^n}{n!}$

eg: $P_{4,0}(-1) = 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} = .375$

$R_{4,0,-1} = e^{-1} - .375 \in \left[\frac{-1}{120}, \frac{-1}{120e}\right]$

both good!

$e^{-1} \approx .368$

$P_{4,0}(2) = 7$ $\quad R_{4,0,2} \in \left[\frac{4}{15}, e^2\frac{4}{15}\right]$

so $e^2 \approx 7$ with error

$f(x) = \frac{1}{x}$ so $f^{(n)}(x) = \frac{(-1)^n}{x^{n+1}} n!$

so $P_{n,c}(x) = \frac{1}{c} + \frac{(x-c)}{c^2} + \frac{(x-c)^2}{c^3} + \dots + \frac{(x-c)^n}{c^{n+1}}$

try $n=4$, $c=1$, $x=\frac{1}{2}$ or $x=3$

$\frac{1}{2}$ is well approx by $\frac{31}{16}$

$\frac{1}{3}$ is very poorly approx ||

Example! $\underline{n=2}$

$$g'(t) = \left[f(x) - f(t) - f'(t)(x-t) - f''(t)\frac{(x-t)^2}{2} \right]'$$

$$= 0 - \cancel{f'(t)} + \cancel{f'(t)} - f''(t)(x-t) + f''(t)(x-t)$$

$$- f'''(t)\frac{(x-t)^2}{2}$$

cancel

210125

limit of rationals

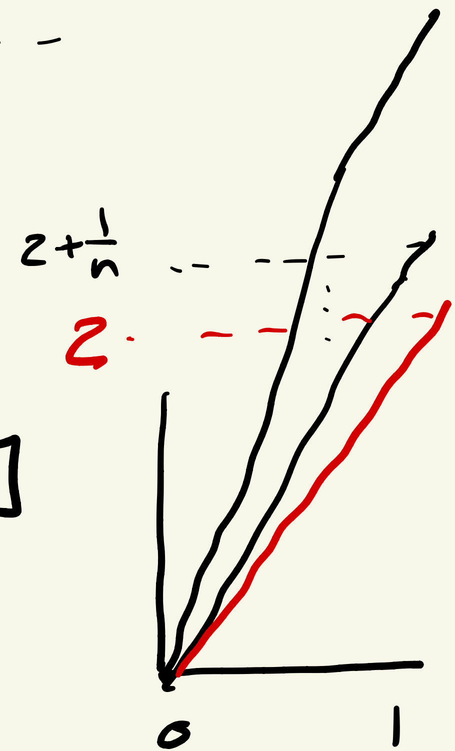
Ex: $e \approx 2.71828\ldots$

$$= 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{4!} + \cdots$$

$$= 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{\ddots}}}}$$

Ex: $f_n(x) = \left(2 + \frac{1}{n}\right)x \in \mathrm{Fun}[0,1]$

$F(x) = 2x$



Ex: $\|F(x)\|_{sup} = 2$

$\|f_n(x)\|_{sup} = 2 + \frac{1}{n}$

$\|f_n(x) - F(x)\|_{sup} = \frac{1}{n}$

$\frac{1}{n}x$

Ex: $\|F(x)\|_{L^1} = 1$

$\|f_n(x)\|_{L^1} = 1 + \frac{1}{2n}$

$\|f_n(x) - F(x)\|_{L^1} = \frac{1}{2n}$

Ex: $\|F(x)\|_{C^1} = 2 + 2 = 4$

$\|f_n(x)\|_{C^1} = 4 + \frac{2}{n}$

$\|f_n(x) - F(x)\|_{C^1} = \frac{2}{n}$

Ex: $|f_n(x) - F(x)| = \frac{1}{n}x$

(A) $f_n \xrightarrow[n\to\infty]{\text{ptwise}} F$ ✓

(B) $f_n \xrightarrow[n\to\infty]{\text{unif}} F$ ✓

(C) $f_n \xrightarrow[n\to\infty]{\|\cdot\|_{\sup}\text{-norm}} F$ ✓

(D) $f_n \xrightarrow[n\to\infty]{\|\cdot\|_{C^1}\text{-norm}} F$ ✓

	ptwise	unif. = $\|\cdot\|_{sup}$-norm	$\|\cdot\|_{C^1}$-norm
①	$(0,1]$	$[\varepsilon, 1-\varepsilon]$ →	same
		for any ε	
②	$\mathbb{R}$	$\mathbb{R}$	$(-\infty,-\varepsilon], [\varepsilon,\infty)$
③	$\underline{(0,\infty)}$	$[\varepsilon,\infty)$ not $(0,\infty)$	$[\varepsilon,\infty)$ not $(0,\infty)$, $\mathbb{R}$
④	$n\to\infty$ $\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}$
	$n\to-\infty$ $\mathbb{R}$	$(-\infty,0], [\varepsilon,\infty)$ →	same

5	$\mathbb{R}$	$\mathbb{R}$	nowhere.
6	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}$
7	$\underline{\mathbb{R}}$	$(-\infty, k]$ for any k $\longrightarrow$	same
8	$(-1, 1)$	$(-1+\varepsilon, 1-\varepsilon)$ $\longrightarrow$	same.

$\hookrightarrow$ Much more later.

210127 Cauchy sequences:

Def 3.45: If (a_n) is a sequence in $\mathbb{R}$ then (a_n) is <u>$|\cdot|$-Cauchy</u> if

$\forall \varepsilon > 0 \ \exists M \in \mathbb{Z} \ \forall n, m > M$ have $|a_n - a_m| < \varepsilon$

Def: 9.12: If $(f_n(x))$ is a seq in $\operatorname{Fun}(a,b)$ then (f_n) is <u>uniformly-Cauchy</u> if

$\forall \varepsilon > 0 \ \exists M \in \mathbb{Z} \ \forall n, m > M$ have $\forall x \in (a,b]$ have $|f_n(x) - f_m(x)| < \varepsilon$

Def: 13.49: If $(f_n(x))$ is a sequence of functions then (f_n) is $\|\cdot\|$-Cauchy if $\forall \varepsilon > 0 \; \exists M \in \mathbb{Z} \; \forall n, m > M$ have $\|f_n - f_m\| < \varepsilon$.

Lemma: A sequence (f_n) in $\mathrm{Fun}(a,b)$ is $\|\cdot\|_{\sup}$-Cauchy iff (f_n) is unif-Cauchy.

Pf: Similar to previous lemma.

Proof of 9.13 (use 3.46).

For one direction Assume $f_n \xrightarrow{\text{unif.}} F$

so $\forall \varepsilon > 0 \ \exists N_{\frac{\varepsilon}{2}} \quad \forall n > N_{\frac{\varepsilon}{2}}$ have $\forall x \in [a,b]$

have $|f_n(x) - F(x)| < \frac{\varepsilon}{2}$.

hence $\forall \varepsilon > 0 \ \exists M_\varepsilon = N_{\frac{\varepsilon}{2}} \quad \forall n, m > M_\varepsilon$ have $\forall x \in [a,b]$

have $|f_n(x) - f_m(x)| \leq |f_n(x) - F(x) + F(x) - f_m(x)|$

$\leq |f_n(x) - F(x)| + |F(x) - f_m(x)| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

Other dir. next.

210129

Rest of pf. of 9.13 is HW 6.2.5.

Thm 9.14: If (f_n) is a segenence in $Bdd(a,b)$ convergin uniformly to $f \in Fun(a,b)$ then $f \in Bdd(a,b)$.

Equivalently: $Bdd(a,b)$ is $\|\cdot\|_{sup}$ - complete.

Proof: Note: $\|f+g\|_{sup} \leq \|f\|_{sup} + \|g\|_{sup}$.
(required of any norm and called the triangle inequality).

If (f_n) conv. unif. to f.

or equiv. conv. in $\|\cdot\|_{sup}$ to f,

then $\forall \varepsilon > 0 \; \exists N_\varepsilon \quad \forall n \geq N_\varepsilon$

have $\|f - f_n\|_{sup} < \varepsilon$.

If also each $f_n \in Bdd(a,b)$ then

$\forall n$ have $\|f_n\|_{sup} < \infty$.

Combining: $\|f\|_{sup} = \|f - f_{N_1} + f_{N_1}\|_{sup}$

$\leq \|f - f_{N_1}\|_{sup} + \|f_{N_1}\|_{sup}$

$\leq \quad 1 \quad + \|f_{N_1}\|_{sup} < \infty$

Def 9.24 (Weierstrass) q~~ed~~.

$$f_n(x) = \sum_{j=1}^{n} 2^{-j} \cos(3^j x)$$

Claim: f_n converges in $\|\cdot\|_{sup}$-norm (or eq. uniformly).

Pf: Need to check: by 9.13 (f_n) is uniformly Cauchy.

that is $\forall \varepsilon > 0 \ \exists M_\varepsilon$ $\forall n, m \geq M_\varepsilon$ have $\|f_n - f_m\|_{sup} < \varepsilon$.

Choose M_ε with $2^{-M_\varepsilon} < \varepsilon$

and compute:

$$\|f_n - f_m\|_{sup} = \left\| \sum_{j=m+1}^{n} 2^{-j} \cos(3^j x) \right\|_{sup}$$

$$\leq \left| \sum_{j=m+1}^{n} 2^{-j} \right|$$

$$= 2^{-m} \sum_{j=1}^{n-m} 2^{-j} < 2^{-m} \sum_{j=1}^{\infty} 2^{-j}$$

$$= 2^{-m}\left(\frac{1}{1-\frac{1}{2}} - 1\right) = 2^{-m}$$

$$\leq 2^{-M_\varepsilon} < \varepsilon.$$

Def: $w(x)$ is the uniform limit of $(f_n(x))$.

Ans: ① $\|w(x)\|_{sup} \leq 1+1=2$

since $\|f_n(x)\|_{sup} \leq \left|\sum_{j=1}^{n} 2^{-j}\right| < \sum_{j=1}^{\infty} 2^{-j} = 1$

② Consider: $g_n(x) = \dfrac{1}{x+\frac{1}{n}} \in \mathrm{Fun}(0,\infty)$

conv. ptwise to $g(x) = \frac{1}{x} \in \mathrm{Fun}(0,\infty)$

and $\|g_n(x)\|_{sup} = \dfrac{1}{\frac{1}{n}} = n$

but $\| \frac{1}{x} \|_{sup} = \infty$ (does not exist).

210201

Thm: If (f_n) is a sequence in $D^1(a,b)$

with ① (f_n') converges uniformly to $g(x)$

and ② for some $c \in (a,b)$ have

$(f_n(c))$ conv. (in $\mathbb{R}$),

then (HW) (f_n) converges uniformly to $f(x)$,

and (9.18) $f'(x) = g(x)$.

Proof ~~plan~~ for 9.19:

Need: $\forall\, p \in (a,b),\ \varepsilon > 0\ \ \exists\, \delta\ \ \forall\, |x-p| < \delta$

have $\left| \dfrac{f(x) - f(p)}{x-p} - g(p) \right| \overset{?}{<} \varepsilon$

So $\forall\, p \in (a,b),\ \varepsilon > 0\ \ \exists\, \delta = \delta_{\underbrace{\underbrace{N_\varepsilon}_{\text{from (1)}},\, p,\, \varepsilon}_{\text{from (4)}}}$

and take $n = N_\varepsilon$

and $\forall\, |x-p| < \delta$

take $m = M_{\varepsilon, p, x}$ and compute:

$$
\begin{aligned}
\left|\frac{f(x)-f(p)}{x-p}-g(p)\right| &\leq \left|\frac{f(x)-f(p)}{x-p}-\frac{f_m(x)-f_m(p)}{x-p}\right| \\
&\quad + \left|\frac{f_m(x)-f_m(p)}{x-p}-\frac{f_n(x)-f_n(p)}{x-p}\right| \\
&\quad + \left|\frac{f_n(x)-f_n(p)}{x-p}-f_n'(p)\right| + \left|f_n'(p)-g(p)\right| \\
&\leq \left|\frac{f(x)-f_m(x)}{x-p}\right| + \left|\frac{f(p)-f_m(p)}{x-p}\right| + \underbrace{\left|f_n'(\xi)-f_m'(\xi)\right|}_{3} \\
&\quad + \left|\frac{f_n(x)-f_n(p)}{x-p}-f_n'(p)\right| + \left|f_n'(p)-g(p)\right|
\end{aligned}
$$

$$\leq \underbrace{\frac{\frac{\varepsilon}{5}|x-p|}{|x-p|} + \frac{\frac{\varepsilon}{6}|x-p|}{|x-p|}}_{2} + \underbrace{\frac{\varepsilon}{5}}_{1} + \underbrace{\frac{\varepsilon}{5}}_{4} + \underbrace{\frac{\varepsilon}{5}}_{1} = \varepsilon$$

210203

Check: triangle ineq. 3 for $\|\cdot\|_{sup}$

If $f, g \in Bdd[a,b]$ then

$$\|f+g\|_{sup} = \sup_{x\in[a,b]} |f(x)+g(x)|$$

$$\leq \sup_{x\in[a,b]} \left[|f(x)| + |g(x)|\right]$$

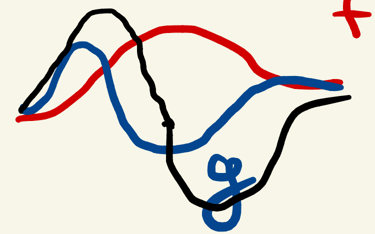
$$\leq \sup_{x\in[a,b]} |f(x)| + \sup_{y\in[a,b]} |g(y)|$$

$$= \|f\|_{sup} + \|g\|_{sup}$$

Examples:

Thm 3.46 $(\mathbb{R}, |\cdot|)$ is a Banach space.

f & g.



① Thm: $(C^0[a,b], \|\cdot\|_{sup})$ is a Banach space.

② Thm: $(D^0[a,b] \cap Bdd[a,b], \|\cdot\|_{sup})$ is a Banach space.

(3) Thm: $(C^1[a,b], \|\cdot\|_{C^1})$ is a Banach space.

Use: Build functions as limits.

Ex: (1) Weierstrass fn:

$$f_n(x) = \sum_{j=0}^{n} 2^{-j}\cos(3^j x)$$

is Cauchy in $(C^0\mathbb{R}, \|\cdot\|_{sup})$.

hence converges to $W(x) \in C^0\mathbb{R}$.

Can check: $W'(0)$ does not exist even though $f_n(x) \in C^1\mathbb{R}$.

(Hence $(C^1\mathbb{R}, \|\cdot\|_{sup})$ is not Banach)

Proof idea for Thm ①:

Checked already that $(C^0[a,b], \|\cdot\|_{sup})$ is a normed linear space. So it remains to check that if

(f_n) is $\|\cdot\|_{sup}$-Cauchy then

(f_n) is $\|\cdot\|_{sup}$-convergent.

but by thm 9.13 (f_n) is

$\|\cdot\|_{sup}$-conv to some $f \in Fun[a,b]$.

also need $f \in C^0[a,b]$

which follows from Thm 9.16.

Proofs of 2 & 3 also follow

from! 9.13, 9.14, 9.16, 9.18

210205)

Proof:

$C^2[0,1]$ is a $\mathbb{R}$-vector space.
by linearity of derivatives.

$\|\cdot\|_{C^2}$ is a norm:

normed lin. space.

- If $\|f\|_{C^2} = 0 = \|f\|_{sup} + \|f'\|_{sup} + \frac{1}{2}\|f''\|_{sup}$

  so $\|f\|_{sup} = 0$ $(= \|f'\|_{sup} = \|f''\|_{sup} = 0)$

  so $f = 0$ ✓

- If $k \in \mathbb{R}$, $f \in C^2[0,1]$ then

$$\|kf\|_{C^2} = |k|\,\|f\|_{sup} + |k|\,\|f'\|_{sup} + \tfrac{1}{2}|k|\,\|f''\|_{sup}$$
$$= |k|\,\|f\|_{C^2}.$$

- If $f, g \in C^2[0,1]$ then

$$\|f+g\|_{C^2} = \|f+g\|_{sup} + \|f'+g'\|_{sup} + \tfrac{1}{2}\|f''+g''\|_{sup}$$

$$\leq \|f\|_{sup} + \|g\|_{sup} + \|f'\|_{sup} + \|g'\|_{sup} + \tfrac{1}{2}\|f''\|_{sup} + \tfrac{1}{2}\|g''\|_{sup}$$

triang ineq for $\|\cdot\|_{sup}$ 3 times

$$= \|f\|_{C^2} + \|g\|_{C^2} \quad \checkmark.$$

Why the $\frac{1}{2}$ in $\|\cdot\|_{C^2}$?

Get Banach algebra:

Note: If $f, g \in C^0[0,1]$ then

$$\|fg\|_{\sup} \leq \|f\|_{\sup} \|g\|_{\sup}$$

Note: If $f, g \in C^1[0,1]$ then

$$\|fg\|_{C^1} = \|fg\|_{\sup} + \|f'g + fg'\|_{\sup}$$

$$\leq \|f\|_{\sup} \|g\|_{\sup} + \|f'\|_{\sup} \|g\|_{\sup} + \|f\|_{\sup} \|g'\|_{\sup}$$

above 3 time

$$\leq (\|f\|_{\sup} + \|f'\|_{\sup})(\|g\|_{\sup} + \|g'\|_{\sup})$$

and triangle ineq.

$$= \|f\|_{C^1} \|g\|_{C^1}$$

one term missing.

Thm (9.21/9.22)

If $(X, \|\cdot\|)$ is a normed linear space

and (f_n) is a sequence in X

and $F_N = \sum_{n=0}^{N} f_n$ then

ⓐ (F_N) is $\|\cdot\|$-Cauchy

iff $\forall \varepsilon\ \exists M\ \forall n,m \geq M$ have $\left\| \sum_{j=n+1}^{m} f_j \right\| < \varepsilon$

ⓑ and (F_N) is $\|\cdot\|$-Cauchy

if (but not only if) $\sum_{n=0}^{\infty} \|f_n\|$ exists $(< \infty)$.

210 208) Power Series and $\mathbb{R}$-analytic functions.

Thm (10.3): If $S(x) \underset{\text{ptwise}}{=} \sum_{n=0}^{\infty} a_n (x-c)^n$

is a power series then there is $R_s = R \in [0, \infty]$ (both possible)

called the radius of convergence with

ⓐ $\forall\ |x-c| > R$ have $S(x)$ diverges.

ⓑ $\forall\ |x-c| < R$ have $S(x)$ converges absolutely.

(so $S(x)$ converges pointwise in $(c-R, c+R)$).

ⓒ $\forall\ r < R$ have $S(x)$ conv. uniformly in $[c-r, c+r]$

(d) $S(c+R)$, $S(c-R)$ are mysterious and usually avoided.

Proof: Uses convergence facts for series in $\mathbb{R}$ from Ch4.

eg Ex4.2: If $0<b<1$ then

$$\sum_{n=0}^{\infty} b^n = \frac{1}{1-b}. \quad \text{(converges)}.$$

For notational convenience take $c=0$.

Recall convergence of series:

4.19 (Comparison).

eg $\left|\sum \frac{\sin(n)}{2^n}\right| \leq \sum \frac{1}{2^n} = \frac{1}{1-\frac{1}{2}} = 2$

4.24 (Ratio test):

$\sum b_n$ converges if $\lim_{n\to\infty} \left|\frac{b_{n+1}}{b_n}\right| < 1$.

(Integral test): If $|f(x)|$ is increasing.

and $\int_1^\infty |f(x)|\,dx < \infty$ (converges).

then $\sum_{n=1}^{\infty} f(n)$ also converges.

eg $\sum_{n=1}^{\infty} \frac{1}{n^2}$ conv since $\int_1^{\infty} \frac{dx}{x^2} = \frac{-2}{x}\Big|_1^{\infty} = 2$.

Ans ①: By ratio test:

$$\lim_{n\to\infty} \left| \frac{x^{n+1}}{x^n} \right| = \lim_{n\to\infty} |x| < 1$$

if $R \geq 1$

but $\sum 1^n$ does not converge so $R = 1$.

② By ratio test again:

$$\lim_{n\to\infty}\left|\frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}}\right| = \lim_{n\to\infty}\left|\frac{x}{n+1}\right| < 1$$

$= 0$ for every x.

so $R = \infty$

③

Algebra:

Thm: If $S(x)$ & $T(x)$ are power series with the same center c with radii of conv $R_S \le R_T$.

Then: $S(x)+T(x)$ has $R_{S+T} \ge R_S$

$S(x)T(x)$ has $R_{ST} \ge R_S$.

$S'(x)$ has $R_{S'} = R_S$

Cor 10.22: If $S(x)$ is a p.s. about c with rad. of conv R then and $0 < r < R$

$$S(x) \in C^{\infty}[c-r, c+r]$$

210210

Cor: If $S(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$ has $R_c > 0$

then this is also the Taylor's series for $S(x)$ about c.

Proof: Taylor series is

$$\sum S^{(n)}(c) \frac{1}{n!} (x-c)^n$$

Need to show: $a_n \overset{?}{=} S^{(n)}(c)\frac{1}{n!}$

but by 10.22

$$S^{(k)}(x) = \sum_{n=0}^{\infty} a_n\, n(n-1)\cdots(n-k+1)\,(x-c)^{n-k}$$

so $$S^{(k)}(c) = \underbrace{a_k\, k!}_{n=k} + \underbrace{0}_{n=k+1} + 0 + \cdots$$ qed

Cor: If $S(x) = \sum a_n (x-c)^n$ with $R_S > 0$

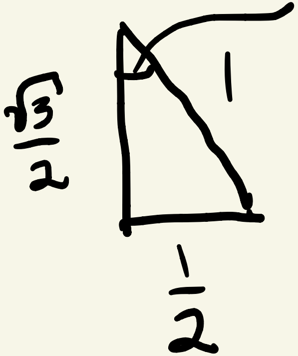
and $T(x) = \sum b_n (x-c)^n$ with $R_T > 0$

and $S(x) = T(x)$ in any neighborhood of c

then every $a_n = b_n$.

Proof: $a_n = S^{(n)}(c) \frac{1}{n!} = T^{(n)}(c) \frac{1}{n!} = b_n$

Cor: Most functions are not limits of power series.



$$\frac{\pi}{6} = \arctan\left(\frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) = \arctan\left(\frac{1}{\sqrt{3}}\right).$$

$$\frac{\pi}{2\sqrt{3}} = \underbrace{\overbrace{1 - \frac{1}{3\cdot 3} + \frac{1}{5\cdot 9} - \frac{1}{7\cdot 27} + \frac{1}{9\cdot 81} - \cdots}^{\sim 3 \text{ decimal places}}}_{\text{sum of rational numbers}}$$

ind.

$$= .9069\ldots$$

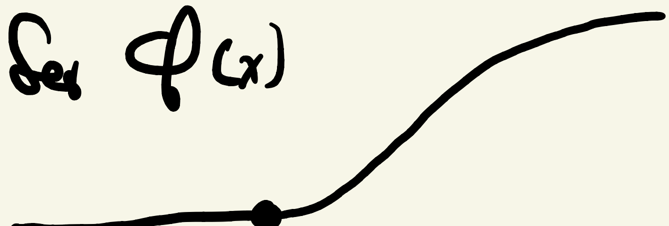
Prop 10.29:

Define $$\varphi(x) = \begin{cases} e^{-\frac{1}{x}} & x > 0 \\ 0 & x \le 0 \end{cases}.$$

then $\varphi(x) \in C^{\infty}(\mathbb{R})$.

and $\varphi^{(n)}(0) = 0$

Hence: the Taylor series for $\varphi(x)$ about $c=0$ is



$$\sum_{n=0}^{\infty} \varphi^{(n)}(0) \frac{1}{n!} (x)^n = 0 \neq \varphi(x).$$

So $\varphi(x)$ is not a limit of a power series about 0.

210217

Notation for Riemann Integrals.

See 11.1 for facts about.

$$\sup_{a\in I} f(a) \quad \text{and} \quad \inf_{a\in I} f(a).$$

Def: If $[a,b]$ is an interval.
then if $P \subseteq [a,b]$ contains a and b
and is finite call P a <u>partition</u>
of $[a,b]$.

Write: $P = \{a = x_0 < x_1 < \dots < x_n = b\}$

with associated intervals: $I_i = [x_{i-1}, x_i]$

of lengths: $\ell_i = x_i - x_{i-1}$

Write $\Pi[a,b]$ is the set of all partitions of $[a,b]$ (so $P \in \Pi[a,b]$)

If $P \subseteq Q$ both in $\Pi[a,b]$ call Q a <u>refinement</u> of P

Note: $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

$\displaystyle\sum_{i=1}^{3} i = 1+2+3=6$

Ans: (a) $I_i = [\frac{i-1}{n}, \frac{i}{n}]$

(b) $l_i = \frac{1}{n}$

(c) $U(f; P_n) = \sum_{i=1}^{n} \frac{1}{n} f(\frac{i}{n}) = \sum_{i=1}^{n} \frac{1}{n}\frac{i}{n} = \frac{1}{n^2}\sum_{i=1}^{n} i = \frac{1}{2} + \frac{1}{2n}$

(d) $L(f; P_n) = \sum_{i=1}^{n} \frac{1}{n} f(\frac{i-1}{n}) = \sum_{i=1}^{n} \frac{1}{n}\frac{i-1}{n} = \frac{1}{n^2}\sum_{i=0}^{n-1} i$

$= \frac{1}{2} - \frac{1}{2n}$

(e) $\frac{1}{2} = \int_0^1 x\,dx$

(f) n a multiple of m

eg $P_6 \supseteq P_3$

Thm 11.20 If $f \in \mathrm{Bdd}[a,b]$,

$P \subseteq Q \in \Pi[a,b]$ partitions.

then (a) $U(f;Q) \leq U(f;P)$

(b) $L(f;Q) \geq L(f;P)$.

Prop 11.21: If $f \in \mathrm{Bdd}[a,b]$

and $P, Q \in \Pi[a,b]$ then

$$L(f;P) \leq U(f;Q)$$

Prop 11.22: If $f \in \mathrm{Bdd}[a,b]$ then $L(f) \leq U(f)$.

210219

Examples for Cauchy criterion:

in $[0,1]$ and write $R_m \in \Pi[0,1]$

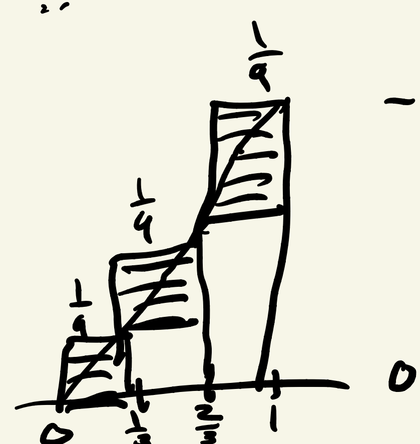
with $R_m = \{0, \frac{1}{m}, \frac{2}{m}, ----, \frac{m-1}{m}, 1\}$

Recall: $U(f, R_m) = \sum_{i=1}^{m} \underbrace{l_i}_{\frac{1}{m}} (\sup_{x \in I_i} f(x))$

$L(f; R_m) = \dots \frac{1}{m} \inf \dots$

Find: $U(f; R_m) - L(f; R_m)$

① $f(x) = x$ $\quad U(f; R_3) - L(f; R_3) = \frac{1}{3}$



$$U(f;R_m) - L(f;R_m) = \frac{1}{m} = \frac{m}{m^2}$$

② $$f(x) = \left\{ \begin{array}{ll} 1 & x = 1 \\ 0 & x \neq 0 \end{array} \right\}$$

③ $$f(x) = \left\{ \begin{array}{ll} 1 & x \geq \frac{1}{2} \\ 0 & x < \frac{1}{2} \end{array} \right\}$$

④ $$f(x) = \left\{ \begin{array}{ll} 1 & x = \frac{a}{b} \\ 0 & \text{otherwise} \end{array} \right\}$$

Proof of 11.28 using the Cauchy criterion.

Recall: Thm 7.42 if $f \in C^0[0,1]$

then f is <u>uniformly</u> continuous.

cts: $\forall x \in [0,1], \varepsilon > 0 \ \exists \delta_{\varepsilon,x} \ \forall$ $|y-x| < \delta$ have $|f(x) - f(y)| < \varepsilon$

unif: cts $\forall \varepsilon > 0 \ \exists \delta_\varepsilon \ \forall x \in [0,1]$, " " " "

To show if $f \in C^0[0,1]$ then $f \in RInt[0,1]$

consider $\forall \varepsilon > 0$ Choose $P_\varepsilon = R_m$ with $m \geq \frac{1}{\delta_\varepsilon}$

$R_m = \{0, \frac{1}{m}, \frac{2}{m}, \ldots, 1\}$

and compute:

For R_m have every $l_i = \frac{1}{m}$

so $\forall x, y \in I_i$ have $|x-y| < \frac{1}{m} = \delta_\varepsilon$

so $|f(x) - f(y)| < \varepsilon$

~~so~~

so $\sup_{x \in I_i} f(x) - \inf_{y \in I_i} f(y) < \varepsilon$

Combining: $U(f; R_m) - L(f; R_m)$

$$= \sum_{i=1}^{m} l_i \left(\sup_{x \in I_i} f(x) - \inf_{y \in I_i} f(y) \right)$$

$$< m \cdot \frac{1}{m} \cdot \varepsilon = \varepsilon.$$

Therefore $f \in \mathrm{RInt}[0,1]$

q.e.d.

Thm 11.30: If $f \in \mathrm{Bdd}[0,1]$ is increasing then $f \in \mathrm{RInt}[0,1]$.

Proof: Use the 11.26 Cauchy criterion.

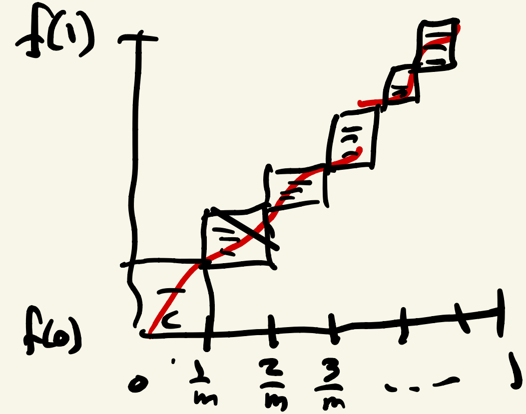
Compute:

$$U(f; R_m) - L(f; R_m)$$

$$= \sum_{i=1}^{m} l_i \left[\sup_{x \in [\frac{i-1}{m}, \frac{i}{m}]} f(x) - \inf_{y} f(y) \right]$$

$$= \sum_{i=1}^{m} l_i \left[f\left(\tfrac{i}{m}\right) - f\left(\tfrac{i-1}{m}\right) \right]$$

since



f is increasing.

$$= \frac{1}{m}\left[f(1) - f\left(\frac{m-1}{m}\right) + f\left(\frac{m-1}{m}\right) - f\left(\frac{m-2}{m}\right) + \cdots \right.$$
$$\left. - - - f(0)\right]$$

$$= \frac{1}{m}\left[f(1) - f(0)\right]$$

other terms cancel.

Cor: If $f \in \mathrm{Bdd}[0,1]$ is decreasing then $f \in \mathrm{RInt}[0,1]$.

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Recall: If $f \in Bdd[a,b]$ and $P \in \Pi[a,b]$ then

$$U(f;P) = \sum_{i=1}^{n} l_i \sup_{x \in I_i} f(x)$$

$$L(f;P) = \sum_{i=1}^{n} l_i \inf_{x \in I_i} f(x)$$

$$\overline{\int_a^b} f(x)\,dx = U(f) = \inf_P U(f;P)$$

$$\underline{\int_a^b} f(x)\,dx = L(f) = \sup_P L(f;P)$$

Note: $L(f;P) \leq L(f) < L(f;P) + \varepsilon$

$$\forall P \qquad \forall \varepsilon \qquad \exists P$$

$$U(f;P) \geq U(f) > U(f;P) - \varepsilon$$

Thm: 11.20 If $P \subseteq Q$ partitions and $f \in \text{Bdd}[a,b]$

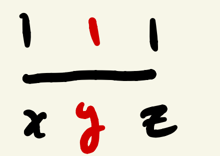
then (a) $U(f;P) \geq U(f;Q)$

(b) $L(f;P) \leq L(f;Q)$

Proof: Sketch (a) ((b) similar).

May assume Q has only one point not in P

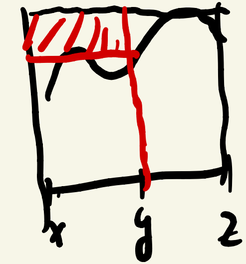
Since only one interval of P is subdiv. by Q,



assume $P = \{x, z\} \subseteq Q = \{x, y, z\}$

Computing:

$$U(f;Q) = (y-x)\sup_{\alpha\in[x,y]} f(\alpha) + (z-y)\sup_{\beta\in[z,y]} f(\beta)$$



$$\le (y-x)\sup_{\alpha\in[x,z]} f(\alpha) + (z-y)\sup_{\beta\in[x,z]} f(\beta)$$

$$= U(f;P).$$

To show $R\,\mathrm{Int}[0,1]$ is a vecter space:

→ • $\forall c \in \mathbb{R}$ and $f \in R\mathrm{Int}$ have $cf \in R\mathrm{Int}$.

→ • $\forall f,g \in R\mathrm{Int}$ have $f+g \in R\mathrm{Int}$.

To show $\int \cdot \, dx$ is linear:

→ • $\forall c \in \mathbb{R}$, $f \in R\mathrm{Int}$ have $\int cf = c\int f$

→ • $\forall f, g \in RInt$ nec $\int(f+g) = \int f + \int g$.

11.32: $\forall c \in \mathbb{R}$ and $f \in RInt[a,b]$ have $cf \in RInt[a,b]$ with $\int_a^b cf(x)dx$
$= c\int_a^b f(x)dx$

Proof: Assume $c > 0$

Compute: $\overline{\int_a^b} cf(x)dx = U(cf) = \inf_P \sum_{i=1}^n l_i \sup_{x \in I_i} [cf(x)]$

$= c \cdot \inf_P \sum_{i=1}^n l_i \sup_{x \in I_i} [f(x)] = c \cdot \overline{\int_a^b} f(x)dx$

similarly $\underline{\int_a^b} cf(x)dx = c \cdot \underline{\int_a^b} f(x)dx$

But $f \in R\,Int[a,b]$ so $\overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx$

so $\overline{\int_a^b} c f(x)\,dx = c \cdot \int_a^b f(x)\,dx = \underline{\int_a^b} c f(x)\,dx$

and $cf \in R\,Int[a,b]$ and $\int cf = c \int f$.

Consider $c < 0$ ... get $\overline{\int_a^b} c f(x)\,dx = c \underline{\int_a^b} f(x)\,dx$..

(Try this).

Thm: 11.39: If $g, f \in Bdd[a,b]$

and $f \leq g$ pointwise then

(a) $\overline{\int_a^b} f(x)\,dx \le \overline{\int_a^b} g(x)\,dx$

(b) $\underline{\int_a^b} f(x)\,dx \le \underline{\int_a^b} g(x)\,dx.$

Proof: (a) ((b) is similar)

Compare:

$$\overline{\int_a^b} f(x)\,dx = \inf_P \sum_{i=1}^{n} l_i \sup_{x\in I_i} f(x)$$

$$\le \inf_P \sum_{i=1}^{n} l_i \sup_{x\in I_i} g(x) = \overline{\int_a^b} g(x)\,dx.$$

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Claim: (scaling): If $c \in \mathbb{R}$ and $f \in RInt[a,b]$ then $\|cf\|_{L^1} = |c| \, \|f\|_{L^1}$

Proof: use: (11.32: $\int cf = c\int f.$),

Compute:

$$\|cf\|_{L^1} = \int_a^b |cf(x)|\,dx = \int_a^b |c||f(x)|\,dx$$

$$= |c| \int_a^b |f(x)|\,dx = |c| \, \|f\|_{L^1}.$$

Claim: (triangle): If $f, g \in RInt[a,b]$.

then $\|f+g\|_{L^1} \leq \|f\|_{L^1} + \|g\|_{L^1}$

Proof: Use: 11.33: $\int(f+g) = \int f + \int g$.

11.39: If $f \leq g$ then $\int f \leq \int g$.

$|f(x)+g(x)| \leq |f(x)|+|g(x)|$

$\underbrace{\|f+g\|_{L^1}}$

$= \int |f(x)+g(x)|\,dx \leq \int (|f(x)|+|g(x)|)\,dx$

$= \int |f(x)|\,dx + \int |g(x)|\,dx$

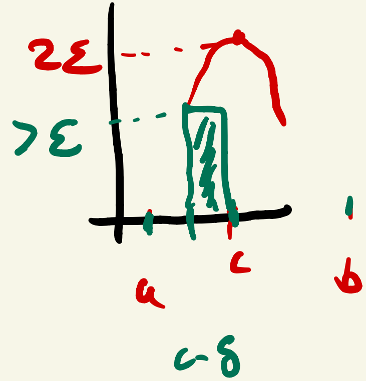
$= \|f\|_{L^1} + \|g\|_{L^1}$

Claim (11.42): If $f \in C^0[a,b]$ and $\|f\|_{L^1} = 0$

then $f = 0$.

Proof: For contradiction assume $f \in C^0[a,b]$, $|f(c)| \neq 0$ and $\|f\|_{L^1} = 0$

$|f(c)| = 2\varepsilon$



Since f is continuous at c
$\exists \delta > 0 \ \forall |x-c| < \delta$ have $\big||f(x)| - |f(c)|\big| < \varepsilon$

hence $|f(x)| > \varepsilon$ so $\sup_{x \in [c-\delta, c]} f(x) > \varepsilon$

Therefore: $0 = L(|f|) \geq L(|f|; P) \geq pos + \delta\varepsilon + pos$
$\geq \delta\varepsilon > 0$

$P = \{a, c-\delta, c, b\}$ or $\{a, c, c+\delta, b\}$

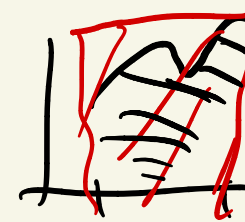
contradiction.

Cor $(C^0[a,b], \|\cdot\|_{L^1})$ is a normed linear space.

Recall: $(C^0[a,b], \|\cdot\|_{sup})$ is another normed lin. space

Note: If $f \in c^0[a,b]$ then

$$\|f\|_{L^1} \leq (b-a)\|f\|_{sup}$$



Proof: $\|f\|_{L^1} = \int_a^b |f(x)|\,dx$

$$= U(|f|) \leq U(|f|; \{a,b\})$$

$$= (b-a) \sup_{x\in[a,b]} |f(x)| = (b-a)\|f\|_{sup}.$$

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Prop 11.50 If $f \in \mathrm{Bdd}[0,1]$ and $\forall \varepsilon > 0$ have $f|_{[\varepsilon,1]} \in \mathrm{RInt}[\varepsilon,1]$ then $f \in \mathrm{RInt}[0,1]$.

Hence: $f(x) = \begin{cases} \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$ is R.Int.

since $f|_{[\varepsilon,1]}$ is cts. hence R.Int.

Proof: Write $M = \|f\|_{\sup}$. (finite since $f \in \mathrm{Bdd}[0,1]$),

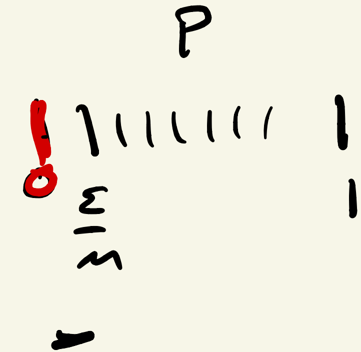
$\forall \varepsilon > 0 \ \exists P \in \Pi[\frac{\varepsilon}{M}, 1]$ with

$$U(f|_{[\frac{\varepsilon}{M},1]}; P) - L(f|_{[\frac{\varepsilon}{M},1]}; P) < \varepsilon$$

Write $\hat{P} = P \cup \{0\} \in \Pi[0,1]$

Compute:

$$U(f)-L(f) \leq U(f;\hat{P}) - L(f;\hat{P})$$



$$= \left(\frac{\varepsilon}{M}-0\right)\left[\sup_{x\in[0,\frac{\varepsilon}{M}]} f(x) - \inf_{x\in[0,\frac{\varepsilon}{M}]} f(x)\right] + U(f|_{[\frac{\varepsilon}{M},1]};P) - L(f|_{[\frac{\varepsilon}{M},1]};P)$$

$$\leq \frac{\varepsilon}{M}(2M) + \varepsilon = 3\varepsilon$$

so $f \in RInt[0,1]$.

qed.

Ch 12: Fundamental Theorem of Calculus.

Def: If $f \in \mathcal{R}\mathrm{Int}[a,b]$ write

$$(Jf)(x) = \int_a^x f(y)\,dy$$

Notes: $(Jf)(x)$ exists since $f|_{[a,x]} \in \mathcal{R}\mathrm{Int}[a,x]$ by 11.44.

$$(Jf)(a) = 0$$

$$|(Jf)(x)| \le (x-a)\,\|f\|_{sup}.$$

so $Jf \in \mathrm{Bdd}[a,b]$.

Thm 12.6 If $f \in RInt[a,b]$ then $Jf \in C^0[a,b]$.

Proof: Need to show: $\forall\ x \in [a,b]$

$$\lim_{h \to 0^+} (Jf)(x+h) \overset{?}{=} (Jf)(x)$$

and the same for $\lim_{h \to 0^-}$

$$\lim_{h \to 0^+} \int_a^{x+h} f(y)\,dy \overset{?}{=} \int_a^x f(y)\,dy$$

$$\lim_{h\to 0^+} \int_a^{x+h} f - \int_a^x f \overset{?}{=} 0$$

$$= \lim_{h\to 0^+} \int_x^{x+h} f(y)\,dy \le \underbrace{h\|f\|_{\sup}}_{\lim_{h\to 0^+}} = 0$$

$\lim_{h\to 0^-}$ is similar q.e.d.

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Thm 12.4: If $f \in RInt[a,b]$ and f is cts at $c \in [a,b]$ then $DJf(c) = f(c)$

Recall notation:

$$Jf(c) = \int_a^c f(t)\,dt$$

$$DF(c) = F'(c).$$

Proof: Since f is cts at c.

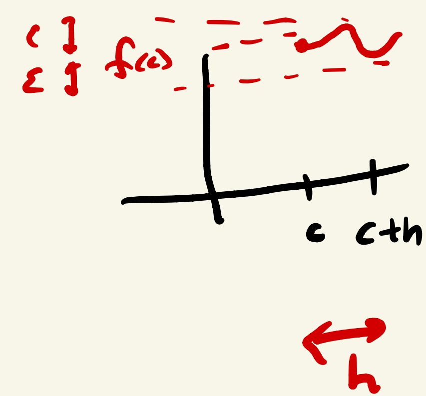
$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall |h| < \delta$

have $f(c+h) \in [f(c)-\varepsilon, \ f(c)+\varepsilon]$

hence $\sup\limits_{|h|<\delta} f(c+h) \in$ " "

and $\inf\limits_{|h|<\delta} f(c+h) \in$ " "

so $\displaystyle\int_c^{c+h} f(t)\,dt \in \Big[h\big[f(c)-\varepsilon\big], \ h\big[f(c)+\varepsilon\big]\Big]$



Computing:

$$DJf(c) = \lim_{h\to 0} \frac{Jf(c+h) - Jf(c)}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\int_a^{c+h} f(t)\,dt - \int_a^{c} f(t)\,dt\right]$$

$$= \lim_{h\to 0} \frac{1}{h} \int_c^{c+h} f(t)\,dt$$

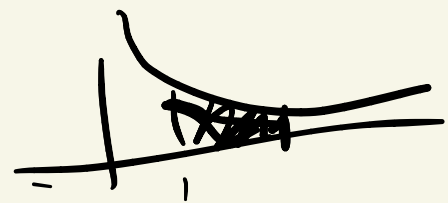
$$\in \left[f(c)-\varepsilon,\ f(c)+\varepsilon\right]$$

Hence $DJf(c) = f(c)$. q.e.d.

Building new functions using integration:

Ex: 12.14: Def:

$$\ln(x) = \int_1^x \frac{dt}{t}$$



Claim: $\ln(xy) = \ln(x) + \ln(y)$.

Pf: $\ln(x) + \ln(y)$

$$= \int_1^x \frac{dt}{t} + \int_1^y \frac{dt}{t}$$

$$= \int_1^x \frac{dt}{t} + \int_x^{xy} \frac{x\,ds}{s\,x}$$

$$= \int_1^{xy} \frac{dt}{t} = \ln(xy)$$

$s = xt$ $\quad$ $t=1$, $s=x$

$\frac{s}{x} = t$ $\quad$ $t=y$

$\frac{ds}{x} = dt$ $\quad$ $s=xy$

Ex: Claim: There is exactly one function $F(x)$ with $F'''(x) = e^{-x^2}$ and $F(0) = F'(0) = F(1) = 0$

Proof: Since e^{-x^2} is cts there is

$g'(x) = e^{-x^2}$ by above corollary.

and every fn with der. e^{-x^2} is $g(x) + c$

$h''(x) = e^{-x^2}$

$k'''(x) = e^{-x^2}$ so every fn. with second der e^{-x^2} is $h(x) + cx + d$

and $F(x) = k(x) + \frac{1}{2}cx^2 + xd + b$

Now use initial conditions:

$0 = F(0) = k(0) + 0 + 0 + b$ so $b = -k(0)$

$0 = F'(0) = k'(0) + 0 + d$ so $d = -k'(0)$

$0 = F(1) = k(1) + \frac{1}{2}c + d + b$ so $\frac{1}{2}c = -k(1) + k'(0) + k(0)$.

and there is a unique such $F(x)$.

Question: Since $g(x) = \begin{cases} x^2 \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$

has $g'(x) = \begin{cases} 2x\sin\frac{1}{x} - \cos\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$

which is not cts at 0.

Is it still true that: $\int_0^x g'(x)\,dx \overset{?}{=} g(x) - g(0)$
$= g(x)$.

Yes: $g'(x) \in RInt[-1,1]$ since: $disc(g') = \{0\}$ which is finite.

Thm 12.1: If $f(x) \in D'(a,b)$
and $f(x) \in C^0[a,b]$

and $f'(x) \in RInt[a,b]$

then $\mathcal{J}Df(x) = f(x) - f(a)$

$\overset{\text{def}}{=} \int_a^x f'(t)\,dt$

Proof of 12.1:

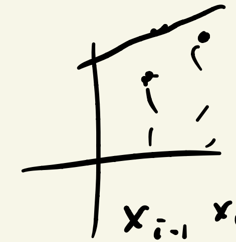
$\forall \varepsilon > 0 \quad \exists P \in \Pi[a,b]$ with

$\int_a^x f'(t)\,dt = Lf' \leq L(f';P) + \varepsilon \leq f(x) - f(a) + \varepsilon$

$= Uf' \geq U(f';P - \varepsilon \geq f(x) - f(a) - \varepsilon$

and

$$L(f_0'; P) = \sum_{i=1}^{n} l_i \inf_{t \in I_i} \left(f'(t)\right)$$



$$\underset{\text{MVT}}{\le} \sum_{i=1}^{n} l_i \underbrace{\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}}_{= f'(c) \text{ by MVT}}$$

$$= \sum_{i=1}^{n} \left(f(x_i) - f(x_{i-1}) \right) = f(x) - f(a)$$

Similarly $U(f_0'; P) \ge f(x) - f(a)$

qed.

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Cor: If $f, F \in RInt[a,b]$ and $F = f'$ then $\forall g \in C^1[a,b]$ with $g(a) = g(b) = 0$ have $\int_a^b f(t)g'(t)dt = -\int_a^b F(t)g(t)dt$

Def: If $f, F \in RInt[a,b]$ call F a <u>weak</u> <u>derivative</u> of f if $\forall g \in C^1[a,b]$ with $g(a) = g(b) = 0$ have $\int_a^b f(t)g'(t)dt = -\int_a^b F(t)g(t)dt$.

Thm 12.12: (change of variables):

If g is differentiable in (a,b)
continuos in $[a,b]$

and g' is R-Integ. in $[a,b]$

and f is cts in Image(g).

then: $$\int_a^b f(g(t))\,g'(t)\,dt = \int_{g(a)}^{g(b)} f(y)\,dy$$

(result of subst. $g(t) = y$).

Pf: similar to 12.10 see HW.

Proof: Next Time.

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Proof of Note:

If (f_n) converges in L^1 to f in $RInt[a,b]$.

then $\left(\forall \varepsilon>0\ \exists N\ \forall n\geq N \text{ have } \|f_n - f\|_{L^1} < \frac{\varepsilon}{b-a}\right)$ did not need this here.

hence
$$\left|\int_a^b f(t)\,dt - \lim_{n\to\infty}\int_a^b f_n(t)\,dt\right|$$
$$= \left|\lim_{n\to\infty}\int_a^b [f(t)-f_n(t)]\,dt\right|$$
$$\leq \lim_{n\to\infty}\int_a^b |f(t)-f_n(t)|\,dt$$
$$= \|f-f_n\|_{L^1} \xrightarrow{n\to\infty} 0$$
by assumption

q.e.d.

Thm 12.17: If (f_n) converges uniformly to f in $RInt[a,b]$ then (f_n) converges in L^1 to f.

Proof: $\forall \varepsilon > 0\ \exists N\ \forall n \geq N$ have $\|f_n - f\|_{sup} < \dfrac{\varepsilon}{b-a}$

hence $\forall \varepsilon > 0\ \exists N\ \forall n \geq N$ have

$$\|f_n - f\|_{L^1} = \int_a^b |f_n(x) - f(x)|\,dx \leq U(\overbrace{\cancel{\xi}}^{f_n - f}; \{a,b\})$$

$$= (b-a) \sup_{x \in [a,b]} |f_n - f| = (b-a)\|f_n - f\|_{sup} < \varepsilon$$

L^1- Convergent sequences:

Example:

If $f \in RInt[a,b]$

there are $P_n \in \Pi[a,b]$

with $\int_a^b f(x)\,dx = U(f) = \lim_{n\to\infty} U(f; P_n)$

If $P \in \Pi[a,b]$ write

$$f_{P,u}(x) = \left\{ \sup_{x \in [x_{i-1}, x_i]} f(x) \quad \text{if } x_{i-1} < x \le x_i \right\}$$

Note: $f_{P,u}(x) \geq f(x)$

and $\int_a^b f_{P,u}(x)\,dx = U(f;P)$

so $\|f_{P,u} - f\|_{L^1} = \int_a^b |f_{P,u}(x) - f(x)|\,dx$

$$= \int_a^b (f_{P,u}(x) - f(x))\,dx$$

$$= \int_a^b f_{P,u}(x)\,dx - \int_a^b f(x)\,dx$$

$$= U(f;P) - U(f)$$

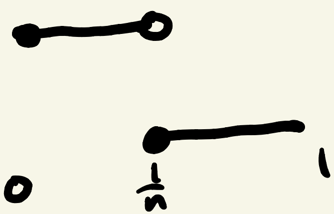
Hence using the sequence P_n above get $(f_{P_n,u})$ conv. in L^1 to f.

Examples:

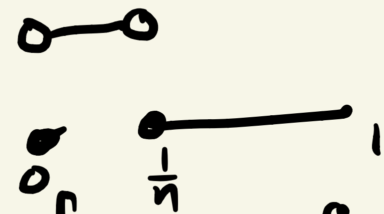
12.22 (a) $f_n(x) = \begin{cases} 1 & 0 \le x < \frac{1}{n} \\ 0 & \text{else} \end{cases} \in Bdd[0,1]$

(b) $g_n(x) = \begin{cases} 1 & 0 < x < \frac{1}{n} \\ 0 & \text{else} \end{cases} \in$ " "

f_n:



g_n:



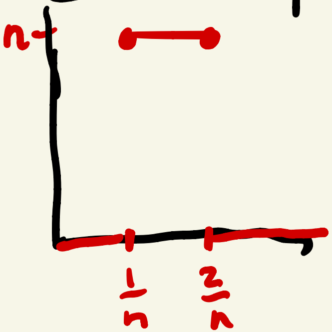
Compare convergence:

	f_n	g_n
pointwise	to $\begin{cases} 1 & x=0 \\ 0 & \text{else} \end{cases}$	to 0
uniform	not unif.	not unif.
L^1	L^1 conv	L^1 conv.

$\|g_n - 0\|_{sup} = 1 \not\longrightarrow 0$

$\|g_n - 0\|_{L^1} = \frac{1}{n} \cdot 1 \longrightarrow 0$

Example: $h_n(x) = \begin{cases} n & \frac{1}{n} \leq x \leq \frac{2}{n} \\ 0 & \text{else} \end{cases}$



	h_n
pointwise	to 0
uniform	not unif conv.
L1	not L^1 conv

$\|h_n - 0\|_{\text{unif}} = n$

$\|h_n - 0\|_{L^1} = \int h_n(x)\,dx$

$= \frac{1}{n} \cdot n = 1$

$\not\to 0$

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Ex 12.25:

$$\int_0^1 \frac{dx}{x^{1-\alpha}} = \lim_{\varepsilon \to 0^+} \int_\varepsilon^1 \frac{dx}{x^{1-\alpha}}$$

$$= \lim_{\varepsilon \to 0^+} \left[\frac{x^\alpha}{\alpha} \right]_\varepsilon^1$$

$$= \begin{cases} \frac{1}{\alpha} - \lim_{\varepsilon \to 0^+} \frac{\varepsilon^\alpha}{\alpha} = \frac{1}{\alpha} & \text{if } \alpha > 0 \\ \text{does not exist} & \text{otherwise} \end{cases}$$

Ex 12.27

$$\int_0^1 \frac{dx}{x^{1-\alpha}} = \int_1^{\infty} y^{1-\alpha} \frac{dy}{y^2} = \int_1^{\infty} \frac{dy}{y^{1+\alpha}}$$

$y = \frac{1}{x} \quad x = \frac{1}{y}$

$dx = -\frac{dy}{y^2}$

$x = 0 \quad y = \infty$

$x = 1 \quad y = 1$

$$= \begin{cases} \frac{1}{\alpha} & \text{if } \alpha > 0 \\ \text{d.n.e.} & \text{otherwise} \end{cases}$$

Examples: $\frac{1}{x^{1-\varepsilon}}$ on $[0,1]$ is abs. imp. int.

if $\varepsilon > 0$

and $\frac{1}{x^{1+\varepsilon}}$ on $[1,\infty)$ is also.

Example: 12.36 $\displaystyle\int_0^1 \frac{\sin\frac{1}{x}}{x^{1-\varepsilon}}\,dx$ with $\varepsilon > 0$

exists since: $\left|\dfrac{\sin\frac{1}{x}}{\underbrace{x^{1-\varepsilon}}}\right| < \dfrac{1}{x^{1-\varepsilon}} = \left|\dfrac{1}{x^{1-\varepsilon}}\right|$

so by 12.33 is abs. imp. int by ⓐ

and imp. int. by ⓑ.

Similarly: $\displaystyle\int_1^\infty \frac{\sin(x)}{x^{1+\varepsilon}}\,dx$ with $\varepsilon > 0$

$\parallel$

above by change of var $y = \frac{1}{x}$.

Example: $\displaystyle\int_0^\infty \frac{\sin(x)}{x}\,dx = \frac{\pi}{2}$.
12.36

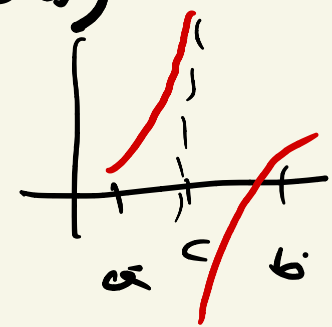
is not absolutely improperly integrable.

but it is improperly integrable.
(Dirichlet).

Call this conditionally convergent.

Def 12.39: Principal values:

If $f \in \text{Fun}([a,b] - \{c\})$ and $\forall \varepsilon > 0$ have $f \in \text{RInt}([a,b] - (c-\varepsilon, c+\varepsilon))$



write

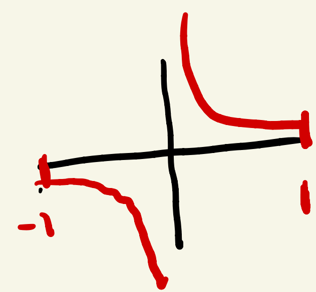
$$\underbrace{\text{P.V.}}_{\text{principal value}} \int_a^b f(x)\,dx = \lim_{\varepsilon \to 0} \left[\int_a^{c-\varepsilon} f(x)\,dx + \int_{c+\varepsilon}^b f(x)\,dx \right]$$

Example: $\frac{1}{x}$ is not abs impr. int. on $[-1,0]$ or $[0,1]$

also not impr. int. on either $[-1,0]$ or $[0,1]$

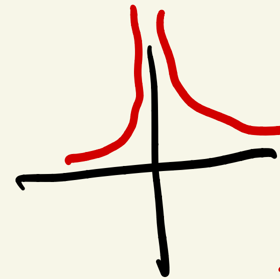
but $$\text{P.V.}\int_{-1}^{1}\frac{dx}{x}=0$$ since $\frac{1}{x}$ is odd

So for every $\varepsilon>0$ $$\int_{-1}^{-\varepsilon}\frac{dx}{x}+\int_{\varepsilon}^{1}\frac{dx}{x}=0$$



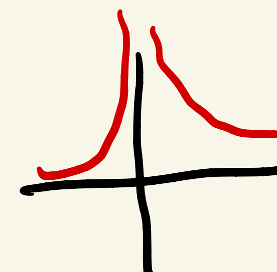
Ex: $\frac{1}{x^2}$ is not abs. impr. int. or imp. int. on $[-1,0]$ or $[0,1]$.

and $\text{P.V.} \int_{-1}^{1} \frac{dx}{x^2}$ does not exist.



Ex: $\frac{1}{\sqrt{|x|}}$ is abs. imp. int. on $[-1,0]$ and $[0,1]$

$\text{P.V.} \int_{-1}^{1} \frac{dx}{|x|^{\frac{1}{2}}}$ hence exists



Ex: $\text{p.v.} \int_{-\infty}^{1} \frac{e^t}{t}\,dt$

$$= \int_{-\infty}^{-1} \frac{e^t}{t}\,dt + \text{p.v.} \int_{-1}^{1} \frac{e^t}{t}\,dt$$

$$= \lim_{n\to\infty} \int_{-n}^{-1} \frac{e^t}{t}\,dt + \lim_{\varepsilon\to 0^+} \left[\int_{-1}^{-\varepsilon} \frac{e^t}{t}\,dt + \int_{\varepsilon}^{1} \frac{e^t}{t}\,dt \right]$$

Compute the second part:

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$$\lim_{\varepsilon \to 0^+} \left[\int_{-1}^{-\varepsilon} \frac{e^t}{t}\,dt + \int_{\varepsilon}^{1} \frac{e^t}{t}\,dt - \underbrace{\left[\int_{-1}^{-\varepsilon} \frac{1}{t}\,dt + \int_{\varepsilon}^{1} \frac{1}{t}\,dt \right]}_{0} \right]$$

$$= \lim_{\varepsilon \to 0^+} \left[\int_{-1}^{-\varepsilon} \frac{e^t - 1}{t}\,dt + \int_{\varepsilon}^{1} \frac{e^t-1}{t}\,dt \right]$$

$= \int_{-1}^{1} \tilde{f}(t)\,dt$ which exists since $\tilde{f}$ is ctc and hence integrable.

Taylor polymials again:

Recall: If f is cts on $[a,b]$ and $a<c<b$ and $f^{(n+1)} \in \text{Fun}[a,b]$,

then write: $P_n(x) = \sum_{r=0}^{n} \frac{f^{(r)}(c)}{r!}(x-c)^r$

and $R_n(x) = f(x) - P_n(x)$

eg If $e^t = f(t)$ and $c=0$

then $R_n(t) = e^t - 1 - t - \frac{t^2}{2} - \cdots - \frac{t^n}{n!}$

Old: Thm 8.46 (Lagrange Error)

$\forall x : \exists \xi$ between c and x with

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-c)^{n+1}$$

New: Thm 12.48 (integral Error)

then $R_n(x) = \frac{1}{n!}\int_c^x f^{(n+1)}(t)\,(x-t)^n\,dt$

if also $f^{(n+1)}(t) \in RInt\,[a,b]$

Assume = holds for n

show = holds for $n+1$

Assume $$R_n(x) = \frac{1}{n!}\int_c^x \underbrace{f^{(n+1)}(t)}_{u}\,\underbrace{(x-t)^n}_{\frac{-v'}{n+1}}\,dt$$

compute:

$$R_{n+1}(x) = f(x) - P_{n+1}(x)$$

$$= f(x) - \left[P_n(x) + \frac{\overbrace{f^{(n+1)}(c)}^{u}}{(n+1)!}\,\overbrace{(x-c)^{n+1}}^{v}\right]$$

$$= R_n(x) \qquad - \qquad '' \qquad\qquad ''$$

.

$$\underset{\text{goal}}{\overset{?}{=}} \frac{1}{(n+1)!}\int_c^x \underbrace{f^{(n+2)}(t)}_{u'}\, \underbrace{(x-t)^{n+1}}_{v}\, dt$$

take $\quad u(t) = f^{(n+1)}(t) \qquad v(t) = (x-t)^{n+1}$

so $\quad u'(t) = f^{(n+2)}(t) \qquad v'(t) = -(n+1)(x-t)^{n}$

and hence:

$$\frac{1}{(n+1)!}\int_c^x u'(t)\, v(t)\, dt = -\frac{1}{n!}\int_c^x u(t)\, v'(t)\, \frac{1}{n+1}$$

$$+ \frac{1}{(n+1)!}\, u(t)\, v(t)\Big|_c^x$$

q.e.d.

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Convergence of sequences of functions:

$\|\cdot\|_{C^1}$

$\Downarrow$

$\|\cdot\|_{unif}$

$\swarrow \quad \searrow$

pointwise $\quad \|\cdot\|_{L^1}$

Cauchy sequences for any of the three $\|\cdot\|$'s,

Banach spaces: Cauchy $\Rightarrow$ convergent.

$(Bdd[0,1], \|\cdot\|_{unif})$

$(Bdd[0,1] \cap D^0[0,1], \|\cdot\|_{unif})$

Asside!

$L^1[0,1] = \{f \in Fun[0,1] \mid$

$\exists (f_n) \xrightarrow{ptwise} f$, (f_n) is $\|\cdot\|_{L^1}$-Cauchy

$(C^0[0,1], \|\cdot\|_{unif})$

$(C^1[0,1], \|\cdot\|_{C^1})$

Normed Lin. Sp:

$(C^0[0,1], \|\cdot\|_{L^1})$

$f_n \in C^0[0,1] \}$ / $\sim$

$f \sim g$ if with f_n, g_n as above $\|f_n - g_n\|_{L^1} \to 0$

$(L^1[0,1], \|\cdot\|_{L^1})$ is Banach

Problem: MVT:

Show that $f_a(x) = \begin{cases} \sin^2 \frac{1}{x} & x \neq 0 \\ a & x = 0 \end{cases}$

for exactly one value of a has

$f \in D^0 \mathbb{R}$.

Ans: (At most one such value a)

If $f_a, f_b \in D^0 \mathbb{R}$

then $f_a - f_b \in D^0 \mathbb{R}$ since $D^0\mathbb{R}$ is a v. sp.

but $f_a(x) - f_b(x) = \begin{cases} 0 & x \neq 0 \\ a-b & x=0 \end{cases}$

which is not in $D^0\mathbb{R}$ by Darboux's Thm,

(At least one such value a)

Note: $\begin{cases} x^2 \sin\frac{1}{x} \cos\frac{1}{x}, & x \neq 0 \end{cases} = g(x)$

then LO $x=0$ }

$$g'(x) = \begin{cases} 2x\sin\frac{1}{x}\cos\frac{1}{x} - \cos^2\frac{1}{x} + \sin^2\frac{1}{x} & \\ 0 & x=0 \end{cases}$$

] use rules,

] compute the using def of deriv.

Note: $1 = \begin{cases} \cos^2\frac{1}{x} + \sin^2\frac{1}{2} & x \neq 0 \\ 1 & x=0 \end{cases}$

and

$$h(x) = \begin{cases} 2x\sin\frac{1}{x}\cos\frac{1}{x} & x \neq 0 \\ 0 & x=0 \end{cases}$$

is cts, compute using def.

hence $\left\lfloor \frac{g'(x) - h(x) + 1}{2} \right. = \left\{ \begin{array}{ll} \sin^2 \frac{1}{x} & x \neq 0 \\ \frac{1}{2} & x = 0 \end{array} \right\}$

$= f_{\frac{1}{2}}(x)$

but $h(x)$ and 1 are cts hence in $D^0 \mathbb{R}$ so $f_{\frac{1}{2}}(x) \in D^0 \mathbb{R}$,

---

Problem!

Find an L^1-Cauchy sequence of differentiable functions which converge pointwise to the Dirichlet function $D(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & \text{else} \end{cases}$

in $[0,1]$.

Ans: Idea: ~~If differentiable~~ is not required:

Enumerate the rationals: all rationals in $[0,1]$.

$q_1, q_2, \ldots$

$$f_n = \begin{cases} 1 & x = q_n \\ 0 & \text{else} \end{cases},$$

$$F_n = \sum_{r=1}^{n} f_n$$

then $F_n \xrightarrow{n\to\infty} 0$ pointwise.

and $\|f_n\|_{L^1} = \int_0^1 f_n(x)\,dx = 0$

$\|F_n\|_{L^1} = 0$

and $\|F_n - F_m\|_{L^1} = 0$ so (F_n) is L^1-Cauchy.