MAT201C Lecture Notes: Introduction to Sobolev Spaces

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These notes, intended for the third quarter of the graduate *Analysis* sequence at UC Davis, should be viewed as a very short introduction to Sobolev space theory, and the rather large collection of topics which are foundational for its development. This includes the theory of L^p spaces, the Fourier series and the Fourier transform, the notion of weak derivatives and distributions, and a fair amount of differential analysis (the theory of differential operators). Sobolev spaces and other very closely related functional frameworks have proved to be indispensable topologies for answering very basic questions in the fields of partial differential equations, mathematical physics, differential geometry, harmonic analysis, scientific computation, and a host of other mathematical specialities. These notes provide only a brief introduction to the material, essentially just enough to get going with the basics of Sobolev spaces. As the course progresses, I will add some additional topics and/or details to these notes. In the meantime, a good reference is *Analysis* by Lieb and Loss, and of course *Applied Analysis* by Hunter and Nachtergaele, particularly Chapter 12, which serves as a nice compendium of the material to be presented.

If only I had the theorems! Then I should find the proofs easily enough.
–Bernhard Riemann (1826-1866)

Facts are many, but the truth is one.
–Rabindranath Tagore (1861-1941)

Contents

1 L^p spaces

1.1 Notation

We will usually use Ω to denote an open and smooth domain in $\mathbb{R}^d$, for $d = 1, 2, 3, ...$ In this chapter on L^p spaces, we will sometimes use X to denote a more general measure space, but the reader can usually think of a subset of Euclidean space.

$C^k(\Omega)$ is the space of functions which are k times differentiable in Ω for integers $k \geq 0$.

$C^0(\Omega)$ then coincides with $C(\Omega)$, the space of continuous functions on Ω.

$C^\infty(\Omega) = \cap_{k\geq 0} C^k(\Omega)$.

$\operatorname{spt} f$ denotes the support of a function f, and is the closure of the set $\{x \in \Omega \mid f(x) \neq 0\}$.

$C_0(\Omega) = \{u \in C(\Omega) \mid \operatorname{spt} u \text{ compact in } \Omega\}$.

$C_0^k(\Omega) = C^k(\Omega) \cap C_0(\Omega)$.
$C_0^\infty(\Omega) = C^\infty(\Omega) \cap C_0(\Omega)$. We will also use $\mathcal{D}(\Omega)$ to denote this space, which is known as the *space of test functions* in the theory of distributions.

1.2 Definitions and basic properties

Definition 1.1. *Let $0 < p < \infty$ and let $(X, \mathcal{M}, \mu)$ denote a measure space. If $f : X \to \mathbb{R}$ is a measurable function, then we define*

$$\|f\|_{L^p(X)} := \left(\int_X |f|^p dx \right)^{\frac{1}{p}} \quad \textit{and} \quad \|f\|_{L^\infty(X)} := \operatorname{ess\,sup}_{x\in X} |f(x)| \,.$$

Note that $\|f\|_{L^p(X)}$ may take the value ∞. Unless stated otherwise, we will usually consider X to be a smooth, open subset Ω of $\mathbb{R}^d$, and we will assume that all functions under consideration are measurable.

Definition 1.2. *The space $L^p(X)$ is the set*

$$L^p(X) = \left\{ f : X \to \mathbb{R} \mid \|f\|_{L^p(X)} < \infty \right\}.$$

The space $L^p(X)$ satisfies the following vector space properties:

1. For each $\alpha \in \mathbb{R}$, if $f \in L^p(X)$ then $\alpha f \in L^p(X)$;
2. If $f, g \in L^p(X)$, then
$$|f + g|^p \leq 2^{p-1}(|f|^p + |g|^p)\,,$$
so that $f + g \in L^p(X)$.

3. The triangle inequality is valid if $p \geq 1$.

The most interesting cases are $p = 1, 2, \infty$, while all of the L^p arise often in *nonlinear* estimates.

Definition 1.3. *The space l^p, called "little L^p", will be useful when we introduce Sobolev spaces on the torus and the Fourier series. For $1 \leq p < \infty$, we set*

$$l^p = \left\{ \{x_n\}_{n\in\mathbb{Z}} \mid \sum_{n=-\infty}^{\infty} |x_n|^p < \infty \right\} ,$$

where $\mathbb{Z}$ denotes the integers.

1.3 Basic inequalities

Convexity is fundamental to L^p spaces for $p \in [1, \infty)$.

Lemma 1.4. *For $\lambda \in (0,1)$, $x^\lambda \leq (1-\lambda) + \lambda x$.*

Proof. Set $f(x) = (1-\lambda)+\lambda x - x^\lambda$; hence, $f'(x) = \lambda - \lambda x^{\lambda-1} = 0$ if and only if $\lambda(1-x^{\lambda-1}) = 0$ so that $x = 1$ is the critical point of f. In particular, the minimum occurs at $x = 1$ with value

$$f(1) = 0 \leq (1-\lambda) + \lambda x - x^\lambda .$$

□

Lemma 1.5. *For $a, b \geq 0$ and $\lambda \in (0,1)$, $a^\lambda b^{1-\lambda} \leq \lambda a + (1-\lambda) b$ with equality if $a = b$.*

Proof. If either $a = 0$ or $b = 0$, then this is trivially true, so assume that $a, b > 0$. Set $x = a/b$, and apply Lemma 1 to obtain the desired inequality. □

Theorem 1.6 (Hölder's inequality)**.** *Suppose that $1 \leq p \leq \infty$ and $1 < q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. If $f \in L^p$ and $g \in L^q$, then $fg \in L^1$. Moreover,*

$$\|fg\|_{L^1} \leq \|f\|_{L^p} \|g\|_{L^q} .$$

Note that if $p = q = 2$, then this is the Cauchy-Schwarz inequality since $\|fg\|_{L^1} = |(f,g)_{L^2}|$.

Proof. We use Lemma 1.5. Let $\lambda = 1/p$ and set

$$a = \frac{|f|^p}{\|f\|^p_{L^p}} , \text{ and } \quad b = \frac{|g|^q}{\|g\|^q_{L^p}}$$

for all $x \in X$. Then $a^\lambda b^{1-\lambda} = a^{1/p} b^{1-1/p} = a^{1/p} b^{1/q}$ so that

$$\frac{|f| \cdot |g|}{\|f\|_{L^p} \|g\|_{L^q}} \le \frac{1}{p} \frac{|f|^p}{\|f\|^p_{L^p}} + \frac{1}{q} \frac{|g|^q}{\|g\|^q_{L^q}} \,.$$

Integrating this inequality yields

$$\int_X \frac{|f| \cdot |g|}{\|f\|_{L^p} \|g\|_{L^q}} dx \le \int_X \left(\frac{1}{p} \frac{|f|^p}{\|f\|^p_{L^p}} + \frac{1}{q} \frac{|g|^q}{\|g\|^q_{L^q}} \right) dx = \frac{1}{p} + \frac{1}{q} = 1 \,.$$

□

Definition 1.7. *The exponent* $q = \frac{p}{p-1}$ *(or* $\frac{1}{q} = 1 - \frac{1}{p}$*) is called the conjugate exponent of* p*.*

Theorem 1.8 (Minkowski's inequality)**.** *If* $1 \le p \le \infty$ *and* $f, g \in L^p$ *then*

$$\|f + g\|_{L^p} \le \|f\|_{L^p} + \|g\|_{L^p} \,.$$

Proof. If $f + g = 0$ a.e., then the statement is trivial. Assume that $f + g \neq 0$ a.e. Consider the equality

$$|f + g|^p = |f + g| \cdot |f + g|^{p-1} \le (|f| + |g|) |f + g|^{p-1} \,,$$

and integrate over X to find that

$$\begin{aligned} \int_X |f + g|^p dx &\le \int_X \left[(|f| + |g|) |f + g|^{p-1} \right] dx \\ &\overset{\text{Hölder's}}{\le} \left(\|f\|_{L^p} + \|g\|_{L^p} \right) \left\| |f + g|^{p-1} \right\|_{L^q} \,. \end{aligned}$$

Since $q = \frac{p}{p-1}$,

$$\left\| |f + g|^{p-1} \right\|_{L^q} = \left(\int_X |f + g|^p dx \right)^{\frac{1}{q}} \,,$$

from which it follows that

$$\left(\int_X |f + g|^p dx \right)^{1 - \frac{1}{q}} \le \|f\|_{L^p} + \|g\|_{L^q} \,,$$

which completes the proof, since $\frac{1}{p} = 1 - \frac{1}{q}$. □

Corollary 1.9. *For* $1 \le p \le \infty$*,* $L^p(X)$ *is a normed linear space.*

Example 1.10. *Let Ω denote a subset of $\mathbb{R}^n$ whose Lebesgue measure is equal to one. If $f \in L^1(\Omega)$ satisfies $f(x) \geq M > 0$ for almost all $x \in \Omega$, then $\log(f) \in L^1(\Omega)$ and satisfies*

$$\int_\Omega \log f dx \leq \log(\int_\Omega f dx) .$$

To see this, consider the function $g(t) = t - 1 - \log t$ for $t > 0$. Compute $g'(t) = 1 - \frac{1}{t} = 0$ so $t = 1$ is a minimum (since $g''(1) > 0$). Thus, $\log t \leq t - 1$ and letting $t \mapsto \frac{1}{t}$ we see that

$$1 - \frac{1}{t} \leq \log t \leq t - 1 . \tag{1.1}$$

Since $\log x$ is continuous and f is measurable, then $\log f$ is measurable for $f > 0$. Let $t = \frac{f(x)}{\|f\|_{L^1}}$ in (1.1) to find that

$$1 - \frac{\|f\|_{L^1}}{f(x)} \leq \log f(x) - \log \|f\|_{L^1} \leq \frac{f(x)}{\|f\|_{L^1}} - 1 . \tag{1.2}$$

Since $g(x) \leq \log f(x) \leq h(x)$ for two integrable functions g and h, it follows that $\log f(x)$ is integrable. Next, integrate (1.2) to finish the proof, as $\int_X \left(\frac{f(x)}{\|f\|_{L^1}} - 1 \right) dx = 0$.

1.4 The space $(L^p(X), \|\cdot\|_{L^p}(X)$ is complete

Recall the a normed linear space is a Banach space if every Cauchy sequence has a limit in that space; furthermore, recall that a sequence $x_n \to x$ in X if $\lim_{n\to\infty} \|x_n - x\|_X = 0$.

The proof of completeness makes use of the following two lemmas which are restatements of the Monotone Convergence Theorem and the Dominated Convergence Theorem, respectively (see the Appendix for this chapter).

Lemma 1.11 (MCT)**.** *If $f_n \in L^1(X)$, $0 \leq f_1(x) \leq f_2(x) \leq \cdots$, and $\|f_n\|_{L^1(X)} \leq C < \infty$, then $\lim_{n\to\infty} f_n(x) = f(x)$ with $f \in L^1(X)$ and $\|f_n - f\|_{L^1} \to 0$ as $n \to 0$.*

Lemma 1.12 (DCT)**.** *If $f_n \in L^1(X)$, $\lim_{n\to\infty} f_n(x) = f(x)$ a.e., and if $\exists\, g \in L^1(X)$ such that $|f_n(x)| \leq |g(x)|$ a.e. for all n, then $f \in L^1(X)$ and $\|f_n - f\|_{L^1} \to 0$.*

Proof. Apply the Dominated Convergene Theorem to the sequence $h_n = |f_n - f| \to 0$ a.e., and note that $|h_n| \leq 2g$. □

Theorem 1.13. *If 1$\leq p < \infty$ then $L^p(X)$ is a Banach space.*

Proof. **Step 1. The Cauchy sequence.** Let $\{f_n\}_{n=1}^\infty$ denote a Cauchy sequence in L^p, and assume without loss of generality (by extracting a subsequence if necessary) that $\|f_{n+1} - f_n\|_{L^p} \leq 2^{-n}$.

Step 2. Conversion to a convergent monotone sequence. Define the sequence $\{g_n\}_{n=1}^\infty$ as

$$g_1 = 0, \quad g_n = |f_1| + |f_2 - f_1| + \cdots + |f_n - f_{n-1}| \quad \text{for} \quad n \geq 2\,.$$

It follows that

$$0 \leq g_1 \leq g_2 \leq \cdots \leq g_n \leq \cdots$$

so that g_n is a monotonically increasing sequence. Furthermore, $\{g_n\}$ is uniformly bounded in L^p as

$$\int_X g_n^p dx = \|g_n\|_{L^p}^p \leq \left(\|f_1\|_{L^p} + \sum_{i=2}^\infty \|f_i - f_{i-1}\|_{L^p} \right)^p \leq (\|f_1\|_{L^p} + 1)^p\,;$$

thus, by the Monotone Convergence Theorem, $g_n^p \nearrow g^p$ a.e., $g \in L^p$, and $g_n \leq g$ a.e.

Step 3. Pointwise convergence of $\{f_n\}$. For all $k \geq 1$,

$$\begin{aligned} |f_{n+k} - f_n| &= |f_{n+k} - f_{n+k-1} + f_{n+k-1} + \cdots - f_{n+1} + f_{n+1} - f_n| \\ &\leq \sum_{i=n+1}^{n+k+1} |f_i - f_{i-1}| = g_{n+k} - g_n \longrightarrow 0 \text{ a.e.} \end{aligned}$$

Therefore, $f_n \to f$ a.e. Since

$$|f_n| \leq |f_1| + \sum_{i=2}^n |f_i - f_{i-1}| \leq g_n \leq g \text{ for all } n \in \mathbb{N}\,,$$

it follows that $|f| \leq g$ a.e. Hence, $|f_n|^p \leq g^p$, $|f|^p \leq g^p$, and $|f - f_n|^p \leq 2g^p$, and by the Dominated Convergence Theorem,

$$\lim_{n\to\infty} \int_X |f - f_n|^p dx = \int_X \lim_{n\to\infty} |f - f_n|^p dx = 0\,.$$

□

1.5 Convergence criteria for L^p functions

If $\{f_n\}$ is a sequence in $L^p(X)$ which converges to f in $L^p(X)$, then there exists a subsequence $\{f_{n_k}\}$ such that $f_{n_k}(x) \to f(x)$ for almost every $x \in X$ (denoted by a.e.), but it is in general *not true* that the entire sequence itself will converge pointwise a.e. to the limit f, without some further conditions holding.

Example 1.14. *Let $X = [0,1]$, and consider the subintervals*

$$\left[0,\frac{1}{2}\right], \left[\frac{1}{2},1\right], \left[0,\frac{1}{3}\right], \left[\frac{1}{3},\frac{2}{3}\right], \left[\frac{2}{3},1\right], \left[0,\frac{1}{4}\right], \left[\frac{1}{4},\frac{2}{4}\right], \left[\frac{2}{4},\frac{3}{4}\right], \left[\frac{3}{4},1\right], \left[0,\frac{1}{5}\right], \cdots$$

Let f_n denote the indicator function of the n^{th} interval of the above sequence. Then $\|f_n\|_{L^p} \to 0$, but $f_n(x)$ does not converge for any $x \in [0,1]$.

Example 1.15. *Set $X = \mathbb{R}$, and for $n \in \mathbb{N}$, set $f_n = \mathbf{1}_{[n,n+1]}$. Then $f_n(x) \to 0$ as $n \to \infty$, but $\|f_n\|_{L^p} = 1$ for $p \in [1,\infty)$; thus, $f_n \to 0$ pointwise, but not in L^p.*

Example 1.16. *Set $X = [0,1]$, and for $n \in \mathbb{N}$, set $f_n = n\mathbf{1}_{[0,\frac{1}{n}]}$. Then $f_n(x) \to 0$ a.e. as $n \to \infty$, but $\|f_n\|_{L^1} = 1$; thus, $f_n \to 0$ pointwise, but not in L^1.*

Theorem 1.17. *For $1 \le p < \infty$, suppose that $\{f_n\} \subset L^p(X)$ and that $f_n(x) \to f(x)$ a.e. If $\lim_{n\to\infty} \|f_n\|_{L^p(X)} = \|f\|_{L^p(X)}$, then $f_n \to f$ in $L^p(X)$.*

Proof. Given $a, b \ge 0$, convexity implies that $\left(\frac{a+b}{2}\right)^p \le \frac{1}{2}(a^p + b^p)$ so that $(a+b)^p \le 2^{p-1}(a^p + b^p)$, and hence $|a-b|^p \le 2^{p-1}(|a|^p + |b|^p)$. Set $a = f_n$ and $b = f$ to obtain the inequality

$$0 \le 2^{p-1}\left(|f_n|^p + |f|^p\right) - |f_n - f|^p .$$

Since $f_n(x) \to f(x)$ a.e.,

$$2^p \int_X |f|^p dx = \int_X \lim_{n\to\infty} \left(2^{p-1}(|f_n|^p + |f|^p) - |f_n - f|^p\right) dx .$$

Thus, Fatou's lemma asserts that

$$\begin{aligned} 2^p \int_X |f|^p dx &\le \liminf_{n\to\infty} \int_X \left(2^{p-1}(|f_n|^p + |f|^p) - |f_n - f|^p\right) dx \\ &= 2^{p-1} \int_X |f|^p dx + 2^{p-1} \lim_{n\to\infty} \int_X |f_n|^p + \liminf_{n\to\infty} \left(- \int_X |f_n - f|^p dx \right) \\ &= 2^{p-1} \int_X |f|^p dx - \limsup_{n\to\infty} \int_X |f_n - f|^p dx . \end{aligned}$$

As $\int_X |f|^p dx < \infty$, the last inequality shows that $\limsup_{n\to\infty} \int_X |f_n - f|^p dx \le 0$. It follows that $\limsup_{n\to\infty} \int_X |f_n - f|^p dx = \liminf_{n\to\infty} \int_X |f_n - f|^p dx = 0$, so that $\lim_{n\to\infty} \int_X |f_n - f|^p dx = 0$. □

1.6 The space $L^\infty(X)$

Definition 1.18. *With $\|f\|_{L^\infty(X)} = \inf\{M \geq 0 \mid |f(x)| \leq M \quad a.e.\}$, we set*

$$L^\infty(X) = \{f : X \to \mathbb{R} \mid \|f\|_{L^\infty(X)} < \infty\}\,.$$

Theorem 1.19. *$(L^\infty(X), \|\cdot\|_{L^\infty(X)})$ is a Banach space.*

Proof. Let f_n be a Cauchy sequence in $L^\infty(X)$. It follows that $|f_n - f_m| \leq \|f_n - f_m\|_{L^\infty(X)}$ a.e. and hence $f_n(x) \to f(x)$ a.e., where f is measurable and essentially bounded.

Choose $\epsilon > 0$ and $N(\epsilon)$ such that $\|f_n - f_m\|_{L^\infty(X)} < \epsilon$ for all $n, m \geq N(\epsilon)$. Since $|f(x) - f_n(x)| = \lim_{m\to\infty} |f_m(x) - f_n(x)| \leq \epsilon$ holds a.e. $x \in X$, it follows that $\|f - f_n\|_{L^\infty(X)} \leq \epsilon$ for $n \geq N(\epsilon)$, so that $\|f_n - f\|_{L^\infty(X)} \to 0$. □

Remark 1.20. *In general, there is no relation of the type $L^p \subset L^q$. For example, suppose that $X = (0,1)$ and set $f(x) = x^{-\frac{1}{2}}$. Then $f \in L^1(0,1)$, but $f \notin L^2(0,1)$. On the other hand, if $X = (1,\infty)$ and $f(x) = x^{-1}$, then $f \in L^2(1,\infty)$, but $f \notin L^1(1,\infty)$.*

Lemma 1.21 (L^p comparisons)**.** *If $1 \leq p < q < r \leq \infty$, then (a) $L^p \cap L^r \subset L^q$, and (b) $L^q \subset L^p + L^r$.*

Proof. We begin with (b). Suppose that $f \in L^q$, define the set $E = \{x \in X \ : \ |f(x)| \geq 1\}$, and write f as

$$\begin{aligned} f &= f\mathbf{1}_E + f\mathbf{1}_{E^c} \\ &= g + h\,. \end{aligned}$$

Our goal is to show that $g \in L^p$ and $h \in L^r$. Since $|g|^p = |f|^p\mathbf{1}_E \leq |f|^q\mathbf{1}_E$ and $|h|^r = |f|^r\mathbf{1}_{E^c} \leq |f|^q\mathbf{1}_{E^c}$, assertion (b) is proven.

For (a), let $\lambda \in [0,1]$ and for $f \in L^q$,

$$\begin{aligned} \|f\|_{L^q} = \left(\int_X |f|^q dx\right)^{\frac{1}{q}} &= \left(\int_X |f|^{\lambda q}|f|^{(1-\lambda)q} d\mu\right)^{\frac{1}{q}} \\ &\leq \left(\|f\|_{L^p}^{\lambda q}\|f\|_{L^r}^{(1-\lambda)q}\right)^{\frac{1}{q}} = \|f\|_{L^p}^{\lambda}\|f\|_{L^r}^{(1-\lambda)}\,. \end{aligned}$$

□

Theorem 1.22. *If $\mu(X) \leq \infty$ and $q > p$, then $L^q \subset L^p$.*

Proof. Consider the case that $q = 2$ and $p = 1$. Then by the Cauchy-Schwarz inequality,

$$\int_X |f|dx = \int_X |f| \cdot 1\, dx \leq \|f\|_{L^2(X)}\sqrt{\mu(X)}\,.$$

□

1.7 Approximation of $L^p(X)$ by simple functions

Lemma 1.23. *If $p \in [1, \infty)$, then the set of simple functions $f = \sum_{i=1}^n a_i \mathbf{1}_{E_i}$, where each E_i is an element of the σ-algebra $\mathcal{A}$ and $\mu(E_i) < \infty$, is dense in $L^p(X, \mathcal{A}, \mu)$.*

Proof. If $f \in L^p$, then f is measurable; thus, there exists a sequence $\{\phi_n\}_{n=1}^\infty$ of simple functions, such that $\phi_n \to f$ a.e. with

$$0 \le |\phi_1| \le |\phi_2| \le \cdots \le |f|,$$

i.e., ϕ_n approximates f from below.

Recall that $|\phi_n - f|^p \to 0$ a.e. and $|\phi_n - f|^p \le 2^p |f|^p \in L^1$, so by the Dominated Convergence Theorem, $\|\phi_n - f\|_{L^p} \to 0$.

Now, suppose that the set E_i are disjoint; then by the definition of the Lebesgue integral,

$$\int_X \phi_n^p dx = \sum_{i=1}^n |a_i|^p \mu(E_i) < \infty \, .$$

If $a_i \neq 0$, then $\mu(E_i) < \infty$. □

1.8 Approximation of $L^p(\Omega)$ by continuous functions

Lemma 1.24. *Suppose that $\Omega \subset \mathbb{R}^n$ is bounded. Then $C^0(\Omega)$ is dense in $L^p(\Omega)$ for $p \in [1, \infty)$.*

Proof. Let K be any compact subset of Ω. The functions

$$F_{K,n}(x) = \frac{1}{1 + n \operatorname{dist}(x, K)} \in C^0(\Omega) \quad \text{satisfy} \quad F_{K,n} \le 1 \, ,$$

and decrease monotonically to the characteristic function $\mathbf{1}_K$. The Monotone Convergence Theorem gives

$$f_{K,n} \to \mathbf{1}_K \text{ in } L^p(\Omega), \quad 1 \le p < \infty \, .$$

Next, let $A \subset \Omega$ be any measurable set, and let λ denote the Lebesgue measure. Then

$$\lambda(A) = \sup\{\mu(K) \ : \ K \subset A, \quad K \text{ compact}\} \, .$$

It follows that there exists an increasing sequence of K_j of compact subsets of A such that $\lambda(A \backslash \cup_j K_j) = 0$. By the Monotone Convergence Theorem, $\mathbf{1}_{K_j} \to \mathbf{1}_A$ in $L^p(\Omega)$ for $p \in [1, \infty)$. According to Lemma 1.23, each function in $L^p(\Omega)$ is a norm limit of simple functions, so the lemma is proved. □

1.9 Approximation of $L^p(\Omega)$ by smooth functions

For $\Omega \subset \mathbb{R}^n$ open, for $\epsilon > 0$ taken sufficiently small, define the open subset of Ω by

$$\Omega_\epsilon := \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) > \epsilon\}.$$

Definition 1.25 (Mollifiers)**.** *Define $\eta \in C^\infty(\mathbb{R}^n)$ by*

$$\eta(x) := \begin{cases} Ce^{(|x|^2-1)^{-1}} & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1 \end{cases},$$

with constant $C > 0$ chosen such that $\int_{\mathbb{R}^n} \eta(x)dx = 1$.

For $\epsilon > 0$, the standard sequence of mollifiers on $\mathbb{R}^n$ is defined by

$$\eta_\epsilon(x) = \epsilon^{-n}\eta(x/\epsilon),$$

and satisfy $\int_{\mathbb{R}^n} \eta_\epsilon(x)dx = 1$ and $\operatorname{spt}(\eta_\epsilon) \subset \overline{B(0,\epsilon)}$.

Definition 1.26. *For $\Omega \subset \mathbb{R}^n$ open, set*

$$L^p_{\text{loc}}(\Omega) = \{u : \Omega \to \mathbb{R} \mid u \in L^p(\tilde{\Omega}) \ \ \forall\, \tilde{\Omega} \subset\subset \Omega\},$$

where $\tilde{\Omega} \subset\subset \Omega$ means that there exists K compact such that $\tilde{\Omega} \subset K \subset \Omega$. We say that $\tilde{\Omega}$ is compactly contained in Ω.

Definition 1.27 (Mollification of L^1)**.** *If $f \in L^1_{\text{loc}}(\Omega)$, define its mollification*

$$f^\epsilon = \eta_\epsilon * f \ \text{ in } \Omega_\epsilon,$$

so that

$$f^\epsilon(x) = \int_\Omega \eta_\epsilon(x-y)f(y)dy = \int_{B(0,\epsilon)} \eta_\epsilon(y)f(x-y)dy \quad \forall x \in \Omega_\epsilon.$$

Theorem 1.28 (Mollification of $L^p(\Omega)$)**.**

(A) $f^\epsilon \in C^\infty(\Omega_\epsilon)$.

(B) $f^\epsilon \to f$ a.e. as $\epsilon \to 0$.

(C) If $f \in C^0(\Omega)$, then $f^\epsilon \to f$ uniformly on compact subsets of Ω.

(D) If $p \in [1,\infty)$ and $f \in L^p_{\text{loc}}(\Omega)$, then $f^\epsilon \to f$ in $L^p_{\text{loc}}(\Omega)$.

Proof. **Part (A).** We rely on the difference quotient approximation of the partial derivative. Fix $x \in \Omega_\epsilon$, and choose h sufficiently small so that $x+he_i \in \Omega_\epsilon$ for $i = 1, ..., n$, and compute the difference quotient of f^ϵ:

$$\begin{aligned}\frac{f^\epsilon(x+he_i)-f(x)}{h} &= \epsilon^{-n}\int_\Omega \frac{1}{h}\left[\eta\left(\frac{x+he_i-y}{\epsilon}\right)-\eta\left(\frac{x-y}{\epsilon}\right)\right]f(y)dy \\ &= \epsilon^{-n}\int_{\tilde\Omega} \frac{1}{h}\left[\eta\left(\frac{x+he_i-y}{\epsilon}\right)-\eta\left(\frac{x-y}{\epsilon}\right)\right]f(y)dy\end{aligned}$$

for some open set $\tilde\Omega \subset\subset \Omega$. On $\tilde\Omega$,

$$\lim_{h\to 0}\frac{1}{h}\left[\eta\left(\frac{x+he_i-y}{\epsilon}\right)-\eta\left(\frac{x-y}{\epsilon}\right)\right] = \frac{1}{\epsilon}\frac{\partial\eta}{\partial x_i}\left(\frac{x-y}{\epsilon}\right),$$

so by the Dominated Convergence Theorem,

$$\frac{\partial f_\epsilon}{\partial x_i}(x) = \int_\Omega \frac{\partial\eta_\epsilon}{\partial x_i}(x-y)f(y)dy\,.$$

A similar argument for higher-order partial derivatives proves (A).

Step 2. Part (B). By the Lebesgue differentiation theorem,

$$\lim_{\epsilon\to 0}\frac{1}{|B(x,\epsilon)|}\int_{B(x,\epsilon)}|f(y)-f(x)|dy = 0 \text{ for a.e. } x\in\Omega\,.$$

Choose $x \in \Omega$ for which this limit holds. Then

$$\begin{aligned}|f_\epsilon(x)-f(x)| &\le \int_{B(x,\epsilon)}\eta_\epsilon(x-y)|f(y)-f(x)|dy \\ &= \frac{1}{\epsilon^n}\int_{B(x,\epsilon)}\eta((x-y)/\epsilon)|f(y)-f(x)|dy \\ &\le \frac{C}{|B(x,\epsilon)|}\int_{B(x,\epsilon)}|f(x)-f(y)|dy \longrightarrow 0 \quad \text{as} \quad \epsilon\to 0\,.\end{aligned}$$

Step 3. Part (C). For $\tilde\Omega \subset \Omega$, the above inequality shows that if $f \in C^0(\Omega)$ and hence uniformly continuous on $\tilde\Omega$, then $f^\epsilon(x) \to f(x)$ uniformly on $\tilde\Omega$.

Step 4. Part (D). For $f \in L^p_{\text{loc}}(\Omega)$, $p \in [1, \infty)$, choose open sets $U \subset\subset D \subset\subset \Omega$; then, for $\epsilon > 0$ small enough,

$$\|f^\epsilon\|_{L^p(U)} \le \|f\|_{L^p(D)}\,.$$

To see this, note that

$$\begin{aligned}|f^\epsilon(x)| &\le \int_{B(x,\epsilon)} \eta_\epsilon(x-y)|f(y)|dy\\ &= \int_{B(x,\epsilon)} \eta_\epsilon(x-y)^{(p-1)/p}\eta_\epsilon(x-y)^{1/p}|f(y)|dy\\ &\le \left(\int_{B(x,\epsilon)} \eta_\epsilon(x-y)dy\right)^{(p-1)/p}\left(\int_{B(x,\epsilon)} \eta_\epsilon(x-y)|f(y)|^p dy\right)^{1/p},\end{aligned}$$

so that for $\epsilon > 0$ sufficiently small

$$\begin{aligned}\int_U |f^\epsilon(x)|^p dx &\le \int_U \int_{B(x,\epsilon)} \eta_\epsilon(x-y)|f(y)|^p dydx\\ &\le \int_D |f(y)|^p \left(\int_{B(y,\epsilon)} \eta_\epsilon(x-y)dx\right) dy \le \int_D |f(y)|^p dy\,.\end{aligned}$$

Since $C^0(D)$ is dense in $L^p(D)$, choose $g \in C^0(D)$ such that $\|f-g\|_{L^p(D)} < \delta$; thus

$$\begin{aligned}\|f^\epsilon - f\|_{L^p(U)} &\le \|f^\epsilon - g^\epsilon\|_{L^p(U)} + \|g^\epsilon - g\|_{L^p(U)} + \|g-f\|_{L^p(U)}\\ &\le 2\|f-g\|_{L^p(D)} + \|g^\epsilon - g\|_{L^p(U)} \le 2\delta + \|g^\epsilon - g\|_{L^p(U)}\,.\end{aligned}$$

□

1.10 Continuous linear functionals on $L^p(X)$

Let $L^p(X)'$ denote the dual space of $L^p(X)$. For $\phi \in L^p(X)'$, the *operator norm* of ϕ is defined by $\|\phi\|_{\text{op}} = \sup_{L^p(X)=1} |\phi(f)|$.

Theorem 1.29. *Let $p \in (1,\infty]$, $q = \frac{p}{p-1}$. For $g \in L^q(X)$, define $F_g : L^p(X) \to \mathbb{R}$ as*

$$F_g(f) = \int_X fg dx\,.$$

Then F_g is a continuous linear functional on $L^p(X)$ with operator norm $\|F_g\|_{\text{op}} = \|g\|_{L^q(X)}$.

Proof. The linearity of F_g again follows from the linearity of the Lebesgue integral. Since

$$|F_g(f)| = \left|\int_X fg dx\right| \le \int_X |fg|\, dx \le \|f\|_{L^p} \|g\|_{L^q}\ ,$$

with the last inequality following from Hölder's inequality, we have that $\sup_{\|f\|_{L^p}=1} |F_g(f)| \le \|g\|_{L^q}$.

For the reverse inequality let $f = |g|^{q-1} \operatorname{sgn} g$. f is measurable and in L^p since $|f|^p = |f|^{\frac{q}{q-1}} = |g|^q$ and since $fg = |g|^q$,

$$F_g(f) = \int_X fg dx = \int_X |g|^q dx = \left(\int_X |g|^q \, dx\right)^{\frac{1}{p}+\frac{1}{q}}$$
$$= \left(\int_X |f|^p \, dx\right)^{\frac{1}{p}} \left(\int_X |g^q| \, dx\right)^{\frac{1}{q}} = \|f\|_{L^p} \|g\|_{L^q}$$

so that $\|g\|_{L^q} = \dfrac{F_g(f)}{\|f\|_{L^p}} \leq \|F_g\|_{\text{op}}$.

□

Remark 1.30. *Theorem 1.29 shows that for $1 < p \leq \infty$, there exists a linear isometry $g \mapsto F_g$ from $L^q(X)$ into $L^p(X)'$, the dual space of $L^p(X)$. When $p = \infty$, $g \mapsto F_g : L^1(X) \to L^\infty(X)'$ is rarely onto ($L^\infty(X)'$ is strictly larger than $L^1(X)$); on the other hand, if the measure space X is σ-finite, then $L^\infty(X) = L^1(X)'$.*

1.11 A theorem of F. Riesz

Theorem 1.31 (Representation theorem)**.** *Suppose that $1 < p < \infty$ and $\phi \in L^p(X)'$. Then there exists $g \in L^q(X)$, $q = \frac{p}{p-1}$ such that*

$$\phi(f) = \int_X fg dx \quad \forall f \in L^p(X),$$

and $\|\phi\|_{\text{op}} = \|g\|_{L^q}$.

Corollary 1.32. *For $p \in (1, \infty)$ the space $L^p(X, \mu)$ is reflexive, i.e., $L^p(X)'' = L^p(X)$.*

The proof Theorem 1.31 crucially relies on the Radon-Nikodym theorem, whose statement requires the following definition.

Definition 1.33. *If μ and ν are measure on (X, A) then $\nu \ll \mu$ if $\nu(E) = 0$ for every set E for which $\mu(E) = 0$. In this case, we say that ν is absolutely continuous with respect to μ.*

Theorem 1.34 (Radon-Nikodym)**.** *If μ and ν are two finite measures on X, i.e., $\mu(X) < \infty, \nu(X) < \infty$, and $\nu \ll \mu$, then*

$$\int_X F(x) \, d\nu(x) = \int_X F(x)h(x)d\mu(x) \tag{1.3}$$

holds for some nonnegative function $h \in L^1(X, \mu)$ and every positive measurable function F.

Proof. Define measures $\alpha = \mu + 2\nu$ and $\omega = 2\mu + \nu$, and let $\mathcal{H} = L^2(X, \alpha)$ (a Hilbert space) and suppose $\phi : L^2(X, \alpha) \to \mathbb{R}$ is defined by $\phi(f) = \int_X f d\omega$. We show that ϕ is a bounded linear functional since

$$|\phi(f)| = \left| \int_X f \, d(2\mu + \nu) \right| \leq \int_X |f| \, d(2\mu + 4\nu) = 2 \int_X |f| \, d\alpha \\ \leq \|f\|_{L^2(x,\alpha)} \sqrt{\alpha(X)} \, .$$

Thus, by the Riesz representation theorem, there exists $g \in L^2(X, \alpha)$ such that

$$\phi(f) = \int_X f \, d\omega = \int_X f g \, d\alpha \, ,$$

which implies that

$$\int_X f(2g - 1) d\nu = \int_X f(2 - g) d\mu \, . \tag{1.4}$$

Given $0 \leq F$ a measurable function on X, if we set $f = \frac{F}{2g-1}$ and $h = \frac{2-g}{2g-1}$ then $\int_X F d\nu = \int_X F h \, dx$ which is the desired result, if we can prove that $1/2 \leq g(x) \leq 2$. Define the sets

$$E_n^1 = \left\{ x \in X \mid g(x) < \frac{1}{2} - \frac{1}{n} \right\} \quad \text{and} \quad E_n^2 = \left\{ x \in X \mid g(x) > 2 + \frac{1}{n} \right\} .$$

By substituting $f = \mathbf{1}_{E_n^j}$, $j = 1, 2$ in (1.4), we see that

$$\mu(E_n^j) = \nu(E_n^j) = 0 \text{ for } j = 1, 2 \, ,$$

from which the bounds $1/2 \leq g(x) \leq 2$ hold. Also $\mu(\{x \in X \mid g(x) = 1/2\}) = 0$ and $\nu(\{x \in X \mid g(x) = 2\}) = 0$. Notice that if $F = 1$, then $h \in L^1(X)$. $\square$

Remark 1.35. *The more general version of the Radon-Nikodym theorem. Suppose that $\mu(X) < \infty$, ν is a finite signed measure (by the Hahn decomposition, $\nu = \nu^- + \nu^+$) such that $\nu \ll \mu$; then, there exists $h \in L^1(X, \mu)$ such that $\int_X F \, d\nu = \int_X F h \, d\mu$.*

Lemma 1.36 (Converse to Hölder's inequality)**.** *Let $\mu(X) < \infty$. Suppose that g is measurable and $fg \in L^1(X)$ for all simple functions f. If*

$$M(g) = \sup_{\|f\|_{L^p}=1} \left\{ \left| \int_X f g \, d\mu \right| \ : \ f \text{ is a simple function} \right\} < \infty \, , \tag{1.5}$$

then $g \in L^q(X)$, and $\|g\|_{L^q(X)} = M(g)$.

Proof. Let ϕ_n be a sequence of simple functions such that $\phi_n \to g$ a.e. and $|\phi_n| \leq |g|$. Set

$$f_n = \frac{|\phi_n|^{q-1} \operatorname{sgn}(\phi_n)}{\|\phi_n\|_{L^q}^{q-1}}$$

so that $\|f_n\|_{L^p} = 1$ for $p = q/(q-1)$. By Fatou's lemma,

$$\|g\|_{L^q(X)} \leq \liminf_{n\to\infty} \|\phi_n\|_{L^q(X)} = \liminf_{n\to\infty} \int_X |f_n \phi_n| d\mu \,.$$

Since $\phi_n \to g$ a.e., then

$$\|g\|_{L^q(X)} \leq \liminf_{n\to\infty} \int_X |f_n \phi_n| d\mu \leq \liminf_{n\to\infty} \int_X |f_n g| d\mu \leq M(g)\,.$$

The reverse inequality is implied by Hölder's inequality. □

Proof of the $L^p(X)'$ representation theorem. We have already proven that there exists a natural inclusion $\iota : L^q(X) \to L^p(X)'$ which is an isometry. It remains to show that ι is surjective.

Let $\phi \in L^p(X)'$ and define a set function ν on measurable subsets $E \subset X$ by

$$\nu(E) = \int_X \mathbf{1}_E d\nu =: \phi(\mathbf{1}_E)\,.$$

Thus, if $\mu(E) = 0$, then $\nu(E) = 0$. Then

$$\int_X f\, d\nu =: \phi(f)$$

for all simple functions f, and by Lemma 1.23, this holds for all $f \in L^p(X)$. By the Radon-Nikodym theorem, there exists $0 \leq g \in L^1(X)$ such that

$$\int_X f\, d\nu = \int_X f g\, d\mu \ \ \forall\, f \in L^p(X)\,.$$

But

$$\phi(f) = \int_X f\, d\nu = \int_X f g\, d\mu \tag{1.6}$$

and since $\phi \in L^p(X)'$, then $M(g)$ given by (1.5) is finite, and by the converse to Hölder's inequality, $g \in L^q(X)$, and $\|\phi\|_{\rm op} = M(g) = \|g\|_{L^q(X)}$. □

1.12 Weak convergence

The importance of the Representation Theorem 1.31 is in the use of the weak-* topology on the dual space $L^p(X)'$. Recall that for a Banach space $\mathbb{B}$ and for any sequence ϕ_j in the dual space $\mathbb{B}'$, $\phi_j \overset{*}{\rightharpoonup} \phi$ in $\mathbb{B}'$ weak-*, if $\langle \phi_j, f\rangle \to \langle \phi, f\rangle$ for each $f \in \mathbb{B}$, where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $\mathbb{B}'$ and $\mathbb{B}$.

Theorem 1.37 (Alaoglu's Lemma)**.** *If $\mathbb{B}$ is a Banach space, then the closed unit ball in $\mathbb{B}'$ is compact in the weak -* topology.*

Definition 1.38. *For $1 \le p < \infty$, a sequence $\{f_n\} \subset L^p(X)$ is said to weakly converge to $f \in L^p(X)$ if*

$$\int_X f_n(x)\phi(x)dx \to \int_X f(x)\phi(x)dx \quad \forall \phi \in L^q(X), q = \frac{p}{p-1}\,.$$

We denote this convergence by saying that $f_n \rightharpoonup f$ in $L^p(X)$ weakly.

Given that $L^p(X)$ is reflexive for $p \in (1, \infty)$, a simple corollary of Alaoglu's Lemma is the following

Theorem 1.39 (Weak compactness for L^p, $1 < p < \infty$)**.** *If $1 < p < \infty$ and $\{f_n\}$ is a bounded sequence in $L^p(X)$, then there exists a subsequence $\{f_{nk}\}$ such that $f_{nk} \rightharpoonup f$ in $L^p(X)$ weakly.*

Definition 1.40. *A sequence $\{f_n\} \subset L^\infty(X)$ is said to converge weak-* to $f \in L^\infty(X)$ if*

$$\int_X f_n(x)\phi(x)dx \to \int_X f(x)\phi(x)dx \quad \forall \phi \in L^1(X)\,.$$

We denote this convergence by saying that $f_n \overset{*}{\rightharpoonup} f$ in $L^\infty(X)$ weak-.*

Theorem 1.41 (Weak-* compactness for L^∞)**.** *If $\{f_n\}$ is a bounded sequence in $L^\infty(X)$, then there exists a subsequence $\{f_{nk}\}$ such that $f_{nk} \overset{*}{\rightharpoonup} f$ in $L^\infty(X)$ weak-*.*

Lemma 1.42. *If $f_n \to f$ in $L^p(X)$, then $f_n \rightharpoonup f$ in $L^p(X)$.*

Proof. By Hölder's inequality,

$$\left| \int_X g(f_n - f)dx \right| \le \|f_n - f\|_{L^p} \|g\|_{L^q}\,.$$

□

Note that if f_n is weakly convergent, in general, this does not imply that f_n is strongly convergent.

Example 1.43. *If $p = 2$, let f_n denote any orthonormal sequence in $L^2(X)$. From Bessel's inequality*

$$\sum_{n=1}^{\infty} \left| \int_X f_n g dx \right| \leq \|g\|^2_{L^2(X)} ,$$

we see that $f_n \rightharpoonup 0$ in $L^2(X)$.

This example shows that the map $f \mapsto \|f\|_{L^p}$ is continuous, but not weakly continuous. It is, however, weakly lower-semicontinuous.

Theorem 1.44. *If $f_n \rightharpoonup f$ weakly in $L^p(X)$, then $\|f\|_{L^p} \leq \liminf_{n\to\infty} \|f_n\|_{L^p}$.*

Proof. As a consequence of Theorem 1.31,

$$\begin{aligned} \|f\|_{L^p(X)} &= \sup_{\|g\|_{L^q(X)}=1} \left| \int_X f g dx \right| = \sup_{\|g\|_{L^q(X)}=1} \lim_{n\to\infty} \left| \int_X f_n g dx \right| \\ &\leq \sup_{\|g\|_{L^q(X)}=1} \liminf_{n\to\infty} \|f_n\|_{L^p} \|g\|_{L^q} . \end{aligned}$$

□

Theorem 1.45. *If $f_n \rightharpoonup f$ in $L^p(X)$, then f_n is bounded in $L^p(X)$.*

Theorem 1.46. *Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded. Suppose that*

$$\sup_n \|f_n\|_{L^p(\Omega)} \leq M < \infty \quad \text{and} \quad f_n \to f \quad a.e.$$

If $1 < p < \infty$, then $f_n \rightharpoonup f$ in $L^p(\Omega)$.

Proof. Egoroff's theorem states that for all $\epsilon > 0$, there exists $E \subset \Omega$ such that $\mu(E) < \epsilon$ and $f_n \to f$ uniformly on E^c. By definition, $f_n \rightharpoonup f$ in $L^p(\Omega)$ for $p \in (1, \infty)$ if $\int_\Omega (f_n - f) g dx \to 0$ for all $g \in L^q(\Omega)$, $q = \frac{p}{p-1}$. We have the inequality

$$\int_\Omega (f_n - f) g dx \leq \int_E |f_n - f| \, |g| \, dx + \int_{E^c} |f_n - f| \, |g| \, dx .$$

Choose $n \in \mathbb{N}$ sufficiently large, so that $|f_n(x) - f(x)| \leq \delta$ for all $x \in E^c$. By Hölder's inequality,

$$\int_{E^c} |f_n - f| \, |g| \, dx \leq \|f_n - f\|_{L^p(E^c)} \|g\|_{L^q(E^c)} \leq \delta \mu(E^c) \|g\|_{L^q(\Omega)} \leq C\delta$$

for a constant $C < \infty$.

By the Dominated Convergence Theorem, $\|f_n - f\|_{L^p(\Omega)} \le 2M$ so by Hölder's inequality, the integral over E is bounded by $2M\|g\|_{L^q(E)}$. Next, we use the fact that the integral is continuous with respect to the measure of the set over which the integral is taken. In particular, if $0 \le h$ is integrable, then for all $\delta > 0$, there exists $\epsilon > 0$ such that if the set E_ϵ has measure $\mu(E_\epsilon) < \epsilon$, then $\int_{E_\epsilon} h dx \le \delta$. To see this, either approximate h by simple functions, or use the Dominated Convergence theorem for the integral $\int_\Omega \mathbf{1}_{E_\epsilon}(x)h(x)dx$. □

Remark 1.47. *The proof of Theorem 1.46 does not work in the case that $p = 1$, as Hölder's inequality gives*

$$\int_E |f_n - f|\, |g|\, dx \le \|f_n - f\|_{L^1(\Omega)} \|g\|_{L^\infty(E)}\,,$$

so we lose the smallness of the right-hand side.

Remark 1.48. *Suppose that $E \subset X$ is bounded and measurable, and let $g = \mathbf{1}_E$. If $f_n \rightharpoonup f$ in $L^p(X)$, then*

$$\int_E f_n(x)dx \to \int_E f(x)dx;$$

hence, if $f_n \rightharpoonup f$, then the average of f_n converges to the average of f pointwise.

1.13 Integral operators

If $u : \mathbb{R}^n \to \mathbb{R}$ satisfies certain integrability conditions, then we can define the operator K acting on the function u as follows:

$$Ku(x) = \int_{\mathbb{R}^n} k(x,y)u(y)dy\,,$$

where $k(x, y)$ is called the *integral kernel.*The mollification procedure, introduced in Definition 1.27, is one example of the use of integral operators; the Fourier transform is another.

Definition 1.49. *Let $\mathcal{L}(L^p(\mathbb{R}^n), L^p(\mathbb{R}^n))$ denote the space of bounded linear operators from $L^p(\mathbb{R}^n)$ to itself. Using the Representation Theorem 1.31, the natural norm on $\mathcal{L}(L^p(\mathbb{R}^n), L^p(\mathbb{R}^n))$ is given by*

$$\|K\|_{\mathcal{L}(L^p(\mathbb{R}^n),L^p(\mathbb{R}^n))} = \sup_{\|f\|_{L^p}=1} \sup_{\|g\|_{L^q}=1} \left| \int_{\mathbb{R}^n} Kf(x)g(x)dx \right|.$$

Theorem 1.50. *Let $1 \le p < \infty$, $Ku(x) = \int_{\mathbb{R}^n} k(x,y)u(y)dy$, and suppose that*

$$\int_{\mathbb{R}^n} |k(x,y)|dx \le C_1 \ \forall y \in \mathbb{R}^n \ \textit{and} \ \int_{\mathbb{R}^n} |k(x,y)|dy \le C_2 \ \forall x \in \mathbb{R}^n\,,$$

where $0 < C_1, C_2 < \infty$. Then $K : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is bounded and

$$\|K\|_{\mathcal{L}(L^p(\mathbb{R}^n),L^p(\mathbb{R}^n))} \le C_1^{\frac{1}{p}} C_2^{\frac{p-1}{p}}\,.$$

In order to prove Theorem 1.50, we will need another well-known inequality.

Lemma 1.51 (Cauchy-Young Inequality). *If* $\frac{1}{p}+\frac{1}{q}=1$, *then for all* $a,b\geq 0$,

$$ab \leq \frac{a^p}{p}+\frac{b^q}{q}\,.$$

Proof. Suppose that $a,b>0$, otherwise the inequality trivially holds.

$$\begin{aligned} ab &= \exp(\log(ab)) = \exp(\log a + \log b) \quad (\text{since } a,b>0) \\ &= \exp\left(\frac{1}{p}\log a^p + \frac{1}{q}\log b^q\right) \\ &\leq \frac{1}{p}\exp(\log a^p) + \frac{1}{q}\exp(\log b^q) \quad (\text{using the convexity of } \exp) \\ &= \frac{a^p}{p}+\frac{b^q}{q} \end{aligned}$$

where we have used the condition $\frac{1}{p}+\frac{1}{q}=1$. □

Lemma 1.52 (Cauchy-Young Inequality with δ). *If* $\frac{1}{p}+\frac{1}{q}=1$, *then for all* $a,b\geq 0$,

$$ab \leq \delta\, a^p + C_\delta b^q\,, \qquad \delta>0\,,$$

with $C_\delta=(\delta p)^{-q/p}q^{-1}$.

Proof. This is a trivial consequence of Lemma 1.51 by setting

$$ab = a\cdot(\delta p)^{1/p} \quad \frac{b}{(\delta p)^{1/p}}\,.$$

□

Proof of Theorem 1.50. According to Lemma 1.51, $|f(y)g(x)|\leq \frac{|f(y)|^p}{p}+\frac{|g(x)|^q}{q}$ so that

$$\begin{aligned} &\left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} k(x,y)f(y)g(x)dydx\right| \\ &\quad\leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|k(x,y)|}{p}dx|f(y)|^p dy + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|k(x,y)|}{q}dy|g(x)|^q dx \\ &\quad\leq \frac{C_1}{p}\|f\|^p_{L^p}+\frac{C_2}{q}\|g\|^q_{L^q}\,. \end{aligned}$$

To improve this bound, notice that

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} k(x,y)f(y)g(x)dydx\right| \\
&\qquad \le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|k(x,y)|}{p}dx|tf(y)|^p dy + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|k(x,y)|}{q}dy|t^{-1}g(x)|^q dx \\
&\qquad \le \frac{C_1 t^p}{p}\|f\|^p_{L^p} + \frac{C_2 t^{-q}}{q}\|g\|^q_{L^q} =: F(t)\,.
\end{aligned}$$

Find the value of t for which $F(t)$ has a minimum to establish the desired bounded. □

Theorem 1.53 (Simple version of Young's inequality)**.** *Suppose that $k \in L^1(\mathbb{R}^n)$ and $f \in L^p(\mathbb{R}^n)$. Then*

$$\|k * f\|_{L^p} \le \|k\|_{L^1}\|f\|_{L^p}\,.$$

Proof. Define

$$K_k(f) = k * f := \int_{\mathbb{R}^n} k(x-y)f(y)dy\,.$$

Let $C_1 = C_2 = \|k\|_{L^1(\mathbb{R}^n)}$. Then according to Theorem 1.50, $K_k : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ and $\|K_k\|_{\mathcal{L}(L^p(\mathbb{R}^n),L^p(\mathbb{R}^n))} \le C_1$. □

Theorem 1.50 can easily be generalized to the setting of integral operators $K : L^q(\mathbb{R}^n) \to L^r(\mathbb{R}^n)$ built with kernels $k \in L^p(\mathbb{R}^n)$ such that $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. Such a generalization leads to

Theorem 1.54 (Young's inequality)**.** *Suppose that $k \in L^p(\mathbb{R}^n)$ and $f \in L^q(\mathbb{R}^n)$. Then*

$$\|k * f\|_{L^r} \le \|k\|_{L^p}\|f\|_{L^q} \quad \text{for} \quad 1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}\,.$$

1.14 Appendix 1: The monotone and dominated convergence theorems and Fatou's lemma

Let $\Omega \subset \mathbb{R}^d$ denote an open and smooth subset. The domain Ω is called smooth whenever its boundary $\partial\Omega$ is a smooth $(d-1)$-dimensional hypersurface.

Theorem 1.55 (Monotone Convergence Theorem)**.** *Let $f_n : \Omega \to \mathbb{R} \cup \{+\infty\}$ denote a sequence of functions, $f_n \ge 0$, and suppose that the sequence f_n is monotonically increasing, i.e.,*

$$f_1 \le f_2 \le f_3 \le \cdots$$

Then

$$\lim_{n\to\infty}\int_\Omega f_n(x)dx = \int_\Omega \lim_{n\to\infty} f_n(x)dx\,.$$

Lemma 1.56 (Fatou's Lemma)**.** *Suppose the sequence $f_n : \Omega \to \overline{\mathbb{R}}$ and $f_n \geq 0$. Then*

$$\int_\Omega \liminf_{n\to\infty} f_n(x)dx \leq \liminf_{n\to\infty} \int_\Omega f_n(x)dx\,.$$

Example 1.57. *Consider $\Omega = (0,1) \subset \mathbb{R}$ and suppose that $f_n = n\mathbf{1}_{(0,1/n)}$. Then $\int_0^1 f_n(x)dx = 1$ for all $n \in \mathbb{N}$, but $\liminf_{n\to\infty} \int_0^1 f_n(x)dx = 0$.*

Theorem 1.58 (Dominated Convergence Theorem)**.** *Suppose the sequence $f_n : \Omega \to \mathbb{R}$, $f_n(x) \to f(x)$ almost everywhere (with respect to Lebesgue measure), and furthermore, $|f_n| \leq g \in L^1(\Omega)$. Then $f \in L^1(\Omega)$ and*

$$\lim_{n\to\infty} \int_\Omega f_n(x)dx = \int_\Omega f(x)dx\,.$$

Equivalently, $f_n \to f$ in $L^1(\Omega)$ so that $\lim_{n\to\infty} \|f_n - f\|_{L^1(\Omega)} = 0$.

In the exercises, you will be asked to prove that the Monotone Convergence Theorem implies Fatou's Lemma which, in turn, implies the Dominated Convergence Theorem.

1.15 Appendix 2: The Fubini and Tonelli Theorems

Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ denote two fixed measure spaces. The product σ-algebra $\mathcal{A} \times \mathcal{B}$ of subsets of $X \times Y$ is defined by

$$\mathcal{A} \times \mathcal{B} = \{A \times B \ : \ A \in \mathcal{A}, B \in \mathcal{B}\}.$$

The set function $\mu \times \nu : \mathcal{A} \times \mathcal{B} \to [0, \infty]$ defined by

$$(\mu \times \nu)(A \times B) = \mu(A) \cdot \nu(B)$$

for each $A \times B \in \mathcal{A} \times \mathcal{B}$ is a measure.

Theorem 1.59 (Fubini)**.** *Let $f : X \times Y \to \mathbb{R}$ be a $\mu \times \nu$-integrable function. Then both iterated integrals exist and*

$$\int_{X\times Y} f\, d(\mu \times \nu) = \int_Y \int_X f\, d\mu d\nu = \int_X \int_Y f\, d\nu d\mu\,.$$

The existence of the iterated integrals is by no means enough to ensure that the function is integrable over the product space. As an example, let $X = Y = [0,1]$ and $\mu = \nu = \lambda$ with λ the Lebesgue measure. Set

$$f(x,y) = \begin{cases} \frac{x^2-y^2}{(x^2+y^2)^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}\,.$$

Then a standard computation shows that

$$\int_0^1 \int_0^1 f(x,y)dxdy = -\frac{\pi}{4}, \qquad \int_0^1 \int_0^1 f(x,y)dydx = \frac{\pi}{4}\,.$$

Fubini's theorem shows, of course, that f is not integrable over $[0,1]^2$

There is a converse to Fubini's theorem, however, according to which the existence of one of the iterated integrals is sufficient for the integrability of the function over the product space. The theorem is known as Tonelli's theorem, and this result is often used.

Theorem 1.60 (Tonelli)**.** *Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ denote two σ-finite measure spaces, and let $f : X \times Y \to \mathbb{R}$ be a $\mu \times \nu$-measurable function. If one of the iterated integrals $\int_X \int_Y |f| d\nu d\mu$ or $\int_Y \int_X |f| d\mu d\nu$ exists, then the function f is $\mu \times \nu$-integrable and hence, the other iterated integral exists and*

$$\int_{X \times Y} f\, d(\mu \times \nu) = \int_Y \int_X f\, d\mu d\nu = \int_X \int_Y f\, d\nu d\mu\,.$$

1.16 Exercises

Problem 1.1. *Use the Monotone Convergence Theorem to prove Fatou's Lemma.*

Problem 1.2. *Use Fatou's Lemma to prove the Dominated Convergence Theorem.*

Problem 1.3. *Let $\Omega \subset \mathbb{R}^d$ denote an open and smooth subset. Let $(a,b) \subset \mathbb{R}$ be an open interval, and let $f : (a,b) \times \Omega \to \mathbb{R}$ be a function such that for each $t \in (a,b)$, $f(t, \cdot) : \Omega \to \mathbb{R}$ is integrable and $\frac{df}{dt}(t,x)$ exists for each $(t,x) \in (a,b) \times \Omega$. Futhermore, assume that there is an integrable function $g : \Omega \to [0, \infty)$ such that $\sup_{t \in (a,b)} |\frac{df}{dt}(t,x)| \le g(x)$ for all $x \in \Omega$. Show that the function h defined by $h(t) \equiv \int_\Omega f(t,x)dx$ is differentiable and that the derivative is given by*

$$\frac{dh}{dt}(t) = \frac{d}{dt} \int_\Omega f(t,x)dx = \int_\Omega \frac{df}{dt}(t,x)dx$$

for each $t \in (a,b)$. **Hint:** *You will need to use the definition of the derivative for a real valued function function $r : (a,b) \to \mathbb{R}$ which is $\frac{dr}{dt}(t_0) = \lim_{h \to 0} \frac{r(t_0+h) - r(t_0)}{h}$, as well as the Mean Value Theorem from calculus which states the following: Let $(t_1, t_2) \subset \mathbb{R}$ and let $q : (t_1, t_2) \to \mathbb{R}$ be differentiable on (t_1, t_2). Then $\frac{|q(t_2) - q(t_1)|}{t_2 - t_1} = \frac{dq}{dt}(t')$ where t_1 is some point between t_1 and t_2.*

Problem 1.4. *Let Ω denote an open subset of $\mathbb{R}^n$. If $f \in L^1(\Omega) \cap L^\infty(\Omega)$, show that $f \in L^p(\Omega)$ for $1 < p < \infty$. If Ω is bounded, then show that $\lim_{p \nearrow \infty} \|f\|_{L^p} = \|f\|_{L^\infty}$. (***Hint:** *For $\epsilon > 0$, you can prove that the set $E = \{x \in \Omega \ : \ |f(x)| > \|f\|_{L^\infty} - \epsilon\}$ has positive Lebesgue measure, and the inequality $[\|f\|_{L^\infty} - \epsilon]\, \mathbf{1}_E \le |f|$ holds.)*

Problem 1.5. *Theorem 1.17 states that if* $1 \le p < \infty$, $f \in L^p$, $\{f_n\} \subset L^p$, $f_n \to f$ *a.e., and* $\lim_{n\to\infty} \|f_n\|_{L^p} = \|f\|_{L^p}$, *then* $\lim_{n\to\infty} \|f_n - f\|_{L^p} \to 0$. *Show by an example that this theorem is false when* $p = \infty$.

Problem 1.6. *Show that equality holds in the inequality*

$$a^\lambda b^{1-\lambda} \le \lambda a + (1-\lambda)b, \quad \lambda \in (0,1), a, b \ge 0$$

if and only if $a = b$. *Use this to show that if* $f \in L^p$ *and* $g \in L^q$ *for* $1 < p, q < \infty$ *and* $\frac{1}{p} + \frac{1}{q} = 1$, *then*

$$\int_\Omega |fg| dx = \|f\|_{L^p} \ \|g\|_{L^q}$$

holds if and only if there exists two constants C_1 *and* C_2 *(not both zero) such that* $C_1|f|^p = C_2|g|^q$ *holds.*

Problem 1.7. *Use the result of Problem 1.6 to prove that if* $f, g \in L^3(\Omega)$ *satisfy*

$$\|f\|_{L^3} = \|g\|_{L^3} = \int_\Omega f^2 \, g \, dx = 1 \,,$$

then $g = |f|$ *a.e.*

Problem 1.8. *Given* $f \in L^1(\mathbb{S}^1)$, $0 < r < 1$, *define*

$$P_r f(\theta) = \sum_{n=-\infty}^{\infty} \hat{f}_n r^{|n|} e^{in\theta}, \quad \hat{f}_n = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) e^{-in\theta} d\theta \,.$$

Show that

$$P_r f(\theta) = p_r * f(\theta) = \frac{1}{2\pi} \int_0^{2\pi} p_r(\theta - \phi) f(\phi) d\phi \,,$$

where

$$p_r(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta} = \frac{1 - r^2}{1 - 2r\cos\theta + r^2} \,.$$

Show that $\frac{1}{2\pi} \int_0^{2\pi} p_r(\theta) d\theta = 1$.

Problem 1.9. *If $f \in L^p(\mathbb{S}^1)$, $1 \le p < \infty$, show that*

$$P_r f \to f \quad in \quad L^p(\mathbb{S}^1) \quad as \quad r \nearrow 1\,.$$

Problem 1.10. *Suppose that $Y = [0,1]^2$ is the unit square in $\mathbb{R}^2$ and let $a(y)$ denote a Y-periodic function in $L^\infty(\mathbb{R}^2)$. For $\epsilon > 0$, let $a_\epsilon(x) = a(\frac{x}{\epsilon})$, and let $\bar{a} = \int_Y a(y)dy$ denote the average value of a. Prove that $a_\epsilon \overset{*}{\rightharpoonup} \bar{a}$ as $\epsilon \to 0$.*

Problem 1.11. *Let $f_n = \sqrt{n}\mathbf{1}_{(0,\frac{1}{n})}$. Prove that $f_n \rightharpoonup 0$ in $L^2(0,1)$, that $f_n \to 0$ in $L^1(0,1)$, but that f_n does not converge strongly in $L^2(0,1)$.*

2 The Sobolev spaces $H^k(\Omega)$ for integers $k \geq 0$

2.1 Weak derivatives

Definition 2.1 (Test functions)**.** *For $\Omega \subset \mathbb{R}^n$, set*

$$C_0^\infty(\Omega) = \{u \in C^\infty(\Omega) \mid \operatorname{spt}(u) \subset V \subset\subset \Omega\},$$

the smooth functions with compact support. Traditionally $\mathcal{D}(\Omega)$ is often used to denote $C_0^\infty(\Omega)$, and $\mathcal{D}(\Omega)$ is often referred to as the space of test functions.

For $u \in C^1(\mathbb{R})$, we can define $\frac{du}{dx}$ by the integration-by-parts formula; namely,

$$\int_{\mathbb{R}} \frac{du}{dx}(x)\phi(x)dx = -\int_{\mathbb{R}} u(x)\frac{d\phi}{dx}(x)dx \ \forall \phi \in C_0^\infty(\mathbb{R})\,.$$

Notice, however, that the right-hand side is well-defined, whenever $u \in L^1_{\text{loc}}(\mathbb{R})$

Definition 2.2. *An element $\alpha \in \mathbb{Z}^n_+$ (nonnegative integers) is called a multi-index. For such an $\alpha = (\alpha_1, ..., \alpha_n)$, we write $D^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}$ and $|\alpha| = \alpha_1 + \cdots \alpha_n$.*

Example 2.3. *Let $n = 2$. If $|\alpha| = 0$, then $\alpha = (0,0)$; if $|\alpha| = 1$, then $\alpha = (1,0)$ or $\alpha = (0,1)$. If $|\alpha| = 2$, then $\alpha = (1,1)$.*

Definition 2.4 (Weak derivative)**.** *Suppose that $u \in L^1_{\text{loc}}(\Omega)$. Then $v^\alpha \in L^1_{\text{loc}}(\Omega)$ is called the α^{th} weak derivative of u, written $v^\alpha = D^\alpha u$, if*

$$\int_\Omega u(x)\, D^\alpha \phi(x)dx = (-1)^{|\alpha|} \int_\Omega v^\alpha(x)\phi(x)dx \ \forall \phi \in C_0^\infty(\Omega)\,.$$

Example 2.5. *Let $n = 1$ and set $\Omega = (0,2)$. Define the function*

$$u(x) = \begin{cases} x, & 0 \leq x < 1 \\ 1, & 1 \leq x \leq 2 \end{cases} .$$

Then the function

$$v(x) = \begin{cases} 1, & 0 \leq x < 1 \\ 0, & 1 \leq x \leq 2 \end{cases}$$

is the weak derivative of u. To see this, note that for $\phi \in C_0^\infty(0,2)$,

$$\begin{aligned} \int_0^2 u(x)\frac{d\phi}{dx}(x)dx &= \int_0^1 x\frac{d\phi}{dx}(x)dx + \int_1^2 \frac{d\phi}{dx}(x)dx \\ &= -\int_0^1 \phi(x)dx + x\phi|_0^1 + \phi|_1^2 = -\int_0^1 \phi(x)dx \\ &= -\int_0^2 v(x)\phi(x)dx\,. \end{aligned}$$

Example 2.6. *Let $n = 1$ and set $\Omega = (0, 2)$. Define the function*

$$u(x) = \begin{cases} x, & 0 \leq x < 1 \\ 2, & 1 \leq x \leq 2 \end{cases} .$$

Then the weak derivative does not exist!

To prove this, assume for the sake of contradiction that there exists $v \in L^1_{\rm loc}(\Omega)$ such that for all $\phi \in C_0^\infty(0, 2)$,

$$\int_0^2 v(x)\phi(x)dx = -\int_0^2 u(x)\frac{d\phi}{dx}(x)dx \,.$$

Then

$$\begin{aligned} \int_0^2 v(x)\phi(x)dx &= -\int_0^1 x\frac{d\phi}{dx}(x)dx - 2\int_1^2 \frac{d\phi}{dx}(x)dx \\ &= \int_0^1 \phi(x)dx - \phi(1) + 2\phi(1) \\ &= \int_0^1 \phi(x)dx + \phi(1) \,. \end{aligned}$$

Suppose that ϕ_j is a sequence in $C_0^\infty(0, 2)$ such that $\phi_j(1) = 1$ and $\phi_j(x) \to 0$ for $x \neq 1$. Then

$$1 = \phi_j(1) = \int_0^2 v(x)\phi_j(x)dx - \int_0^1 \phi_j(x)dx \to 0 \,,$$

which provides the contradiction.

Definition 2.7. *For $p \in [1, \infty]$, define $W^{1,p}(\Omega) = \{u \in L^p(\Omega) \mid$ weak derivative exists $, Du \in L^p(\Omega)\}$, where Du is the weak derivative of u.*

Example 2.8. *Let $n = 1$ and set $\Omega = (0, 1)$. Define the function $f(x) = \sin(1/x)$. Then $u \in L^1(0, 1)$ and $\frac{du}{dx} = -\cos(1/x)/x^2 \in L^1_{\rm loc}(0, 1)$, but $u \notin W^{1,p}(\Omega)$ for any p.*

Definition 2.9. *In the case $p = 2$, we set $H^1(\Omega) = W^{1,p}(\Omega)$.*

Example 2.10. *Let $\Omega = B(0, 1) \subset \mathbb{R}^2$ and set $u(x) = |x|^{-\alpha}$. We want to determine the values of α for which $u \in H^1(\Omega)$.*

Since $|x|^{-\alpha} = \sum_{j=1}^3 (x_j x_j)^{-\alpha/2}$, then $\partial_{x_i}|x|^{-\alpha} = -\alpha|x|^{-\alpha-2}x_i$ is well-defined away from $x = 0$.

Step 1. We show that $u \in L^1_{\rm loc}(\Omega)$. To see this, note that $\int_\Omega |x|^{-\alpha}dx = \int_0^{2\pi}\int_0^1 r^{-\alpha} r dr d\theta < \infty$ whenever $\alpha < 2$.

Step 2. *Set the vector $v(x) = -\alpha|x|^{-\alpha-2}x$ (so that each component is given by $v_i(x) = -\alpha|x|^{-\alpha-2}x_i$). We show that*

$$\int_{B(0,1)} u(x)D\phi(x)dx = -\int_{B(0,1)} v(x)\phi(x)dx \quad \forall \phi \in C_0^\infty(B(0,1))\,.$$

To see this, let $\Omega_\delta = B(0,1) - B(0,\delta)$, let n denote the unit normal to $\partial\Omega_\delta$ (pointing toward the origin). Integration by parts yields

$$\int_{\Omega_\delta} |x|^{-\alpha} D\phi(x)dx = \int_0^{2\pi} \delta^{-\alpha}\phi(x)n(x)\delta d\theta + \alpha \int_{\Omega_\delta} |x|^{-\alpha-2}x\,\phi(x)dx\,.$$

Since $\lim_{\delta\to 0}\delta^{1-\alpha}\int_0^{2\pi}\phi(x)n(x)d\theta = 0$ if $\alpha < 1$, we see that

$$\lim_{\delta\to 0}\int_{\Omega_\delta} |x|^{-\alpha} D\phi(x)dx = \lim_{\delta\to 0}\alpha\int_{\Omega_\delta} |x|^{-\alpha-2}x\,\phi(x)dx$$

Since $\int_0^{2\pi}\int_0^1 r^{-\alpha-1}rdrd\theta < \infty$ if $\alpha < 1$, the Dominated Convergence Theorem shows that v is the weak derivative of u.

Step 3. $v \in L^2(\Omega)$, *whenever $\int_0^{2\pi}\int_0^1 r^{-2\alpha-2}rdrd\theta < \infty$ which holds if $\alpha < 0$.*

Remark 2.11. *Note that if the weak derivative exists, it is unique. To see this, suppose that both v_1 and v_2 are the weak derivative of u on Ω. Then $\int_\Omega(v_1 - v_2)\phi dx = 0$ for all $\phi \in C_0^\infty(\Omega)$, so that $v_1 = v_2$ a.e.*

2.2 Definition of Sobolev Spaces

Definition 2.12. *For integers $k \geq 0$ and $1 \leq p \leq \infty$,*

$$W^{k,p}(\Omega) = \{u \in L^1_{\text{loc}}(\Omega)\ |D^\alpha u \text{ exists and is in } L^p(\Omega) \text{ for } |\alpha| \leq k\}.$$

Definition 2.13. *For $u \in W^{k,p}(\Omega)$ define*

$$\|u\|_{W^{k,p}(\Omega)} = \left(\sum_{|\alpha|\leq k}\|D^\alpha u\|^p_{L^p(\Omega)}\right)^{\frac{1}{p}} \quad \text{for } 1 \leq p < \infty\,,$$

and

$$\|u\|_{W^{k,\infty}(\Omega)} = \sum_{|\alpha|\leq k}\|D^\alpha u\|_{L^\infty(\Omega)}\,.$$

The function $\|\cdot\|_{W^{k,p}(\Omega)}$ is clearly a norm since it is a finite sum of L^p norms.

Definition 2.14. *A sequence $u_j \to u$ in $W^{k,p}(\Omega)$ if* $\lim_{j\to\infty} \|u_j - u\|_{W^{k,p}(\Omega)} = 0$.

Theorem 2.15. *$W^{k,p}(\Omega)$ is a Banach space.*

Proof. Let u_j denote a Cauchy sequence in $W^{k,p}(\Omega)$. It follows that for all $|\alpha| \leq k$, $D^\alpha u_j$ is a Cauchy sequence in $L^p(\Omega)$. Since $L^p(\Omega)$ is a Banach space (see Theorem 1.19), for each α there exists $u^\alpha \in L^p(\Omega)$ such that

$$D^\alpha u_j \to u^\alpha \text{ in } L^p(\Omega)\,.$$

When $\alpha = (0, ..., 0)$ we set $u := u^{(0,\dots,0)}$ so that $u_j \to u$ in $L^p(\Omega)$. We must show that $u^\alpha = D^\alpha u$.

For each $\phi \in C_0^\infty(\Omega)$,

$$\begin{aligned}\int_\Omega u D^\alpha \phi dx &= \lim_{j\to\infty} \int_\Omega u_j D^\alpha \phi dx \\ &= (-1)^{|\alpha|} \lim_{j\to\infty} \int_\Omega D^\alpha u_j \phi dx \\ &= (-1)^{|\alpha|} \int_\Omega u^\alpha \phi dx\,;\end{aligned}$$

thus, $u^\alpha = D^\alpha u$ and hence $D^\alpha u_j \to D^\alpha u$ in $L^p(\Omega)$ for each $|\alpha| \leq k$, which shows that $u_j \to u$ in $W^{k,p}(\Omega)$. □

Definition 2.16. *For integers $k \geq 0$ and $p = 2$, we define*

$$H^k(\Omega) = W^{k,2}(\Omega)\,.$$

$H^k(\Omega)$ is a Hilbert space with inner-product $(u, v)_{H^k(\Omega)} = \sum_{|\alpha|\leq k}(D^\alpha u, D^\alpha v)_{L^2(\Omega)}$.

2.3 A simple version of the Sobolev embedding theorem

For two Banach spaces $\mathbb{B}_1$ and $\mathbb{B}_2$, we say that $\mathbb{B}_1$ is embedded in $\mathbb{B}_2$ if $\|u\|_{\mathbb{B}_2} \leq C\|u\|_{\mathbb{B}_1}$ for some constant C and for $u \in \mathbb{B}_1$. We wish to determine which Sobolev spaces $W^{k,p}(\Omega)$ can be embedded in the space of continuous functions. To motivate the type of analysis that is to be employed, we study a special case.

Theorem 2.17 (Sobolev embedding in 2-D)**.** *For $kp \geq 2$,*

$$\max_{x\in\mathbb{R}^2} |u(x)| \leq C\|u\|_{W^{k,p}(\mathbb{R}^2)} \quad \forall u \in C_0^\infty(\Omega)\,. \tag{2.1}$$

Proof. Given $u \in C_0^\infty(\Omega)$, we prove that for all $x \in \operatorname{spt}(u)$,

$$|u(x)| \leq C\|D^\alpha u(x)\|_{L^p(\Omega)} \quad \forall |\alpha| \leq k\,.$$

By choosing a coordinate system centered about x, we can assume that $x = 0$; thus, it suffices to prove that

$$|u(0)| \leq C\|D^\alpha u(x)\|_{L^p(\Omega)} \quad \forall |\alpha| \leq k\,.$$

Let $g \in C^\infty([0,\infty))$ with $0 \leq g \leq 1$, such that $g(x) = 1$ for $x \in [0, \frac{1}{2}]$ and $g(x) = 0$ for $x \in [\frac{3}{4}, \infty)$.

By the fundamental theorem of calculus,

$$\begin{aligned}
u(0) &= -\int_0^1 \partial_r[g(r)u(r,\theta)]dr = -\int_0^1 \partial_r(r)\, \partial_r[g(r)u(r,\theta)]dr \\
&= \int_0^1 r\, \partial_r^2[g(r)u(r,\theta)]dr \\
&= \frac{(-1)^k}{(k-1)!}\int_0^1 r^{k-1}\, \partial_r^k[g(r)u(r,\theta)]dr = \frac{(-1)^k}{(k-1)!}\int_0^1 r^{k-2}\, \partial_r^k[g(r)u(r,\theta)]rdr
\end{aligned}$$

Integrating both sides from 0 to 2π, we see that

$$u(0) = \frac{(-1)^k}{2\pi(k-1)!}\int_0^{2\pi}\int_0^1 r^{k-2}\, \partial_r^k[g(r)u(r,\theta)]rdrd\theta\,.$$

The change of variables from Cartesian to polar coordinates is given by

$$x(r,\theta) = r\cos\theta\,, \quad y(r,\theta) = r\sin\theta\,.$$

By the chain-rule,

$$\begin{aligned}
\partial_r u(x(r,\theta), y(r,\theta)) &= \partial_x u \cos\theta + \partial_y u \sin\theta\,, \\
\partial_r^2 u(x(r,\theta), y(r,\theta)) &= \partial_x^2 u \cos^2\theta + 2\partial_{xy}^2 u \cos\theta\sin\theta + \partial_y^2 u \sin^2\theta \\
&\;\;\vdots
\end{aligned}$$

It follows that $\partial_r^k = \sum_{|\alpha|\leq k} a_\alpha(\theta) D^\alpha$, where a_α consists of trigonometric polynomials of θ, so that

$$\begin{aligned}
u(0) &= \frac{(-1)^k}{2\pi(k-1)!}\int_{B(0,1)} r^{k-2} \sum_{|\alpha|\leq k} a_\alpha(\theta) D^\alpha[g(r)u(x)]dx \\
&\leq C\|r^{k-2}\|_{L^q(B(0,1))} \sum_{|\alpha|\leq k} \|D^\alpha(gu)\|_{L^p(B(0,1))} \\
&\leq C\left(\int_0^1 r^{\frac{p(k-2)}{p-1}} rdr\right)^{\frac{p-1}{p}} \|u\|_{W^{k,p}(\mathbb{R}^2)}\,.
\end{aligned}$$

Hence, we require $\frac{p(k-2)}{p-1} + 1 > -1$ or $kp > 2$. $\square$

2.4 Approximation of $W^{k,p}(\Omega)$ by smooth functions

Recall that $\Omega_\epsilon = \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) > \epsilon\}$.

Theorem 2.18. *For integers $k \geq 0$ and $1 \leq p < \infty$, let*

$$u^\epsilon = \eta_\epsilon * u \text{ in } \Omega_\epsilon\,,$$

where η_ϵ is the standard mollifier defined in Definition 1.25. Then

(A) $u^\epsilon \in C^\infty(\Omega_\epsilon)$ for each $\epsilon > 0$, and

(B) $u^\epsilon \to u$ in $W^{k,p}_{\mathrm{loc}}(\Omega)$ as $\epsilon \to 0$.

Definition 2.19. *A sequence $u_j \to u$ in $W^{k,p}_{\mathrm{loc}}(\Omega)$ if $u_j \to u$ in $W^{k,p}(\tilde\Omega)$ for each $\tilde\Omega \subset\subset \Omega$.*

Proof of Theorem 2.18. Theorem 1.28 proves part (A). Next, let v^α denote the the αth weak partial derivative of u. To prove part (B), we show that $D^\alpha u^\epsilon = \eta_\epsilon * v^\alpha$ in Ω_ϵ. For $x \in \Omega_\epsilon$,

$$\begin{aligned} D^\alpha u^\epsilon(x) &= D^\alpha \int_\Omega \eta_\epsilon(x-y) u(y) dy \\ &= \int_\Omega D^\alpha_x \eta_\epsilon(x-y) u(y) dy \\ &= (-1)^{|\alpha|} \int_\Omega D^\alpha_y \eta_\epsilon(x-y) u(y) dy \\ &= \int_\Omega \eta_\epsilon(x-y) v^\alpha(y) dy = (\eta_\epsilon * v^\alpha)(x)\,. \end{aligned}$$

By part (D) of Theorem 1.28, $D^\alpha u^\epsilon \to v^\alpha$ in $L^p_{\mathrm{loc}}(\Omega)$. □

It is possible to refine the above *interior* approximation result all the way to the boundary of Ω. We record the following theorem without proof.

Theorem 2.20. *Suppose that $\Omega \subset \mathbb{R}^n$ is a smooth, open, bounded subset, and that $u \in W^{k,p}(\Omega)$ for some $1 \leq p < \infty$ and integers $k \geq 0$. Then there exists a sequence $u_j \in C^\infty(\overline{\Omega})$ such that*

$$u_j \to u \quad \text{in} \quad W^{k,p}(\Omega)\,.$$

It follows that the inequality (2.1) holds for all $u \in W^{k,p}(\mathbb{R}^2)$.

2.5 Hölder Spaces

Recall that for $\Omega \subset \mathbb{R}^n$ open and smooth, the class of Lipschitz functions $u : \Omega \to \mathbb{R}$ satisfies the estimate

$$|u(x) - u(y)| \leq C|x - y| \quad \forall x, y \in \Omega$$

for some constant C.

Definition 2.21 (Classical derivative)**.** *A function $u : \Omega \to \mathbb{R}$ is differentiable at $x \in \Omega$ if there exists $f : \Omega \to \mathcal{L}(\mathbb{R}^n; \mathbb{R}^n)$ such that*

$$\frac{|u(x) - u(y) - f(x) \cdot (x - y)|}{|x - y|} \to 0\,.$$

We call $f(x)$ the classical derivative (or gradient) of $u(x)$, and denote it by $Du(x)$.

Definition 2.22. *If $u : \Omega \to \mathbb{R}$ is bounded and continuous, then*

$$\|u\|_{C^0(\overline{\Omega})} = \max_{x \in \Omega} |u(x)|\,.$$

If in addition u has a continuous and bounded derivative, then

$$\|u\|_{C^1(\overline{\Omega})} = \|u\|_{C^0(\overline{\Omega})} + \|Du\|_{C^0(\overline{\Omega})}\,.$$

The Hölder spaces *interpolate* between $C^0(\overline{\Omega})$ and $C^1(\overline{\Omega})$.

Definition 2.23. *For $0 < \gamma \leq 1$, the space $C^{0,\gamma}(\overline{\Omega})$ consists of those functions for which*

$$\|u\|_{C^{0,\gamma}(\overline{\Omega})} := \|u\|_{C^0(\overline{\Omega})} + [u]_{C^{0,\gamma}(\overline{\Omega})} < \infty\,,$$

where the γth Hölder semi-norm $[u]_{C^{0,\gamma}(\overline{\Omega})}$ is defined as

$$[u]_{C^{0,\gamma}(\overline{\Omega})} = \max_{\substack{x,y \in \Omega \\ x \neq y}} \left(\frac{|u(x) - u(y)|}{|x - y|^\gamma} \right)\,.$$

The space $C^{0,\gamma}(\overline{\Omega})$ is a Banach space.

2.6 Morrey's inequality

We can now offer a refinement and extension of the simple version of the Sobolev Embedding Theorem 2.17.

Theorem 2.24 (Morrey's inequality)**.** *For $n < p \leq \infty$, let $B(x, r) \subset \mathbb{R}^n$ and let $y \in B(x, r)$. Then*

$$|u(x) - u(y)| \leq C r^{1 - \frac{n}{p}} \|Du\|_{L^p(B(x,2r))} \forall u \in C^1(\mathbb{R}^n)\,.$$

In fact, Morrey's inequality holds for all $u \in W^{1,p}(B(x,2r))$ (see Problem 2.19 in the Exercises).

Notation 2.25 (Averaging)**.** *Let $B(0,1) \subset \mathbb{R}^n$. The volume of $B(0,1)$ is given by $\alpha_n = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}$ and the surface area is $|\mathbb{S}^{n-1}| = n\alpha_n$. We define*

$$\fint_{B(x,r)} f(y)dy = \frac{1}{\alpha_n r^n} \int_{B(x,r)} f(y)dy$$
$$\fint_{\partial B(x,r)} f(y)dS = \frac{1}{n\alpha_n r^{n-1}} \int_{\partial B(x,r)} f(y)dS\,.$$

Lemma 2.26. *For $B(x,r) \subset \mathbb{R}^n$, $y \in B(x,r)$ and $u \in C^1(\overline{B(x,r)})$,*

$$\fint_{B(x,r)} |u(y) - u(x)|dy \leq C \int_{B(x,r)} \frac{|Du(y)|}{|x-y|^{n-1}} dy\,.$$

Proof. For some $0 < s < r$, let $y = x + s\omega$ where $\omega \in \mathbb{S}^{n-1} = \partial B(0,1)$. By the fundamental theorem of calculus, for $0 < s < r$,

$$\begin{aligned} u(x+s\omega) - u(x) &= \int_0^s \frac{d}{dt} u(x+t\omega)dt \\ &= \int_0^s Du(x+t\omega)\,\omega dt\,. \end{aligned}$$

Since $|\omega| = 1$, it follows that

$$|u(x+s\omega) - u(x)| \leq \int_0^s |Du(x+t\omega)|dt\,.$$

Thus, integrating over $\mathbb{S}^{n-1}$ yields

$$\begin{aligned} \int_{\mathbb{S}^{n-1}} |u(x+s\omega) - u(x)|d\omega &\leq \int_0^s \int_{\mathbb{S}^{n-1}} |Du(x+t\omega)|d\omega dt \\ &\leq \int_0^s \int_{\mathbb{S}^{n-1}} |Du(x+t\omega)| \frac{t^{n-1}}{t^{n-1}} d\omega dt \\ &\leq \int_{B(x,r)} \frac{|Du(y)|}{|x-y|^{n-1}} dy\,, \end{aligned}$$

where we have set $y = x + t\omega$.

Multipling the above inequality by s^{n-1} and integrating s from 0 to r shows that

$$\begin{aligned} \int_0^r \int_{\mathbb{S}^{n-1}} |u(x+s\omega) - u(x)|d\omega s^{n-1} ds &\leq \frac{r^n}{n} \int_{B(x,r)} \frac{|Du(y)|}{|x-y|^{n-1}} dy \\ &\leq C\alpha_n r^n \int_{B(x,r)} \frac{|Du(y)|}{|x-y|^{n-1}} dy\,, \end{aligned}$$

which proves the lemma. □

Proof of Theorem 2.24. Assume first that $u \in C^1(\overline{B(x,2r)})$. Let $D = B(x,r) \cap B(y,r)$ and set $r = |x-y|$. Then

$$\begin{aligned} |u(x)-u(y)| &= \fint_D |u(x)-u(y)|dz \\ &\leq \fint_D |u(x)-u(z)|dz + \fint_D |u(y)-u(z)|dz \,. \end{aligned}$$

Since D equals the intersection of two balls of radius r, it is clear that can choose a constant C, depending only on the dimension n, such that

$$\frac{|D|}{|B(x,r)|} = \frac{|D|}{|B(y,r)|} = C\,.$$

It follows that

$$\begin{aligned} |u(x)-u(y)| &\leq \fint_D |u(x)-u(z)|dz + \fint_D |u(y)-u(z)|dz \\ &\leq \frac{C}{|B(x,r)|}\left(\int_D |u(x)-u(z)|dz + \int_D |u(y)-u(z)|dz\right) \\ &\leq C\fint_{B(x,r)} |u(x)-u(z)|dz + C\fint_{B(y,r)} |u(y)-u(z)|dz\,. \end{aligned}$$

Thus, by Lemma 2.26,

$$\fint_{B(x,r)} |u(x)-u(z)|dz \leq C\int_{B(x,r)} |x-z|^{1-n}|Du(z)|dz \leq C\int_{B(x,2r)} |x-z|^{1-n}|Du(z)|dz$$

and

$$\fint_{B(y,r)} |u(x)-u(z)|dz \leq C\int_{B(y,r)} |x-z|^{1-n}|Du(z)|dz \leq C\int_{B(x,2r)} |x-z|^{1-n}|Du(z)|dz$$

so that

$$|u(x)-u(y)| \leq C\int_{B(x,2r)} |x-z|^{1-n}|Du(z)|dz \tag{2.2}$$

and by Hölder's inequality,

$$|u(x)-u(y)| \leq C\left(\int_{B(0,2r)} s^{\frac{p(1-n)}{p-1}} s^{n-1} ds d\omega\right)^{\frac{p-1}{p}} \left(\int_{B(x,2r)} |Du(z)|^p dz\right)^{\frac{1}{p}}$$

□

Morrey's inequality implies the following embedding theorem.

Theorem 2.27 (Sobolev embedding theorem for $k = 1$)**.** *There exists a constant $C = C(p,n)$ such that*

$$\|u\|_{C^{0,1-\frac{n}{p}}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(\mathbb{R}^n)} \quad \forall u \in W^{1,p}(\mathbb{R}^n)\,.$$

Proof. First assume that $u \in C^1_0(\mathbb{R}^n)$. Given Morrey's inequality, it suffices to show that $\max|u| \leq C\|u\|_{W^{1,p}(\mathbb{R}^n)}$. Using Lemma 2.26, for all $x \in \mathbb{R}^n$,

$$\begin{aligned} |u(x)| &\leq \fint_{B(x,1)} |u(x)-u(y)|dy + \fint_{B(x,1)} |u(y)|dy \\ &\leq C\int_{B(x,1)} \frac{|Du(y)|}{|x-y|^{n-1}}dy + C\|u\|_{L^p(\mathbb{R}^n)} \\ &\leq C\|u\|_{W^{1,p}(\mathbb{R}^n)}\,, \end{aligned}$$

the last inequality following whenever $p > n$.

Thus,

$$\|u\|_{C^{0,1-\frac{n}{p}}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(\mathbb{R}^n)} \quad \forall u \in C^1(\mathbb{R}^n)\,. \tag{2.3}$$

By the density of $C^\infty_0(\mathbb{R}^n)$ in $W^{1,p}(\mathbb{R}^n)$, there is a sequence $u_j \in C^\infty_0(\mathbb{R}^n)$ such that

$$u_j \to u \in W^{1,p}(\mathbb{R}^n)\,.$$

By (2.3), for $j,k \in \mathbb{N}$,

$$\|u_j - u_k\|C^{0,1-\frac{n}{p}}(\mathbb{R}^n) \leq C\|u_j - u_k\|_{W^{1,p}(\mathbb{R}^n)}\,.$$

Since $C^{0,1-\frac{n}{p}}(\mathbb{R}^n)$ is a Banach space, there exists a $U \in C^{0,1-\frac{n}{p}}(\mathbb{R}^n)$ such that

$$u_j \to U \ \text{ in } \ C^{0,1-\frac{n}{p}}(\mathbb{R}^n)\,.$$

It follows that $U = u$ a.e. in Ω. By the continuity of norms with respect to strong convergence, we see that

$$\|U\|_{C^{0,1-\frac{n}{p}}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(\mathbb{R}^n)}$$

which completes the proof. □

In proving the above embedding theorem, we established that that for $p > n$, we have the inequality

$$\|u\|_{L^\infty(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(\mathbb{R}^n)}\,. \tag{2.4}$$

We will see later that (2.4), via a scaling argument, leads to the following important *interpolation inequality*: for $p > n$,

$$\|u\|_{L^\infty(\mathbb{R}^n)} \leq C(n,p)\|Du\|^{\frac{n}{p}}_{L^p(\mathbb{R}^n)}\|u\|^{\frac{p-n}{p}}_{L^p(\mathbb{R}^n)}\,.$$

Another important consequence of Morrey's inequality is the relationship between the weak and classical derivative of a function. We begin by recalling the definition of classical differentiability. A function $u : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at a point x if there exists a linear operator $L : \mathbb{R}^n \to \mathbb{R}^m$ such that for each $\epsilon > 0$, there exists $\delta > 0$ with $|y - x| < \delta$ implying that

$$\|u(y) - u(x) - L(y - x)\| \leq \epsilon\|y - x\|\,.$$

When such an L exists, we write $Du(x) = L$ and call it the classical derivative.

As a consequence of Morrey's inequality, we extract information about the classical differentiability properties of weak derivatives.

Theorem 2.28 (Differentiability a.e.)**.** *If $\Omega \subset \mathbb{R}^n$, $n < p \leq \infty$ and $u \in W^{1,p}_{\text{loc}}(\Omega)$, then u is differentiable a.e. in Ω, and its gradient equals its weak gradient almost everywhere.*

Proof. We first restrict $n < p < \infty$. By a version Lebesgue's differentiation theorem, for almost every $x \in \Omega$,

$$\lim_{r\to 0} \fint_{B(x,r)} |Du(x) - Du(z)|^p dz = 0\,, \tag{2.5}$$

where Du denotes the weak derivative of u. Thus, for $r > 0$ sufficiently small, we see that

$$\fint_{B(x,r)} |Du(x) - Du(z)|^p dz < \epsilon\,.$$

Fix a point $x \in \Omega$ for which (2.5) holds, and define the function

$$w_x(y) = u(y) - u(x) - Du(x) \cdot (y - x)\,.$$

Notice that $w_x(x) = 0$ and that

$$D_y w_x(y) = Du(y) - Du(x)\,.$$

Set $r = |x - y|$. Since $|u(y) - u(x) - Du(x) \cdot (y - x)| = |w_x(y) - w_x(x)|$, an application of the inequality (2.2) that we obtained in the proof of Morrey's inequality then yields the estimate

$$
\begin{aligned}
|u(y)-u(x)-Du(x)\cdot(y-x)| &\leq C\int_{B(x,2r)}\frac{|D_z w_x(z)|}{|x-z|^{n-1}}dz \\
&= C\int_{B(x,2r)}\frac{|Du(z)-Du(x)|}{|x-z|^{n-1}}dz \\
&\leq Cr^{1-\frac{n}{p}}\left(\int_{B(x,2r)}|Du(z)-Du(x)|^p dz\right)^{\frac{1}{p}} \\
&\leq Cr\left(\fint_{B(x,2r)}|Du(z)-Du(x)|^p dz\right)^{\frac{1}{p}} \\
&\leq C|x-y|\epsilon\,,
\end{aligned}
$$

from which it follows that $Du(x)$ is the classical derivative of u at the point x.

The case that $p=\infty$ follows from the inclusion $W^{1,\infty}_{\rm loc}(\Omega) \subset W^{1,p}_{\rm loc}(\Omega)$ for all $1 \leq p < \infty$. □

2.7 The Gagliardo-Nirenberg-Sobolev inequality

In the previous section, we considered the embedding for the case that $p > n$.

Theorem 2.29 (Gagliardo-Nirenberg inequality)**.** *For $1 \leq p < n$, set $p^* = \frac{np}{n-p}$. Then*

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C_{p,n}\|Du\|_{L^p(\mathbb{R}^n)} \quad \forall u \in W^{1,p}(\mathbb{R}^n)\,.$$

Proof for the case $n=2$. Suppose first that $p=1$ in which case $p^*=2$, and we must prove that

$$\|u\|_{L^2(\mathbb{R}^2)} \leq C\|Du\|_{L^1(\mathbb{R}^2)} \quad \forall u \in C^1_0(\mathbb{R}^2)\,. \tag{2.6}$$

Since u has compact support, by the fundamental theorem of calculus,

$$u(x_1,x_2) = \int_{-\infty}^{x_1}\partial_1 u(y_1,x_2)dy_1 = \int_{-\infty}^{x_2}\partial_2 u(x_1,y_2)dy_2$$

so that

$$|u(x_1,x_2)| \leq \int_{-\infty}^{\infty}|\partial_1 u(y_1,x_2)|dy_1 \leq \int_{-\infty}^{\infty}|Du(y_1,x_2)|dy_1$$

and

$$|u(x_1,x_2)| \leq \int_{-\infty}^{\infty}|\partial_2 u(x_1,y_2)|dy_2 \leq \int_{-\infty}^{\infty}|Du(x_1,y_2)|dy_2\,.$$

Hence, it follows that

$$|u(x_1,x_2)|^2 \leq \int_{-\infty}^{\infty} |Du(y_1,x_2)|dy_1 \int_{-\infty}^{\infty} |Du(x_1,y_2)|dy_2$$

Integrating over $\mathbb{R}^2$, we find that

$$\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |u(x_1,x_2)|^2 dx_1 dx_2 \\
&\qquad \leq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} |Du(y_1,x_2)|dy_1 \int_{-\infty}^{\infty} |Du(x_1,y_2)|dy_2\right) dx_1 dx_2 \\
&\qquad \leq \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |Du(x_1,x_2)|dx_1 dx_2\right)^2
\end{aligned}$$

which is (2.6).

Next, if $1 \leq p < 2$, substitute $|u|^\gamma$ for u in (2.6) to find that

$$\begin{aligned}
\left(\int_{\mathbb{R}^2} |u|^{2\gamma} dx\right)^{\frac{1}{2}} &\leq C\gamma \int_{\mathbb{R}^2} |u|^{\gamma-1}|Du|dx \\
&\leq C\gamma \|Du\|_{L^p(\mathbb{R}^2)} \left(\int_{\mathbb{R}^2} |u|^{\frac{p(\gamma-1)}{p-1}} dx\right)^{\frac{p-1}{p}}
\end{aligned}$$

Choose γ so that $2\gamma = \frac{p(\gamma-1)}{p-1}$; hence, $\gamma = \frac{p}{2-p}$, and

$$\left(\int_{\mathbb{R}^2} |u|^{\frac{2p}{2-p}} dx\right)^{\frac{2-p}{2p}} \leq C\gamma \|Du\|_{L^p(\mathbb{R}^2)}\,,$$

so that

$$\|u\|_{L^{\frac{2p}{2-p}}(\mathbb{R}^n)} \leq C_{p,n} \|Du\|_{L^p(\mathbb{R}^n)} \tag{2.7}$$

for all $u \in C_0^1(\mathbb{R}^2)$.

Since $C_0^\infty(\mathbb{R}^2)$ is dense in $W^{1,p}(\mathbb{R}^2)$, there exists a sequence $u_j \in C_0^\infty(\mathbb{R}^2)$ such that

$$u_j \to u \text{ in } W^{1,p}(\mathbb{R}^2)\,.$$

Hence, by (2.7), for all $j,k \in \mathbb{N}$,

$$\|u_j - u_k\|_{L^{\frac{2p}{2-p}}(\mathbb{R}^n)} \leq C_{p,n} \|Du_j - Du_k\|_{L^p(\mathbb{R}^n)}$$

so there exists $U \in L^{\frac{2p}{2-p}}(\mathbb{R}^n)$ such that

$$u_j \to U \text{ in } L^{\frac{2p}{2-p}}(\mathbb{R}^n)\,.$$

Hence $U = u$ a.e. in $\mathbb{R}^2$, and by continuity of the norms, (2.7) holds for all $u \in W^{1,p}(\mathbb{R}^2)$. □

Proof for the general case of dimension n. Following the proof for $n = 2$, we see that

$$|u(x)|^{\frac{n}{n-1}} \leq \Pi_{i=1}^n \left(\int_{-\infty}^{\infty} |Du(x_1, ..., y_i, ..., x_n)| dy_i \right)^{\frac{1}{n-1}}$$

so that

$$\begin{aligned}
\int_{-\infty}^{\infty} |u(x)|^{\frac{n}{n-1}} dx_1 \leq & \int_{-\infty}^{\infty} \Pi_{i=1}^n \left(\int_{-\infty}^{\infty} |Du(x_1, ..., y_i, ..., x_n)| dy_i \right)^{\frac{1}{n-1}} dx_1 \\
& \left(\int_{-\infty}^{\infty} |Du| dy_1 \right)^{\frac{1}{n-1}} \int_{-\infty}^{\infty} \Pi_{i=2}^n \left(\int_{-\infty}^{\infty} |Du| dy_i \right)^{\frac{1}{n-1}} dx_1 \\
& \left(\int_{-\infty}^{\infty} |Du| dy_1 \right)^{\frac{1}{n-1}} \Pi_{i=2}^n \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |Du| dx_1 dy_i \right)^{\frac{1}{n-1}} ,
\end{aligned}$$

where the last inequality follows from Hölder's inequality.

Integrating the last inequality with respect to x_2, we find that

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |u(x)|^{\frac{n}{n-1}} dx_1 dx_2 < \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |Du| dx_1 dy_2 \right)^{\frac{1}{n-1}} \int_{-\infty}^{\infty} \Pi_{\substack{i=1 \\ i \neq 2}}^n I_i^{\frac{1}{n-1}} dx_2 ,$$

where

$$I_1 = \int_{-\infty}^{\infty} |Du| dy_1 , \quad I_i = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |Du| dx_1 dy_i \text{ for } i = 3, ..., n .$$

Applying Hölder's inequality, we find that

$$\begin{aligned}
& \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |u(x)|^{\frac{n}{n-1}} dx_1 dx_2 \\
& \quad \leq \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |Du| dx_1 dy_2 \right)^{\frac{1}{n-1}} \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |Du| dy_1 dx_2 \right)^{\frac{1}{n-1}} \\
& \qquad \Pi_{i=3}^n \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |Du| dx_1 dx_2 dy_i \right)^{\frac{1}{n-1}} .
\end{aligned}$$

Next, continue to integrate with respect to $x_3, ..., x_n$ to find that

$$\begin{aligned}
\int_{\mathbb{R}^n} |u|^{\frac{n}{n-1}} dx & \leq \Pi_{i=1}^n \left(\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} |Du| dx_1 ... dy_i ... dx_n \right)^{\frac{1}{n-1}} \\
& = \left(\int_{\mathbb{R}^n} |Du| dx \right)^{\frac{n}{n-1}} .
\end{aligned}$$

This proves the case that $p = 1$. The case that $1 < p < n$ follows identically as in the proof of $n = 2$. □

It is common to employ the Gagliardo-Nirenberg inequality for the case that $p = 2$; as stated, the inequality is not well-defined in dimension two, but in fact, we have the following theorem.

Theorem 2.30. *Suppose that $u \in H^1(\mathbb{R}^2)$. Then for all $1 \leq q < \infty$,*

$$\|u\|_{L^q(\mathbb{R}^2)} \leq C\sqrt{q}\|u\|_{H^1(\mathbb{R}^2)} .$$

Proof. Let x and y be points in $\mathbb{R}^2$, and write $r = |x - y|$. Let $\theta \in \mathbb{S}^1$. Introduce spherical coordinates (r, θ) with origin at x, and let g be the same cut-off function that was used in the proof of Theorem 2.17. Define $U := g(r)u(r, \theta)$. Then

$$u(x) = -\int_0^1 \frac{\partial U}{\partial r}(r,\theta)dr - \int_0^1 |x-y|^{-1}\frac{\partial U}{\partial r}(r,\theta)rdr$$

and

$$|u(x)| \leq \int_0^1 |x-y|^{-1}|DU(r,\theta)|rdr .$$

Integrating over $\mathbb{S}^1$, we obtain:

$$|u(x)| \leq \frac{1}{2\pi}\int_{\mathbb{R}^2} \mathbf{1}_{B(x,1)}|x-y|^{-1}|DU(y)|dy := K * |DU| ,$$

where the integral kernel $K(x) = \frac{1}{2\pi}\mathbf{1}_{B(0,1)}|x|^{-1}$.

Using Young's inequality from Theorem 1.54, we obtain the estimate

$$\|K * f\|_{L^q(\mathbb{R}^2)} \leq \|K\|_{L^k(\mathbb{R}^2)}\|f\|_{L^2(\mathbb{R}^2)} \quad \text{for} \quad \frac{1}{k} = \frac{1}{q} - \frac{1}{2} + 1 . \tag{2.8}$$

Using the inequality (2.8) with $f = |DU|$, we see that

$$\begin{aligned} \|u\|_{L^q(\mathbb{R}^2)} &\leq C\|DU\|_{L^2(\mathbb{R}^2)} \left[\int_{B(0,1)} |y|^{-k}dy\right]^{\frac{1}{k}} \\ &\leq C\|DU\|_{L^2(\mathbb{R}^2)} \left[\int_0^1 r^{1-k}dr\right]^{\frac{1}{k}} \\ &= C\|u\|_{H^1(\mathbb{R}^2)} \left[\frac{q+2}{4}\right]^{\frac{1}{k}} . \end{aligned}$$

When $q \to \infty$, $\frac{1}{k} \to \frac{1}{2}$, so

$$\|u\|_{L^q(\mathbb{R}^2)} \leq Cq^{\frac{1}{2}}\|u\|_{H^1(\mathbb{R}^2)} .$$

□

Evidently, it is not possible to obtain the estimate $\|u\|_{L^\infty(\mathbb{R}^n)} \leq C\|u\|_{W^{1,n}(\mathbb{R}^n)}$ with a constant $C < \infty$. The following provides an example of a function in this borderline situation.

Example 2.31. *Let $\Omega \subset \mathbb{R}^2$ denote the open unit ball in $\mathbb{R}^2$. The unbounded function $u = \log\log\left(1 + \frac{1}{|x|}\right)$ belongs to $H^1(B(0,1))$.*

First, note that

$$\int_\Omega |u(x)|^2 dx = \int_0^{2\pi}\int_0^1 \Big[\log\log\Big(1+\frac{1}{r}\Big)\Big]^2 r dr d\theta\,.$$

The only potential singularity of the integrand occurs at $r = 0$, but according to L'Hospital's rule,

$$\lim_{r\to 0} r\Big[\log\log\Big(1+\frac{1}{r}\Big)\Big]^2 = 0, \tag{2.9}$$

so the integrand is continuous and hence $u \in L^2(\Omega)$.

In order to compute the partial derivatives of u, note that

$$\frac{\partial}{\partial x_j}|x| = \frac{x_j}{|x|}\,, \text{ and } \frac{d}{dz}|f(z)| = \frac{f(x)\frac{df}{dz}}{|f(z)|}\,,$$

where $f : \mathbb{R} \to \mathbb{R}$ is differentiable. It follows that for x away from the origin,

$$Du(x) = \frac{-x}{\log(1+\frac{1}{|x|})(|x|+1)|x|^2}\,, \quad (x \neq 0)\,.$$

Let $\phi \in C_0^\infty(\Omega)$ and fix $\epsilon > 0$. Then

$$\int_{\Omega - B_\epsilon(0)} u(x)\frac{\partial\phi}{\partial x_i}(x)dx = -\int_{\Omega - B(0,\epsilon)} \frac{\partial u}{\partial x_i}(x)\phi(x)dx + \int_{\partial B(0,\epsilon)} u\phi N_i dS\,,$$

where $N = (N_1, ..., N_n)$ denotes the inward-pointing unit normal on the curve $\partial B(0,\epsilon)$, so that $N\,dS = \epsilon(\cos\theta, \sin\theta)d\theta$. It follows that

$$\begin{aligned}\int_{\Omega - B_\epsilon(0)} u(x)D\phi(x)dx &= -\int_{\Omega - B_\epsilon(0)} Du(x)\phi(x)dx \\ &\quad - \int_0^{2\pi} \epsilon(\cos\theta,\sin\theta)\,\log\log\Big(1+\frac{1}{\epsilon}\Big)\phi(\epsilon,\theta)d\theta\,.\end{aligned} \tag{2.10}$$

We claim that $Du \in L^2(\Omega)$ (and hence also in $L^1(\Omega)$), for

$$\begin{aligned}\int_\Omega |Du(x)|^2 dx &= \int_0^{2\pi}\int_0^1 \frac{1}{r(r+1)^2\Big[\log\Big(1+\frac{1}{r}\Big)\Big]^2} dr d\theta \\ &\leq \pi\int_0^{1/2}\frac{1}{r(\log r)^2}dr + \pi\int_{1/2}^1 \frac{1}{r(r+1)^2\Big[\log\Big(1+\frac{1}{r}\Big)\Big]^2}dr\end{aligned}$$

where we use the inequality $\log(1+\frac{1}{r}) \geq \log\frac{1}{r} = -\log r \geq 0$ *for* $0 \leq r \leq 1$. *The second integral on the right-hand side is clearly bounded, while*

$$\int_0^{1/2} \frac{1}{r(\log r)^2} dr \quad = \quad \int_{-\infty}^{-\log 2} \frac{1}{t^2 e^t} e^t dt = \int_{-\infty}^{-\log 2} \frac{1}{x^2} dx < \infty \,,$$

so that $Du \in L^2(\Omega)$. *Letting* $\epsilon \to 0$ *in (2.10) and using (2.9) for the boundary integral, by the Dominated Convergence Theorem, we conclude that*

$$\int_\Omega u(x) D\phi(x) dx = -\int_\Omega Du(x)\phi(x) dx \quad \forall \phi \in C_0^\infty(\Omega) \,.$$

2.8 Local coordinates near $\partial\Omega$

Let $\Omega \subset \mathbb{R}^n$ denote an open, bounded subset with C^1 boundary, and let $\{U_l\}_{l=1}^K$ denote an open covering of $\partial\Omega$, such that for each $l \in \{1, 2, ..., K\}$, with

$$\begin{aligned} &\mathcal{V}_l = B(0, r_l), \text{ denoting the open ball of radius } r_l \text{ centered at the origin and,}\\ &\mathcal{V}_l^+ = \mathcal{V}_l \cap \{x_n > 0\} \,,\\ &\mathcal{V}_l^- = \mathcal{V}_l \cap \{x_n < 0\} \,, \end{aligned}$$

there exist C^1-class *charts* θ_l which satisfy

$$\begin{gathered} \theta_l : \mathcal{V}_l \to U_l \text{ is a } C^1 \text{ diffeomorphism} \,, \\ \theta_l(\mathcal{V}_l^+) = U_l \cap \Omega \,, \\ \theta_l(\mathcal{V}_l \cap \{x_n = 0\}) = U_l \cap \partial\Omega \,. \end{gathered} \tag{2.11}$$

2.9 Sobolev extensions and traces.

Let $\Omega \subset \mathbb{R}^n$ denote an open, bounded domain with C^1 boundary.

Theorem 2.32. *Suppose that* $\tilde{\Omega} \subset \mathbb{R}^n$ *is a bounded and open domain such that* $\Omega \subset\subset \tilde{\Omega}$. *Then for* $1 \leq p \leq \infty$, *there exists a bounded linear operator*

$$E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n)$$

such that for all $u \in W^{1,p}(\Omega)$,

1. $Eu = u$ *a.e. in* Ω;
2. $\operatorname{spt}(Eu) \subset \tilde{\Omega}$;
3. $\|Eu\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(\Omega)}$ *for a constant* $C = C(p, \Omega, \tilde{\Omega})$.

Theorem 2.33. *For $1 \leq p < \infty$, there exists a bounded linear operator*

$$T : W^{1,p}(\Omega) \to L^p(\Omega)$$

such that for all $u \in W^{1,p}(\Omega)$

1. *$Tu = u|_{\partial\Omega}$ for all $u \in W^{1,p}(\Omega) \cup C^0(\overline{\Omega})$;*
2. *$\|Tu\|_{L^p(\partial\Omega)} \leq C\|u\|_{W^{1,p}(\Omega)}$ for a constant $C = C(p, \Omega)$.*

Proof. Suppose that $u \in C^1(\overline{\Omega})$, $z \in \partial\Omega$, and that $\partial\Omega$ is locally flat near z. In particular, for $r > 0$ sufficiently small, $B(z, r) \cup \partial\Omega \subset \{x_n = 0\}$. Let $0 \leq \xi \in C_0^\infty(B(z, r))$ such that $\xi = 1$ on $B(z, r/2)$. Set $\Gamma = \partial\Omega \cup B(z, r/2)$, $B^+(z, r) = B(z, r) \cup \Omega$, and let $dx_h = dx_1 \cdots dx_{n-1}$. Then

$$\begin{aligned}
\int_\Gamma |u|^p dx_h &\leq \int_{\{x_n=0\}} \xi |u|^p dx_h \\
&= -\int_{B^+(z,r)} \frac{\partial}{\partial x_n}(\xi|u|^p) dx \\
&\leq -\int_{B^+(z,r)} \frac{\partial \xi}{\partial x_n}|u|^p dx - p\int_{B^+(z,2\delta)} \xi |u|^{p-2} u \frac{\partial u}{\partial x_n} dx \\
&\leq C \int_{B^+(z,r)} |u|^p dx + C \| |u|^{p-1} \|_{L^{\frac{p}{p-1}}(B^+(z,r))} \left\| \frac{\partial u}{\partial x_n} \right\|_{L^p(B^+(z,r))} \\
&\leq C \int_{B^+(z,r)} (|u|^p + |Du|^p) dx \,. \qquad (2.12)
\end{aligned}$$

On the other hand, if the boundary is not locally flat near $z \in \partial\Omega$, then we use a C^1 diffeomorphism to locally *straighten the boundary.* More specifically, suppose that $z \in \partial\Omega \cup U_l$ for some $l \in \{1, ..., K\}$ and consider the C^1 chart θ_l defined in (2.11). Define the function $U = u \circ \theta_l$; then $U : V_l^+ \to \mathbf{R}$. Setting $\Gamma = V_l \cup \{x_n = 0\|$, we see from the inequality (2.12), that

$$\int_\Gamma |U|^p dx_h \leq C_l \int_{V_l^+} (|U|^p + |DU|^p) dx \,.$$

Using the fact that $D\theta_l$ is bounded and continuous on V_l^+, the change of variables formula shows that

$$\int_{U_l \cup \partial\Omega} |u|^p dS \leq C_l \int_{U_l^+} (|u|^p + |Du|^p) dx \,.$$

Summing over all $l \in \{1, ..., K\}$ shows that

$$\int_{\partial\Omega} |u|^p dS \leq C \int_\Omega (|u|^p + |Du|^p) dx \,. \tag{2.13}$$

The inequality (2.13) holds for all $u \in C^1(\overline{\Omega})$. According to Theorem 2.20, for $u \in W^{1,p}(\Omega)$ there exists a sequence $u_j \in C^\infty(\overline{\Omega})$ such that $u_j \to u$ in $W^{1,p}(\Omega)$. By inequality (2.13),

$$\|Tu_k - Tu_j\|_{L^p(\partial\Omega)} \leq C \|u_k - u_j\|_{W^{1,p}(\Omega)} \,,$$

so that Tu_j is Cauchy in $L^p(\partial\Omega)$, and hence a limit exists in $L^p(\partial\Omega)$ We define the trace operator T as this limit:

$$\lim_{j\to 0} \|Tu - Tu_j\|_{L^p(\partial\Omega)} = 0 \,.$$

Since the sequence u_j converges uniformly to u if $u \in C^0(\overline{\Omega})$, we see that $Tu = u|_{\partial\Omega}$ for all $u \in W^{1,p}(\Omega) \cup C^0(\overline{\Omega})$. □

Sketch of the proof of Theorem 2.32. Just as in the proof of the trace theorem, first suppose that $u \in C^1(\overline{\omega})$ and that near $z \in \partial\Omega$, $\partial\Omega$ is *locally flat*, so that for some $r > 0$, $\partial\Omega \cup B(z,r) \subset \{x_n = 0\}$. Letting $B^+ = B(z,r) \cup \{x_n \geq 0\}$ and $B^- = B(z,r) \cup \{x_n \leq 0\}$, we define the extension of u by

$$\bar{u}(x) = \begin{cases} u(x) & \text{if } x \in B^+ \\ -3u(x_1, ..., x_{n-1}, -x_n) + 4u(x_1, ..., x_{n-1}, -\frac{x_n}{2}) & \text{if } x \in B^- \,. \end{cases}$$

Define $u^+ = \bar{u}|_{B^+}$ and $u^- = \bar{u}|_{B^-}$.

It is clear that $u^+ = u^-$ on $\{x_n = 0\}$, and by the chain-rule, it follows that

$$\frac{\partial u^-}{\partial x_n}(x) = 3\frac{\partial u^-}{\partial x_n}(x_1, ..., -x_n) - 2\frac{\partial u^-}{\partial x_n}(x_1, ..., -\frac{x_n}{2}) \,,$$

so that $\frac{\partial u^+}{\partial x_n} = \frac{\partial u^-}{\partial x_n}$ on $\{x_n = 0\}$. This shows that $\bar{u} \in C^1(B(z,r)$. using the charts θ_l to locally straighten the boundary, and the density of the $C^\infty(\overline{\Omega})$ in $W^{1,p}(\Omega)$, the theorem is proved. □

Later, we will provide a proof for higher-order Sobolev extensions of H^k-type functions.

2.10 The subspace $W_0^{1,p}(\Omega)$

Definition 2.34. *We let $W_0^{1,p}(\Omega)$ denote the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$.*

Theorem 2.35. *Suppose that $\Omega \subset \mathbb{R}^n$ is bounded with C^1 boundary, and that $u \in W^{1,p}(\Omega)$. Then*

$$u \in W_0^{1,p}(\Omega) \quad \textit{iff } Tu = 0 \quad \textit{on} \quad \partial\Omega \,.$$

We can now state the Sobolev embedding theorems for bounded domains Ω.

Theorem 2.36 (Gagliardo-Nirenberg inequality for $W^{1,p}(\Omega)$)**.** *Suppose that $\Omega \subset \mathbb{R}^n$ is open and bounded with C^1 boundary, $1 \leq p < n$, and $u \in W^{1,p}(\Omega)$. Then*

$$\|u\|_{L^{\frac{np}{n-p}}(\Omega)} \leq C\|u\|_{W^{1,p}(\Omega)} \text{ for a constant } C = C(p,n,\Omega)\,.$$

Proof. Choose $\tilde{\Omega} \subset \mathbb{R}^n$ bounded such that $\Omega \subset\subset \tilde{\Omega}$, and let Eu denote the Sobolev extension of u to $\mathbb{R}^n$ such that $Eu = u$ a.e., $\text{spt}(Eu) \subset \tilde{\Omega}$, and $\|Eu\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(\Omega)}$.

Then by the Gagliardo-Nirenberg inequality,

$$\|u\|_{L^{\frac{np}{n-p}}(\Omega)} \leq \|Eu\|_{L^{\frac{np}{n-p}}(\mathbb{R}^n)} \leq C\|D(Eu)\|_{L^p(\mathbb{R}^n)} \leq C\|Eu\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(\Omega)}\,.$$

□

Theorem 2.37 (Gagliardo-Nirenberg inequality for $W_0^{1,p}(\Omega)$)**.** *Suppose that $\Omega \subset \mathbb{R}^n$ is open and bounded with C^1 boundary, $1 \leq p < n$, and $u \in W_0^{1,p}(\Omega)$. Then for all $1 \leq q \leq \frac{np}{n-p}$,*

$$\|u\|_{L^q(\Omega)} \leq C\|Du\|_{L^p(\Omega)} \text{ for a constant } C = C(p,n,\Omega)\,. \tag{2.14}$$

Proof. By definition there exists a sequence $u_j \in C_0^\infty(\Omega)$ such that $u_j \to u$ in $W^{1,p}(\Omega)$. Extend each u_j by 0 on Ω^c. Applying Theorem 2.29 to this extension, and using the continuity of the norms, we obtain $\|u\|_{L^{\frac{pn}{n-p}}(\Omega)} \leq C\|Du\|_{L^p(\Omega)}$. Since Ω is bounded, the assertion follows by Hölder's inequality. □

Theorem 2.38. *Suppose that $\Omega \subset \mathbb{R}^2$ is open and bounded with C^1 boundary, and $u \in H_0^1(\Omega)$. Then for all $1 \leq q < \infty$,*

$$\|u\|_{L^q(\Omega)} \leq C\sqrt{q}\|Du\|_{L^2(\Omega)} \text{ for a constant } C = C(\Omega)\,. \tag{2.15}$$

Proof. The proof follows that of Theorem 2.30. Instead of introducing the cut-off function g, we employ a partition of unity subordinate to the finite covering of the bounded domain Ω, in which case it suffices that assume that $\text{spt}(u) \subset \text{spt}(U)$ with U also defined in the proof Theorem 2.30. □

Remark 2.39. *Inequalities (2.14) and (2.15) are commonly referred to as Poincaré inequalities. They are invaluable in the study of the Dirichlet problem for Poisson's equation, since the right-hand side provides an $H^1(\Omega)$-equivalent norm for all $u \in H_0^1(\Omega)$. In particular, there exists constants C_1, C_2 such that*

$$C_1\|Du\|_{L^2(\Omega)} \leq \|u\|_{H^1(\Omega)} \leq C_2\|Du\|_{L^2(\Omega)}\,.$$

2.11 Weak solutions to Dirichlet's problem

Suppose that $\Omega \subset \mathbb{R}^n$ is an open, bounded domain with C^1 boundary. A classical problem in the linear theory of partial differential equations consists of finding solutions to the *Dirichlet problem*:

$$-\Delta u = f \quad \text{in} \quad \Omega\,, \tag{2.16a}$$
$$u = 0 \quad \text{on} \quad \partial\Omega\,, \tag{2.16b}$$

where $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ denotes the Laplace operator or *Laplacian*. As written, (2.16) is the so-called *strong form* of the Dirichlet problem, as it requires that u to possess certain weak second-order partial derivatives. A major turning-point in the modern theory of linear partial differential equations was the realization that *weak solutions* of (2.16) could be defined, which only require weak first-order derivatives of u to exist. (We will see more of this idea later when we discuss the theory of distributions.)

Definition 2.40. *The dual space of $H_0^1(\Omega)$ is denoted by $H^{-1}(\Omega)$. For $f \in H^{-1}(\Omega)$,*

$$\|f\|_{H^{-1}(\Omega)} = \sup_{\|\psi\|_{H_0^1(\Omega)}=1} \langle f, \psi\rangle\,,$$

where $\langle f, \psi\rangle$ denotes the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.

Definition 2.41. *A function $u \in H_0^1(\Omega)$ is a weak solution of (2.16) if*

$$\int_\Omega Du \cdot Dv\, dx = \langle f, v\rangle \quad \forall v \in H_0^1(\Omega)\,.$$

Remark 2.42. *Note that f can be taken in $H^{-1}(\Omega)$. According to the Sobolev embedding theorem, this implies that when $n = 1$, the forcing function f can be taken to be the Dirac Delta distribution.*

Remark 2.43. *The motivation for Definition 2.41 is as follows. Since $C_0^\infty(\Omega)$ is dense in $H_0^1(\Omega)$, multiply equation (2.16a) by $\phi \in C_0^\infty(\Omega)$, integrate over Ω, and employ the integration-by-parts formula to obtain $\int_\Omega Du \cdot D\phi\, dx = \int_\Omega f\phi\, dx$; the boundary terms vanish because ϕ is compactly supported.*

Theorem 2.44 (Existence and uniqueness of weak solutions)**.** *For any $f \in H^{-1}(\Omega)$, there exists a unique weak solution to (2.16).*

Proof. Using the Poincaré inequality, $\|Du\|_{L^2(\Omega)}$ is an H^1-equivalent norm for all $u \in H_0^1(\Omega)$, and $(Du, Dv)_{L^2(\Omega)}$ defines the inner-product on $H_0^1(\Omega)$. As such, according to the definition of weak solutions to (2.16), we are seeking $u \in H_0^1(\Omega)$ such that

$$(u, v)_{H_0^1(\Omega)} = \langle f, v\rangle \quad \forall v \in H_0^1(\Omega)\,. \tag{2.17}$$

The existence of a unique $u \in H_0^1(\Omega)$ satisfying (2.17) is provided by the Riesz representation theorem for Hilbert spaces. □

Remark 2.45. *Note that the Riesz representation theorem shows that there exists a distribution, denoted by $-\Delta u \in H^{-1}(\Omega)$ such that*

$$\langle -\Delta u, v\rangle = \langle f, v\rangle \quad \forall v \in H_0^1(\Omega)\,.$$

The operator $-\Delta : H_0^1(\Omega) \to H^{-1}(\Omega)$ is thus an isomorphism.

A fundamental question in the theory of linear partial differential equations is commonly referred to as *elliptic regularity*, and can be explained as follows: in order to develop an existence and uniqueness theorem for the Dirichlet problem, we have significantly generalized the notion of solution to the class of weak solutions, which permitted very weak forcing functions in $H^{-1}(\Omega)$. Now suppose that the forcing function is smooth; is the weak solution smooth as well? Furthermore, does the weak solution agree with the classical solution? The answer is yes, and we will develop this regularity theory in Chapter 7, where it will be shown that for integers $k \geq 2$, $-\Delta : H^k(\Omega) \cap H_0^1(\Omega) \to H^{k-2}(\Omega)$ is also an isomorphism. An important consequence of this result is that $(-\Delta)^{-1} : H^{k-2}(\Omega) \to H^k(\Omega) \cap H_0^1(\Omega)$ is a *compact* linear operator, and as such has a countable set of eigenvalues, a fact that is eminently useful in the construction of solutions for heat- and wave-type equations.

For this reason, as well as the consideration of weak limits of nonlinear combinations of sequences, we must develop a compactness theorem, which generalizes the well-known Arzela-Ascoli theorem to Sobolev spaces.

2.12 Strong compactness

In Section 1.12, we defined the notion of weak converence and weak compactness for L^p-spaces. Recall that for $1 \leq p < \infty$, a sequence $u_j \in L^p(\Omega)$ converges *weakly* to $u \in L^p(\Omega)$, denoted $u_j \rightharpoonup u$ in $L^p(\Omega)$, if $\int_\Omega u_j v dx \to \int_\Omega uv dx$ for all $v \in L^q(\Omega)$, with $q = \frac{p}{p-1}$. We can extend this definition to Sobolev spaces.

Definition 2.46. *For $1 \leq p < \infty$, $u_j \rightharpoonup u$ in $W^{1,p}(\Omega)$ provided that $u_j \rightharpoonup u$ in $L^p(\Omega)$ and $Du_j \rightharpoonup Du$ in $L^p(\Omega)$.*

Alaoglu's Lemma (Theorem 1.37) then implies the following theorem.

Theorem 2.47 (Weak compactness in $W^{1,p}(\Omega)$)**.** *For $\Omega \subset \mathbb{R}^n$, suppose that*

$$\sup \|u_j\|_{W^{1,p}(\Omega)} \leq M < \infty \quad \textit{for a constant } M \neq M(j)\,.$$

Then there exists a subsequence $u_{j_k} \rightharpoonup u$ in $W^{1,p}(\Omega)$.

It turns out that weak compactness often does not suffice for limit processes involving nonlinearities, and that the Gagliardo-Nirenberg inequality can be used to obtain the following strong compactness theorem.

Theorem 2.48 (Rellich's theorem on a bounded domain Ω)**.** *Suppose that $\Omega \subset \mathbb{R}^n$ is an open, bounded domain with C^1 boundary, and that $1 \leq p < n$. Then $W^{1,p}(\Omega)$ is compactly embedded in $L^q(\Omega)$ for all $1 \leq q < \frac{np}{n-p}$, i.e. if*

$$\sup \|u_j\|_{W^{1,p}(\Omega)} \leq M < \infty \quad \textit{for a constant } M \neq M(j)\,,$$

then there exists a subsequence $u_{j_k} \to u$ in $L^q(\Omega)$. In the case that $n = 2$ and $p = 2$, $H^1(\Omega)$ is compactly embedded in $L^q(\Omega)$ for $1 \leq q < \infty$.

In order to prove Rellich's theorem, we need two lemmas.

Lemma 2.49 (Arzela-Ascoli Theorem)**.** *Suppose that $u_j \in C^0(\overline{\Omega})$, $\|u_j\|_{C^0(\overline{\Omega})} \leq M < \infty$, and u_j is equicontinuous. Then there exists a subsequence $u_{j_k} \to u$ uniformly on $\overline{\Omega}$.*

Lemma 2.50. *Let $1 \leq r \leq s \leq t \leq \infty$, and suppose that $u \in L^r(\Omega) \cap L^t(\Omega)$. Then for $\frac{1}{s} = \frac{a}{r} + \frac{1-a}{t}$*

$$\|u\|_{L^s(\Omega)} \leq \|u\|^a_{L^r(\Omega)} \|u\|^{1-a}_{L^t(\Omega)}\,.$$

Proof. By Hölder's inequality,

$$\begin{aligned}
\int_\Omega |u|^s dx &= \int_\Omega |u|^{as} |u|^{(1-a)s} dx \\
&\leq \left(\int_\Omega |u|^{as\frac{r}{as}} dx \right)^{\frac{as}{r}} \left(\int_\Omega |u|^{(1-a)s\frac{t}{(1-a)s}} dx \right)^{\frac{(1-a)s}{t}} = \|u\|^{as}_{L^r(\Omega)} \|u\|^{(1-a)s}_{L^t(\Omega)}\,.
\end{aligned}$$

□

Proof of Rellich's theorem. Let $\tilde{\Omega} \subset \mathbb{R}^n$ denote an open, bounded domain such that $\Omega \subset\subset \tilde{\Omega}$. By the Sobolev extension theorem, the sequence u_j satisfies $\text{spt}(u_j) \subset \tilde{\Omega}$, and

$$\sup \|Eu_j\|_{W^{1,p}(\mathbb{R}^n)} \leq CM\,.$$

Denote the sequence Eu_j by $\bar{u}_j$. By the Gagliardo-Nirenberg inequality, if $1 \leq q < \frac{np}{n-p}$,

$$\sup \|u\|_{L^q(\Omega)} \leq \sup \|\bar{u}\|_{L^q(\mathbb{R}^n)} \leq C \sup \|\bar{u}_j\|_{W^{1,p}(\mathbb{R}^n)} \leq CM\,.$$

For $\epsilon > 0$, let η_ϵ denote the standard mollifiers and set $\bar{u}^\epsilon_j = \eta_\epsilon * Eu_j$. By choosing $\epsilon > 0$ sufficiently small, $\bar{u}^\epsilon_j \in C^\infty_0(\tilde{\Omega})$. Since

$$\bar{u}^\epsilon_j = \int_{B(0,\epsilon)} \frac{1}{\epsilon^n} \eta(\frac{y}{\epsilon}) \bar{u}_j(x-y) dy = \int_{B(0,1)} \eta(z) \bar{u}_j(x - \epsilon z) dz\,,$$

and if $\bar{u}_j$ is smooth,

$$\bar{u}_j(x-\epsilon z) - \bar{u}_j(x) = \int_0^1 \frac{d}{dt}\bar{u}_j(x-\epsilon tz)dt = -\epsilon \int_0^1 D\bar{u}_j(x-\epsilon tz)\cdot z\,dt\,.$$

Hence,

$$|\bar{u}_j^\epsilon(x) - \bar{u}_j(x)| = \epsilon \int_{B(0,1)} \eta(z) \int_0^1 |D\bar{u}_j(x-\epsilon tz)|\,dzdt\,,$$

so that

$$\begin{aligned}\int_{\tilde{\Omega}} |\bar{u}_j^\epsilon(x) - \bar{u}_j(x)|dx &= \epsilon \int_{B(0,1)} \eta(z) \int_0^1 \int_{\tilde{\Omega}} |D\bar{u}_j(x-\epsilon tz)|\,dxdzdt \\ &\leq \epsilon \|D\bar{u}_j\|_{L^1(\tilde{\Omega})} \leq \epsilon \|D\bar{u}_j\|_{L^p(\tilde{\Omega})} < \epsilon CM\,.\end{aligned}$$

Using the L^p-interpolation Lemma 2.50,

$$\begin{aligned}\|\bar{u}_j^\epsilon - \bar{u}_j\|_{L^q(\tilde{\Omega})} &\leq \|\bar{u}_j^\epsilon - \bar{u}_j\|^a_{L^1(\tilde{\Omega})} \|\bar{u}_j^\epsilon - \bar{u}_j\|^{1-a}_{L^{\frac{np}{n-p}}(\tilde{\Omega})} \\ &\leq \epsilon CM\, \|D\bar{u}_j^\epsilon - D\bar{u}_j\|^{1-a}_{L^p(\tilde{\Omega})} \\ &\leq \epsilon CMM^{1-a} \end{aligned} \tag{2.18}$$

The inequality (2.18) shows that $\bar{u}_j^\epsilon$ is arbitrarily close to $\bar{u}_j$ in $L^q(\Omega)$ uniformly in $j \in \mathbb{N}$; as such, we attempt to use the smooth sequence $\bar{u}_j^\epsilon$ to construct a convergent subsequence $\bar{u}_{j_k}^\epsilon$. Our goal is to employ the Arzela-Ascoli Theorem, so we show that for $\epsilon > 0$ fixed,

$$\|\bar{u}_j^\epsilon\|_{C^0(\bar{\tilde{\Omega}})} \leq \tilde{M} < \infty \quad \text{and } \bar{u}_j^\epsilon \text{ is equicontinous.}$$

For $x \in \mathbb{R}^n$,

$$\begin{aligned}\sup_j \|\bar{u}_j^\epsilon\|_{C^0(\bar{\tilde{\Omega}})} &\leq \sup_j \int_{B(x,\epsilon)} \eta_\epsilon(x-y)|\bar{u}_j(y)|dy \\ &\leq \|\eta_\epsilon\|_{L^\infty(\mathbb{R}^n)} \sup_j \|\bar{u}_j\|_{L^1(\tilde{\Omega})} \leq C\epsilon^{-n} < \infty\,,\end{aligned}$$

and similarly

$$\sup_j \|\bar{D}u_j^\epsilon\|_{C^0(\bar{\tilde{\Omega}})} \leq \|D\eta_\epsilon\|_{L^\infty(\mathbb{R}^n)} \sup_j \|\bar{u}_j\|_{L^1(\tilde{\Omega})} \leq C\epsilon^{-n-1} < \infty\,.$$

The latter inequality proves equicontinuity of the sequence $\bar{u}_j^\epsilon$, and hence there exists a subsequence u_{j_k} which converges uniformly on $\tilde{\Omega}$, so that

$$\limsup_{k,l\to\infty} \|\bar{u}_{j_k}^\epsilon - \bar{u}_{j_l}^\epsilon\|_{L^q(\tilde{\Omega})} = 0\,.$$

It follows from (2.18) and the triangle inequality that

$$\limsup_{k,l\to\infty} \|\bar{u}_{j_k} - \bar{u}_{j_l}\|_{L^q(\tilde{\Omega})} \leq C\epsilon\,.$$

Letting $C\epsilon = 1, \frac{1}{2}, \frac{1}{3}$, etc., and using the diagonal argument to extract further subsequences, we can arrange to find a subsequence again denoted by $\{\bar{u}_{j_k}\}$ of $\{\bar{u}_j\}$ such that

$$\limsup_{k,l\to\infty} \|\bar{u}_{j_k} - \bar{u}_{j_l}\|_{L^q(\tilde{\Omega})} = 0\,,$$

and hence

$$\limsup_{k,l\to\infty} \|u_{j_k} - u_{j_l}\|_{L^q(\Omega)} = 0\,,$$

The case that $n = p = 2$ follows from Theorem 2.30. □

2.13 Exercises

Problem 2.1. *Suppose that $1 < p < \infty$. If $\tau_y f(x) = f(x-y)$, show that f belongs to $W^{1,p}(\mathbb{R}^n)$ if and only if $\tau_y f$ is a Lipschitz function of y with values in $L^p(\mathbb{R}^n)$, i.e.*

$$\|\tau_y f - \tau_z f\|_{L^p(\mathbb{R}^n)} \leq C|y-z|\,.$$

What happens in the case $p = 1$?

Problem 2.2. *If for $j = 1, 2$ and $p_j \in [1, \infty]$ and $u_j \in L^{p_j}$, show that $u_1 u_2 \in L^r$ provided that $1/r = 1/p_1 + 1/p_2$ and*

$$\|u_1\, u_2\|_{L^r} \leq \|u_1\|_{L^{p_1}} \|u_2\|_{L^{p_2}}\,.$$

Show that this implies that the generalized Hölder's inequality, which states that if for $j = 1, ..., m$ and $p_j \in [1, \infty]$ with $\sum_{j=1}^m \frac{1}{p_j} = 1$, then

$$\int_{\mathbb{R}^n} |u_1 \cdots u_m|\, dx \leq \|u_1\|_{L^{p_1}} \cdots \|u_m\|_{L^{p_m}}\,.$$

Problem 2.3. *Let $f \in L^1(\mathbb{R})$, and set*

$$g(x) = \int_{-\infty}^{x} f(y) dy\,. \tag{*}$$

Continuity of g follows from the Dominated Convergence Theorem. Show that $\partial_1 g = f$. (Hint. Given $\phi \in C_0^\infty(\mathbb{R})$, use (*) to obtain

$$\int_{\mathbb{R}} \phi'(x) g(x) dx = \int_{\mathbb{R}} \int_{-\infty}^{x} \phi'(x) f(y) dy dx \,.$$

Then write this integral as

$$\lim_{h\to 0} \frac{1}{h} \int_{\mathbb{R}} [\phi(x+h) - \phi(x)]\, g(x) dx = -\lim_{h\to 0} \frac{1}{h} \int_{\mathbb{R}} \int_{x}^{x+h} f(y)\phi(x)\, dy dx \,.)$$

Problem 2.4. *Show that $W^{n,1}(\mathbb{R}^n) \subset C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$.*
(Hint. $u(x) = \int_{-\infty}^0 \cdots \int_{-\infty}^0 \partial_1 \cdots \partial_n u(x+y) dy_1 \cdots dy_n$.)

Problem 2.5. *If $u \in W^{1,p}(\mathbb{R}^n)$ for some $p \in [1,\infty)$ and $\partial_j u = 0$ on a connected open set $\Omega \subset \mathbb{R}^n$, for $1 \leq j \leq n$, show that u is equal a.e. to a constant on Ω.*
(Hint. Approximate u using that

$$\phi_i * u \to u \text{ in } W^{1,p}(\mathbb{R}^n) \,,$$

where ϕ_i is a sequence of standard mollifiers. As we showed, given $\epsilon > 0$, we can choose i such that

$$\|\phi_i * u - u\|_{W^{1,p}(\mathbb{R}^n)} < \epsilon \,.$$

Show that $\partial_j(\phi_i * u) = 0$ on $\Omega_i \subset\subset \Omega$, where $\Omega_i \nearrow \Omega$ as $i \to \infty$.)

More generally, if $\partial_j u = f_j \in C(\Omega)$, $1 \leq j \leq n$, show that u is equal a.e. to a function in $C^1(\Omega)$.

Problem 2.6. *In case $n = 1$, deduce from Problems 2.3 and 2.5 that, if $u \in L^1_{loc}(\mathbb{R})$ and if $\partial_1 u = f \in L^1(\mathbb{R})$, then*

$$u(x) = c + \int_{-\infty}^{x} f(y) dy \,, \qquad a.e.\ x \in \mathbb{R} \,,$$

for some constant c.

Problem 2.7. *Let $\Omega := B(0, \frac{1}{2}) \subset \mathbb{R}^2$ denote the open ball of radius $\frac{1}{2}$. For $x = (x_1, x_2) \in \Omega$, let*

$$u(x_1, x_2) = x_1 x_2 \log\left(|\log(|x|)|\right) \quad \text{where } |x| = \sqrt{x_1^2 + x_2^2}\,.$$

(a) Show that $u \in C^1(\bar{\Omega})$;

(b) show that $\frac{\partial^2 u}{\partial x_j^2} \in C(\bar{\Omega})$ for $j = 1, 2$, but that $u \notin C^2(\bar{\Omega})$;

(c) show that $u \in H^2(\Omega)$.

Problem 2.8. *Theorem 2.24 states that for $p > n$, and $y \in B(y, r)$,*

$$|u(x) - u(y)| \leq C r^{1-\frac{n}{p}} \|Du\|_{L^p(\mathbb{R}^n)} \quad \forall u \in C^1(\mathbb{R}^n)\,. \tag{2.19}$$

Prove that the inequality (2.19) in fact holds for all $u \in W^{1,p}(\mathbb{R}^n)$; in particular, show that Du can be taken to be the weak derivative of u.

Problem 2.9. *Let η_ϵ denote the standard mollifier, and for $u \in H^2(\mathbb{R}^3)$, set $u^\epsilon = \eta_\epsilon * u$. Prove that*

$$\|u^\epsilon - u\|_{L^\infty(\mathbb{R}^3)} \leq C\sqrt{\epsilon}\|u\|_{H^2(\mathbb{R}^3)}\,,$$

and that

$$\|u^\epsilon - u\|_{L^\infty(\mathbb{R}^3)} \leq C\epsilon\|u\|_{H^3(\mathbb{R}^3)}\,.$$

Problem 2.10. *Suppose that for $n \geq 2$, $\Omega \subset \mathbb{R}^n$ is a smooth, open, and bounded domain, and let n denote the outward-pointing unit normal vector to the boundary $\partial\Omega$. Suppose that $u \in L^2(\Omega)$ and $\operatorname{div} u \in L^2(\Omega)$. Prove that $u \cdot n \in H^{-1/2}(\partial\Omega)$ and that*

$$\|u \cdot n\|_{H^{-1/2}(\partial\Omega)} \leq C\left(\|u\|_{L^2(\Omega)} + \|\operatorname{div} u\|_{L^2(\Omega)}\right)\,.$$

Problem 2.11. *Let $\Omega \subset \mathbb{R}^2$ denote an open, bounded, subset with smooth boundary. Prove the interpolation inequality:*

$$\|Du\|^2_{L^2(\Omega)} \leq C\|u\|_{L^2(\Omega)}\|D^2u\|_{L^2(\Omega)} \quad \forall\ u \in H^2(\Omega) \cap H^1_0(\Omega)\,,$$

where D^2u denotes the Hessian matrix of u, i.e., the matrix of second partial derivatives $\frac{\partial^2 u}{\partial x_i \partial x_j}$. Use the fact that $C^\infty(\overline{\Omega}) \cap H^1_0(\Omega)$ is dense in $u \in H^2(\Omega) \cap H^1_0(\Omega)$.

Problem 2.12. *Let $D := B(0,1) \subset \mathbb{R}^2$ denote the unit disc, and let*

$$u(x) = \Big[-\log|x| \Big]^\alpha .$$

Prove that the weak derivative of u exists for all $\alpha \geq 0$.

Problem 2.13. *Suppose that $\{f_n\}_{n=1}^\infty$ is a bounded sequence in $H^1(\Omega)$ for $\Omega \subset \mathbb{R}^2$ bounded. Show that there exists an $f \in H^1(\Omega)$ such that for $1 < p < 2$,*

$$f_{n_l}\, Df_{n_l} \rightharpoonup f\, Df \quad \text{weakly in} \quad L^p(\Omega)\,.$$

Problem 2.14. *Suppose that $u_j \rightharpoonup u$ in $W^{1,1}(0,1)$. Show that $u_j \to u$ a.e.*

3 The Fourier Transform

The Fourier transform is one of the most powerful and fundamental tools in linear analysis, converting constant-coefficient linear differential operators into multiplication by polynomials. In this section, we define the Fourier transform, first on $L^1(\mathbb{R}^n)$ functions, next (and miraculously) on $L^2(\mathbb{R}^n)$ functions, and finally on the space of tempered distributions.

3.1 Fourier transform on $L^1(\mathbb{R}^n)$ and the space $\mathcal{S}(\mathbb{R}^n)$

Definition 3.1. *For all $f \in L^1(\mathbb{R}^n)$ the Fourier transform $\mathcal{F}$ is defined by*

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi}dx\,.$$

By Hölder's inequality, $\mathcal{F} : L^1(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)$.

Definition 3.2. *The space of Schwartz functions of rapid decay is denoted by*

$$\mathcal{S}(\mathbb{R}^n) = \{u \in C^\infty(\mathbb{R}^n) \mid x^\beta D^\alpha u \in L^\infty(\mathbb{R}^n) \quad \forall \alpha, \beta \in \mathbb{Z}^n_+\}.$$

It is not difficult to show (as it follows from the definition) that

$$\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)\,,$$

and that

$$\xi^\alpha D^\beta_\xi \hat{f} = (-i)^{|\alpha|}(-1)^{|\beta|}\mathcal{F}(D^\alpha_x x^\beta f)\,.$$

The Schwartz space $\mathcal{S}(\mathbb{R}^n)$ is also known as the space of rapidly decreasing functions; thus, after multiplying by any polynomial functions $P(x)$, $P(x)D^\alpha u(x) \to 0$ as $x \to \infty$ for all $\alpha \in \mathbb{Z}^n_+$. The classical space of test functions $\mathcal{D}(\mathbb{R}^n) := C^\infty_0(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$. The prototype element of $\mathcal{S}(\mathbb{R}^n)$ is $e^{-|x|^2}$ which is not compactly supported, but has rapidly decreasing derivatives.

The reader is encouraged to verify the following basic properties of $\mathcal{S}(\mathbb{R}^n)$ which we will denote by $\mathcal{S}$:

1. $\mathcal{S}$ is a vector space.
2. $\mathcal{S}$ is an algebra under the pointwise product of functions.
3. $P(x)u(x) \in \mathcal{S}$ for all $u \in \mathcal{S}$ and all polynomial functions $P(x)$.
4. $\mathcal{S}$ is closed under differentiation.
5. $\mathcal{S}$ is closed under translations and multiplication by complex exponentials $e^{ix\cdot\xi}$.

6. $\mathcal{S} \subset L^1(\mathbb{R}^n)$ (since $|u(x)| \le C(1+|x|)^{n+1}$ for all $u \in \mathcal{S}$ and $(1+|x|)^{-(n+1)}dx$ decays like $|x|^{-2}$ as $|x| \to \infty$).

Definition 3.3. *For all $f \in L^1(\mathbb{R}^n)$, we define operator $\mathcal{F}^*$ by*

$$\mathcal{F}^* f(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f(\xi) e^{ix\cdot\xi} d\xi \,.$$

Lemma 3.4. *For all $u, v \in \mathcal{S}(\mathbb{R}^n)$,*

$$(\mathcal{F}u, v)_{L^2(\mathbb{R}^n)} = (u, \mathcal{F}^* v)_{L^2(\mathbb{R}^n)} \,.$$

Recall that the $L^2(\mathbb{R}^n)$ inner-product for complex-valued functions is given by $(u,v)_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} u(x)\overline{v(x)}dx$.

Proof. Since $u, v \in \mathcal{S}(\mathbb{R}^n)$, by Fubini's Theorem,

$$\begin{aligned}
(\mathcal{F}u, v)_{L^2(\mathbb{R}^n)} &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} u(x) e^{-ix\cdot\xi} dx\, \overline{v(\xi)}\, d\xi \\
&= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} u(x) \overline{e^{ix\cdot\xi} v(\xi)}\, d\xi\, dx \\
&= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} u(x) \int_{\mathbb{R}^n} \overline{e^{ix\cdot\xi} v(\xi)}\, d\xi\, dx = (u, \mathcal{F}^* v)_{L^2(\mathbb{R}^n)} \,,
\end{aligned}$$

□

Theorem 3.5. *$\mathcal{F}^* \circ \mathcal{F} = \mathrm{Id} = \mathcal{F} \circ \mathcal{F}^*$ on $\mathcal{S}(\mathbb{R}^n)$.*

Proof. We first prove that for all $f \in \mathcal{S}(\mathbb{R}^n)$, $\mathcal{F}^*\mathcal{F} f(x) = f(x)$.

$$\begin{aligned}
\mathcal{F}^*\mathcal{F} f(x) &= (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\xi\cdot x} \left(\int_{\mathbb{R}^n} e^{-iy\cdot\xi} f(y) dy \right) d\xi \\
&= (2\pi)^{-n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} f(y)\, dy\, d\xi \,.
\end{aligned}$$

By the dominated convergence theorem,

$$\mathcal{F}^*\mathcal{F} f(x) = \lim_{\epsilon\to 0} (2\pi)^{-n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-\epsilon|\xi|^2} e^{i(x-y)\cdot\xi} f(y)\, dy\, d\xi \,.$$

For all $\epsilon > 0$, the *convergence factor* $e^{-\epsilon|\xi|^2}$ allows us to interchange the order of integration, so that by Fubini's theorem,

$$\mathcal{F}^*\mathcal{F} f(x) = \lim_{\epsilon\to 0} (2\pi)^{-n} \int_{\mathbb{R}^n} f(y) \left(\int_{\mathbb{R}^n} e^{-\epsilon|\xi|^2} e^{i(y-x)\cdot\xi}\, d\xi \right) dy \,.$$

Define the integral kernel

$$p_\epsilon(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-\epsilon|\xi|^2 + ix\cdot\xi} d\xi$$

Then

$$\mathcal{F}^*\mathcal{F}f(x) = \lim_{\epsilon\to 0} p_\epsilon * f := \int_{\mathbb{R}^n} p_\epsilon(x-y)f(y)dy\,.$$

Let $p(x) = p_1(x) = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{-|\xi|^2+ix\cdot\xi}d\xi$. Then

$$\begin{aligned} p(x/\sqrt{\epsilon}) &= (2\pi)^{-n}\int_{\mathbb{R}^n} e^{-|\xi|^2 + ix\cdot\xi/\sqrt{\epsilon}} d\xi \\ &= (2\pi)^{-n}\int_{\mathbb{R}^n} e^{-|\xi|^2+ix\cdot\xi}\epsilon^{\frac{n}{2}} d\xi = \epsilon^{\frac{n}{2}} p_\epsilon(x)\,. \end{aligned}$$

We claim that

$$p_\epsilon(x) = \frac{1}{(4\pi\epsilon)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4\epsilon}} \quad \text{and that} \quad \int_{\mathbb{R}^n} p(x)dx = 1\,. \tag{3.1}$$

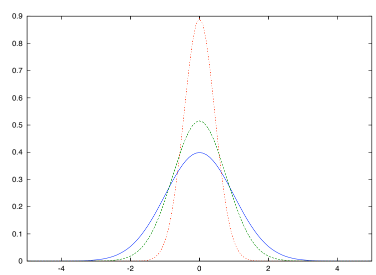


Figure 1: As $\epsilon \to 0$, the sequence of functions p_ϵ becomes more localized about the origin.

Given (3.1), then for all $f \in \mathcal{S}(\mathbb{R}^n)$, $p_\epsilon * f \to f$ uniformly as $\epsilon \to 0$, which shows that $\mathcal{F}^*\mathcal{F} = \text{Id}$, and similar argument shows that $\mathcal{F}\mathcal{F}^* = \text{Id}$. (Note that this follows from the proof of Theorem 1.28, since the standard mollifiers η_ϵ can be replaced by the sequence p_ϵ and all assertions of the theorem continue to hold, for if (3.1) is true, then even though p_ϵ does not have compact support, $\int_{B(0,\delta)^c} p_\epsilon(x)dx \to 0$ as $\epsilon \to 0$ for all $\delta > 0$.)

Thus, it remains to prove (3.1). It suffices to consider the case $\epsilon = \frac{1}{2}$; then by definition

$$\begin{aligned} p_{\frac{1}{2}}(x) &= (2\pi)^{-n}\int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{-\frac{|\xi|^2}{2}} d\xi \\ &= \mathcal{F}\left((2\pi)^{-n/2} e^{-\frac{|\xi|^2}{2}}\right)\,. \end{aligned}$$

In order to prove that $p_{\frac{1}{2}}(x) = (2\pi)^{-n/2} e^{-\frac{|x|^2}{2}}$, we must show that with the Gaussian function $G(x) = (2\pi)^{-n/2} e^{-\frac{|x|^2}{2}}$,

$$G(x) = \mathcal{F}(G(\xi)) .$$

By the multiplicative property of the exponential,

$$e^{-|\xi|^2/2} = e^{-\xi_1^2/2} \cdots e^{-\xi_n^2/2} ,$$

it suffices to consider the case that $n = 1$. Then the Gaussian satisfies the differential equation

$$\frac{d}{dx} G(x) + xG(x) = 0 .$$

Computing the Fourier transform, we see that

$$-i \frac{d}{d\xi} \hat{G}(x) - i\xi \hat{G}(x) = 0 .$$

Thus,

$$\hat{G}(\xi) = C e^{-\frac{\xi^2}{2}} .$$

To compute the constant C,

$$C = \hat{G}(0) = (2\pi)^{-1} \int_{\mathbb{R}} e^{\frac{x^2}{2}} dx = (2\pi)^{-\frac{1}{2}}$$

which follows from the fact that

$$\int_{\mathbb{R}} e^{\frac{x^2}{2}} dx = (2\pi)^{\frac{1}{2}} . \tag{3.2}$$

To prove (3.2), one can again rely on the multiplication property of the exponential to observe that

$$\begin{aligned} \int_{\mathbb{R}} e^{\frac{x_1^2}{2}} dx \int_{\mathbb{R}} e^{\frac{x_2^2}{2}} dx &= \int_{\mathbb{R}^2} e^{\frac{x_1^2 + x_2^2}{2}} dx \\ &= \int_0^{2\pi} \int_0^{\infty} e^{-2r^2} r dr d\theta = 2\pi . \end{aligned}$$

□

It follows from Lemma 3.4 that for all $u, v \in \mathcal{S}(\mathbb{R}^n)$,

$$(\mathcal{F}u, \mathcal{F}v)_{L^2(\mathbb{R}^n)} = (u, \mathcal{F}^* \mathcal{F} v)_{L^2(\mathbb{R}^n)} = (u, v)_{L^2(\mathbb{R}^n)} .$$

Thus, we have established the *Plancheral theorem* on $\mathcal{S}(\mathbb{R}^n)$.

Theorem 3.6 (Plancheral's theorem). $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ *is an isomorphism with inverse* $\mathcal{F}^*$ *preserving the* $L^2(\mathbb{R}^n)$ *inner-product.*

3.2 The topology on $\mathcal{S}(\mathbb{R}^n)$ and tempered distributions

An alternative to Definition 3.2 can be stated as follows:

Definition 3.7 (The space $\mathcal{S}(\mathbb{R}^n)$). *Setting $\langle x\rangle = \sqrt{1+|x|^2}$,*

$$\mathcal{S}(\mathbb{R}^n) = \{u \in C^\infty(\mathbb{R}^n) \mid \langle x\rangle^k |D^\alpha u| \le C_{k,\alpha} \quad \forall k \in \mathbb{Z}_+\}\,.$$

The space $\mathcal{S}(\mathbb{R}^n)$ has a Fréchet topology determined by seminorms.

Definition 3.8 (Topology on $\mathcal{S}(\mathbb{R}^n)$). *For $k \in \mathbb{Z}_+$, define the semi-norm*

$$p_k(u) = \sup_{x\in\mathbb{R}^n, |\alpha|\le k} \langle x\rangle^k |D^\alpha u(x)|\,,$$

and the metric on $\mathcal{S}(\mathbb{R}^n)$

$$d(u,v) = \sum_{k=0}^\infty 2^{-k} \frac{p_k(u-v)}{1+p_k(u-v)}\,.$$

The space $(\mathcal{S}(\mathbb{R}^n), d)$ is a Fréchet space.

Definition 3.9 (Convergence in $\mathcal{S}(\mathbb{R}^n)$). *A sequence $u_j \to u$ in $\mathcal{S}(\mathbb{R}^n)$ if $p_k(u_j - u) \to 0$ as $j \to \infty$ for all $k \in \mathbb{Z}_+$.*

Definition 3.10 (Tempered Distributions). *A linear map $T : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$ is continuous if there exists some $k \in \mathbb{Z}_+$ and constant C such that*

$$|\langle T, u\rangle| \le C p_k(u) \quad \forall u \in \mathcal{S}(\mathbb{R}^n)\,.$$

The space of continuous linear functionals on $\mathcal{S}(\mathbb{R}^n)$ is denoted by $\mathcal{S}'(\mathbb{R}^n)$. Elements of $\mathcal{S}'(\mathbb{R}^n)$ are called tempered distributions.

Definition 3.11 (Convergence in $\mathcal{S}'(\mathbb{R}^n)$). *A sequence $T_j \rightharpoonup T$ in $\mathcal{S}'(\mathbb{R}^n)$ if $\langle T_j, u\rangle \to \langle T, u\rangle$ for all $u \in \mathcal{S}(\mathbb{R}^n)$.*

For $1 \le p \le \infty$, there is a natural injection of $L^p(\mathbb{R}^n)$ into $\mathcal{S}'(\mathbb{R}^n)$ given by

$$\langle f, u\rangle = \int_{\mathbb{R}^n} f(x)u(x)dx \quad \forall u \in \mathcal{S}(\mathbb{R}^n)\,.$$

Any finite measure on $\mathbb{R}^n$ provides an element of $\mathcal{S}'(\mathbb{R}^n)$. The basic example of such a finite measure is the Dirac delta 'function' defined as follows:

$$\langle \delta, u\rangle = u(0) \ \text{ or, more generally, } \ \langle \delta_x, u\rangle = u(x) \quad \forall u \in \mathcal{S}(\mathbb{R}^n)\,.$$

Definition 3.12. *The distributional derivative* $D : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ *is defined by the relation*

$$\langle DT, u\rangle = -\langle T, Du\rangle \quad \forall u \in \mathcal{S}(\mathbb{R}^n)\,.$$

More generally, the α*th distributional derivative exists in* $\mathcal{S}'(\mathbb{R}^n)$ *and is defined by*

$$\langle D^\alpha T, u\rangle = (-1)^{|\alpha|}\langle T, D^\alpha u\rangle \quad \forall u \in \mathcal{S}(\mathbb{R}^n)\,.$$

Multiplication by $f \in \mathcal{S}(\mathbb{R}^n)$ preserves $\mathcal{S}'(\mathbb{R}^n)$; in particular, if $T \in \mathcal{S}'(\mathbb{R}^n)$, then $fT \in \mathcal{S}'(\mathbb{R}^n)$ and is defined by

$$\langle fT, u\rangle = \langle T, fu\rangle \quad \forall u \in \mathcal{S}(\mathbb{R}^n)\,.$$

Example 3.13. *Let* $H := \mathbf{1}_{[0,\infty)}$ *denote the Heavyside function. Then*

$$\frac{dH}{dx} = \delta \quad in \quad \mathcal{S}'(\mathbb{R}^n)\,.$$

This follows since for all $u \in \mathcal{S}(\mathbb{R}^n)$*,*

$$\langle \frac{dH}{dx}, u\rangle = -\langle H, \frac{du}{dx}\rangle = -\int_0^\infty \frac{du}{dx}dx = u(0) = \langle \delta, u\rangle\,.$$

Example 3.14 (Distributional derivative of Dirac measure)**.**

$$\langle \frac{d\delta}{dx}, u\rangle = -\frac{du}{dx}(0) \quad \forall u \in \mathcal{S}(\mathbb{R}^n)\,.$$

3.3 Fourier transform on $\mathcal{S}'(\mathbb{R}^n)$

Definition 3.15. *Define* $\mathcal{F} : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ *by*

$$\langle \mathcal{F}T, u\rangle = \langle T, \mathcal{F}u\rangle \quad \forall u \in \mathcal{S}(\mathbb{R}^n)\,,$$

with the analogous definition for $\mathcal{F}^* : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$.

Theorem 3.16. $\mathcal{F}\mathcal{F}^* = \mathrm{Id} = \mathcal{F}^*\mathcal{F}$ *on* $\mathcal{S}'(\mathbb{R}^n)$.

Proof. By Definition 3.15, for all $u \in \mathcal{S}(\mathbb{R}^n)$

$$\langle \mathcal{F}\mathcal{F}^*T, u\rangle = \langle \mathcal{F}^* w, \mathcal{F}u\rangle = \langle T, \mathcal{F}^*\mathcal{F}u\rangle = \langle T, u\rangle\,,$$

the last equality following from Theorem 3.5. □

Example 3.17 (Fourier transform of δ)**.** *We claim that $\mathcal{F}\delta = (2\pi)^{-\frac{n}{2}}$. According to Definition 3.15, for all $u \in \mathcal{S}(\mathbb{R}^n)$,*

$$\langle \mathcal{F}\delta, u\rangle = \langle \delta, \mathcal{F}u\rangle = \mathcal{F}u(0) = \int_{\mathbb{R}^n} (2\pi)^{-\frac{n}{2}}\, u(x)dx\,,$$

so that $\mathcal{F}\delta = (2\pi)^{-\frac{n}{2}}$.

Example 3.18. *The same argument shows that $\mathcal{F}^*\delta = (2\pi)^{-\frac{n}{2}}$ so that $\mathcal{F}^*[(2\pi)^{\frac{n}{2}}] = 1$. Using Theorem 3.16, we see that $\mathcal{F}(1) = (2\pi)^{-\frac{n}{2}}\delta$. This demonstrates nicely the identity*

$$|\xi^\alpha \hat{u}(\xi)| = |\mathcal{F}(D^\alpha u)(\xi)|.$$

In other the words, the smoother the function $x \mapsto u(x)$ is, the faster $\xi \mapsto \hat{u}(\xi)$ must decay.

3.4 The Fourier transform on $L^2(\mathbb{R}^n)$

In Theorem 1.28, we proved that $C_0^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$ for $1 \le p < \infty$. Since $C_0^\infty(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$, it follows that $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$ as well. Thus, for every $u \in L^2(\mathbb{R}^n)$, there exists a sequence $u_j \in \mathcal{S}(\mathbb{R}^n)$ such that $u_j \to u$ in $L^2(\mathbb{R}^n)$, so that by Plancheral's Theorem 3.6,

$$\|\hat{u}_j - \hat{u}_k\|_{L^2(\mathbb{R}^n)} = \|u_j - u_k\|_{L^2(\mathbb{R}^n)} < \epsilon\,.$$

It follows from the completeness of $L^2(\mathbb{R}^n)$ that the sequence $\hat{u}_j$ *converges* in $L^2(\mathbb{R}^n)$.

Definition 3.19 (Fourier transform on $L^2(\mathbb{R}^n)$)**.** *For $u \in L^2(\mathbb{R}^n)$ let u_j denote an approximating sequence in $\mathcal{S}(\mathbb{R}^n)$. Define the Fourier transform as follows:*

$$\mathcal{F}u = \hat{u} = \lim_{j\to\infty} \hat{u}_j\,.$$

Note well that $\mathcal{F}$ on $L^2(\mathbb{R}^n)$ is well-defined, as the limit is independent of the approximating sequence. In particular,

$$\|\hat{u}\|_{L^2(\mathbb{R}^n)} = \lim_{j\to\infty} \|\hat{u}_j\|_{L^2(\mathbb{R}^n)} = \lim_{j\to\infty} \|u_j\|_{L^2(\mathbb{R}^n)} = \|u\|_{L^2(\mathbb{R}^n)}\,.$$

By the polarization identity

$$(u,v)_{L^2(\mathbb{R}^n)} = \frac{1}{2}\left(\|u+v\|^2_{L^2(\mathbb{R}^n)} - i\|u+iv\|^2_{L^2(\mathbb{R}^n)} - (1-i)\|u\|^2_{L^2(\mathbb{R}^n)} - (1-i)\|v\|^2_{L^2(\mathbb{R}^n)}\right)$$

we have proved the Plancheral theorem[1] on $L^2(\mathbb{R}^n)$:

Theorem 3.20. $(u,v)_{L^2(\mathbb{R}^n)} = (\mathcal{F}u, \mathcal{F}v)_{L^2(\mathbb{R}^n)} \quad \forall u,v \in L^2(\mathbb{R}^n)\,.$

[1] The unitarity of the Fourier transform is often called Parseval's theorem in science and engineering fields, based on an earlier (but less general) result that was used to prove the unitarity of the Fourier series.

3.5 Bounds for the Fourier transform on $L^p(\mathbb{R}^n)$

We have shown that for $u \in L^1(\mathbb{R}^n)$, $\|\hat{u}\|_{L^\infty(\mathbb{R}^n)} \le (2\pi)^{-\frac{n}{2}}\|u\|_{L^1(\mathbb{R}^n)}$, and that for $u \in L^2(\mathbb{R}^n)$, $\|\hat{u}\|_{L^2(\mathbb{R}^n)} = \|u\|_{L^2(\mathbb{R}^n)}$. Interpolating p between 1 and 2 yields the following result.

Theorem 3.21 (Hausdorff-Young inequality)**.** *If $u \in L^p(\mathbb{R}^n)$ for $1 \le p \le 2$, then for $q = \frac{p-1}{p}$, there exists a constant C such that*

$$\|\hat{u}\|_{L^q(\mathbb{R}^n)} \le C\|u\|_{L^p(\mathbb{R}^n)}\,.$$

Returning to the case that $u \in L^1(\mathbb{R}^n)$, not only is $\mathcal{F}u \in L^\infty(\mathbb{R}^n)$, but the transformed function decays at infinity.

Theorem 3.22 (Riemann-Lebesgue "lemma")**.** *For $u \in L^1(\mathbb{R}^n)$, $\mathcal{F}u$ is continuous and $\mathcal{F}u(\xi) \to 0$ as $|\xi| \to \infty$.*

Proof. Let $B_M = B(0,M) \subset \mathbb{R}^n$. Since $f \in L^1(\mathbb{R}^n)$, for each $\epsilon > 0$, we can choose M sufficiently large such that $\hat{f}(\xi) \le \epsilon + \int_{B_M} e^{-ix\cdot\xi}|f(x)|dx$. Using Lemma 1.23, choose a sequence of simple functions $\phi_j(x) \to f(x)$ a.e. on B_M. For $jn\mathbb{N}$ chosen sufficiently large,

$$\hat{f}(\xi) \le 2\epsilon + \int_{B_M} \phi_j(x)e^{-ix\cdot\xi}dx\,.$$

Write $\phi_j(x) = \sum_{l=1}^N C_l \mathbf{1}_{E_l}(x)$ so that

$$\hat{f}(\xi) \le 2\epsilon + \sum_{l=1}^N C_l \int_{E_l} \phi_j(x)e^{-ix\cdot\xi}dx\,.$$

By the regularity of the Lebesgue measure μ, for all $\epsilon > 0$ and each $l \in \{1,...,N\}$, there exists a compact set K_l and an open set O_l such that

$$\mu(O_l) - \epsilon/2 < \mu(E_l) < \mu(K_l) + \epsilon/2\,.$$

Then $O_l = \{\cup_{\alpha\in A_l} V_\alpha^l \mid V_l^\alpha \subset \mathbb{R}^n \text{ is open rectangle }, A_l \text{ arbitrary set }\}$, and $K_l \subset \cup_{j=1}^{N_l} V_j^l \subset O_l$ where $\{1,...,N_l\} \subset A_l$ such that

$$|\mu(E_l) - \mu(\cup_{j=1}^{N_l} V_j^l)| < \epsilon\,.$$

It follows that

$$\left| \int_{E_l} e^{-ix\cdot\xi}dx - \int_{\cup_{j=1}^{N_l} V_j^l} e^{-ix\cdot\xi}dx \right| < \epsilon\,.$$

On the other hand, for each rectangle V_j^l, $\int_{V_j^l} e^{-ix\cdot\xi}dx| \le C/(\xi_1\cdots\xi_n)$, so that

$$\hat{f}(\xi) \le C\left(\epsilon + \frac{1}{\xi_1\cdots\xi_n}\right)\,.$$

Since $\epsilon > 0$ is arbitrary, we see that $\hat{f}(\xi) \to 0$ as $|\xi| \to \infty$. Continuity of $\mathcal{F}u$ follows easily from the dominated convergence theorem. □

3.6 Convolution and the Fourier transform

Theorem 3.23. *If $u, v \in L^1(\mathbb{R}^n)$, then $u * v \in L^1(\mathbb{R}^n)$ and*

$$\mathcal{F}(u * v) = (2\pi)^{\frac{n}{2}} \mathcal{F}u\, \mathcal{F}v\,.$$

Proof. Young's inequality (Theorem 1.53) shows that $u * v \in L^1(\mathbb{R}^n)$ so that the Fourier transform is well-defined. The assertion then follows from a direct computation:

$$\begin{aligned}
\mathcal{F}(u * v) &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi}(u * v)(x) dx \\
&= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} u(x-y)v(y) dy\, e^{-ix\cdot\xi}\, dx \\
&= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} u(x-y) e^{-i(x-y)\cdot\xi}\, dx\, v(x)\, e^{-iy\cdot\xi}\, dy \\
&= (2\pi)^{\frac{n}{2}} \hat{u}\hat{v} \quad \text{(by Fubini's theorem)}\,.
\end{aligned}$$

□

By using Young's inequality (Theorem 1.54) together with the Hausdorff-Young inequality, we can generalize the convolution result to the following

Theorem 3.24. *Suppose that $u \in L^p(\mathbb{R}^n)$ and $v \in L^q(\mathbb{R}^n)$, and let r satisfy $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$ for $1 \le p, q, r \le 2$. Then $\mathcal{F}(u * v) \in L^{\frac{r}{r-1}}(\mathbb{R}^n)$ and*

$$\mathcal{F}(u * v) = (2\pi)^{\frac{n}{2}} \mathcal{F}u\, \mathcal{F}v\,.$$

3.7 An explicit computation with the Fourier Transform

The computation of the Green's function for the Laplace operator is an important application of the Fourier transform. For this purpose, we will compute $\hat{f}$ for the following two cases: (1) $f(x) = e^{-t|x|}$, $t > 0$ and (2) $f(x) = |x|^\alpha$, $-n < \alpha < 0$.

Case (1) In this case, f is rapidly decreasing but not in the Schwartz class $\mathcal{S}(\mathbb{R}^n)$. We begin with $n = 1$. It follows that

$$\begin{aligned}
[\mathcal{F}(e^{-t|x|})](\xi) &= \int_{-\infty}^{\infty} e^{-t|x|} e^{-ix\cdot\xi} d\mu_1(x) = \int_{-\infty}^{0} e^{x(t-i\xi)} d\mu_1(x) + \int_0^{\infty} e^{x(-t-i\xi)} d\mu_1(x) \\
&= \frac{1}{\sqrt{2\pi}} \Big[\frac{e^{x(t-i\xi)}}{t-i\xi}\Big|_{-\infty}^{0} + \frac{e^{x(-t-i\xi)}}{-t-i\xi}\Big|_0^{\infty} \Big] = \sqrt{\frac{2}{\pi}} \frac{t}{t^2+\xi^2}\,.
\end{aligned}$$

By the inversion formula, we then see that $e^{-t|x|} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{t}{t^2+\xi^2}\ e^{ix\xi}d\xi$. Next, when $n>1$ we will show that for some function $g(t,s)$,

$$e^{-t|x|} = \int_0^\infty g(t,s)e^{-s|x|^2}ds\,. \tag{3.3}$$

In order to determine $g(t,s)$, we suppose that (3.3) holds, and compute its Fourier transform:

$$\mathcal{F}(e^{-t|x|}) = \int_0^\infty g(t,s)\mathcal{F}(e^{-s|x|^2})ds = \int_0^\infty g(t,s)\Big(\frac{1}{\sqrt{2s}}\Big)^n e^{\frac{-|\xi|^2}{4s}}ds\,,$$

where we have used the definition of the Fourier transform of the Gaussian function given in the proof of Theorem 3.5. We are thus seeking a function $g(t,s)$ which satisfies

$$e^{-t\lambda} = \int_0^\infty g(t,s)e^{-s\lambda^2}ds, \quad \forall\ \lambda>0\,.$$

We begin by computing

$$\int_0^\infty e^{-st^2}e^{-s\xi^2}ds = \frac{e^{-s(t^2+\xi^2)}}{-(t^2+\xi^2)}\Big|_0^\infty = \frac{1}{t^2+\xi^2}\,. \tag{3.4}$$

With $\lambda=|x|>0$, we use (3.4) to find that

$$\begin{aligned} e^{-t\lambda} &= \frac{1}{\pi}\int_{-\infty}^\infty \frac{t}{t^2+\xi^2}\ e^{i\lambda\xi}d\xi = \frac{1}{\pi}\int_{-\infty}^\infty t\Big(\int_0^\infty e^{-st^2}e^{-s\xi^2}ds\Big)e^{i\lambda\xi}d\xi \\ &= \frac{1}{\pi}\int_0^\infty t\Big(\int_{-\infty}^\infty e^{-s\xi^2}e^{i\lambda\xi}d\xi\Big)e^{-st^2}ds = \int_0^\infty \frac{t}{\sqrt{s\pi}}e^{-st^2}e^{-\frac{|x|^2}{4s}}ds \end{aligned}$$

so that

$$g(t,s) = \sqrt{2s}^n\frac{t}{\sqrt{s\pi}}e^{-st^2}\,,$$

and hence

$$\begin{aligned} \mathcal{F}(e^{-t|x|})(\xi) &= \int_0^\infty g(t,s)\mathcal{F}(e^{-\frac{|x|^2}{4s}})ds = \int_0^\infty \frac{t}{\sqrt{s\pi}}(2s)^{\frac{n}{2}}e^{-s(t^2+|\xi|^2)}ds \\ &= \frac{t}{(t^2+|\xi|^2)^{\frac{n+1}{2}}}\int_0^\infty \frac{1}{\sqrt{\pi s}}(2s)^{\frac{n}{2}}e^{-s}ds = \frac{C(n)t}{(t^2+|\xi|^2)^{\frac{n+1}{2}}}\,, \end{aligned}$$

where the constant $C(n) = \int_0^\infty \frac{1}{\sqrt{\pi s}}(2s)^{\frac{n}{2}}ds = \sqrt{\frac{2^n}{\pi}}\Gamma(\frac{n+1}{2})$, and Γ is the so-called *gamma-function*. It follows that

$$\mathcal{F}^{-1}(e^{-t|\xi|})(x) = \mathcal{F}(e^{-t|\xi|})(-x) = \sqrt{\frac{2^n}{\pi}}\Gamma(\frac{n+1}{2})\frac{t}{(t^2+|x|^2)^{\frac{n+1}{2}}}\,. \tag{3.5}$$

Case (2) For this case, we compute $\mathcal{F}(|x|^\alpha)$, when $-n < \alpha < 0$. Using the definition of $\Gamma(n)$ above, we see that

$$\int_0^\infty s^{-\frac{\alpha}{2}-1} e^{-s|x|^2} ds = |x|^\alpha \int_0^\infty s^{-\frac{\alpha}{2}-1} e^{-s} ds = |x|^\alpha \Gamma(-\frac{\alpha}{2}),$$

Therefore,

$$\begin{aligned}\mathcal{F}(|x|^\alpha) &= \frac{1}{\Gamma(-\frac{\alpha}{2})} \int_0^\infty s^{-\frac{\alpha}{2}-1} \mathcal{F}(e^{-s|x|^2}) ds = \frac{1}{2^{\frac{n}{2}} \Gamma(-\frac{\alpha}{2})} \int_0^\infty s^{-\frac{\alpha}{2}-\frac{n}{2}-1} e^{-\frac{|\xi|^2}{4s}} ds \\ &= \frac{1}{2^{\frac{n}{2}} \Gamma(-\frac{\alpha}{2})} \int_0^\infty \Big(\frac{|\xi|^2}{4s}\Big)^{-\frac{\alpha}{2}-\frac{n}{2}-1} e^{-s} \frac{|\xi|^2}{4s^2} ds = \frac{2^{\alpha+\frac{n}{2}} \Gamma(\frac{\alpha+n}{2})}{\Gamma(-\frac{\alpha}{2})} |\xi|^{-\alpha-n},\end{aligned}$$

where we impose the condition $-n < \alpha < 0$ to ensure the boundedness of the Γ-function. In particular, for $n = 3$ and $\alpha = -1$,

$$\mathcal{F}(|x|^{-1}) = \frac{\sqrt{2}\Gamma(1)}{\Gamma(\frac{1}{2})} |\xi|^{-2} = \sqrt{\frac{2}{\pi}} |\xi|^{-2},$$

from which it follows that

$$\mathcal{F}^{-1}(|\xi|^{-2}) = \sqrt{\frac{\pi}{2}} \frac{1}{|x|}. \tag{3.6}$$

3.8 Applications to the Poisson, Heat, and Wave equations

3.8.1 The Poisson equation on $\mathbb{R}^3$

In Theorem 2.44, we proved the existence of unique weak solutions to the Dirichlet problem on a bounded domain Ω. We will now provide an explicit representation for solutions to the Poisson problem on $\mathbb{R}^3$. The issue of uniqueness in this setting will be of interest.

Given the Poisson problem

$$\Delta u = f \text{ in } \mathcal{S}',$$

we compute the Fourier transform of both sides to obtain that

$$-|\xi|^2 \hat{u}(\xi) = \hat{f}(\xi). \tag{3.7}$$

Distributional solutions to (3.7) are not unique; for example,

$$\hat{u}(\xi) = -\frac{\hat{f}(\xi)}{|\xi|^2} \text{ and } \hat{u}(\xi) = -\frac{\hat{f}(\xi)}{|\xi|^2} + \delta$$

are both solutions. By requiring solutions to have enough decay, such as $u \in L^2(\mathbb{R}^n)$ so that $\hat{u} \in L^2(\mathbb{R}^n)$, then we do obtain uniqueness.

We will find an explicit representation for the solution to the Poisson problem when $n = 3$. If $u \in L^2(\mathbb{R}^3)$, then using (3.6), we see that

$$\hat{u}(\xi) = -\frac{\hat{f}(\xi)}{|\xi|^2} \Rightarrow u(x) = -\mathcal{F}^{-1}\Big(\frac{\hat{f}(\xi)}{|\xi|^2}\Big)(x) = -\Big[\mathcal{F}^{-1}(|\xi|^{-2}) \star \mathcal{F}^{-1}(\hat{f})\Big](x) = (\Phi * f)(x)\,,$$

where $\Phi(x) = -\frac{1}{4\pi|x|}$. The function Φ is the so-called *fundamental solution*; more precisely, it is the distributional solution of the equation

$$\Delta\Phi = \delta \text{ in } \mathcal{S}'\,.$$

Conceptually

$$\Delta(\Phi * f) = \Delta\Phi * f = \delta * f = f \qquad \forall\ f \in C(\mathbb{R}^n) \text{ whenever } \Phi * f \text{ makes sense}\,,$$

where the first equality follows from the fact that

$$\begin{aligned}\langle \Delta(\Phi * f), \hat{\varphi}\rangle &= (2\pi)^{\frac{n}{2}}\langle -|\xi|^2\hat{\Phi}\hat{f}, \varphi\rangle = (2\pi)^{\frac{n}{2}}\langle \mathcal{F}(\Delta\Phi), \hat{f}\varphi\rangle = (2\pi)^{\frac{n}{2}}\langle \Delta\Phi, \mathcal{F}(\hat{f}\varphi)\rangle = \langle \Delta\Phi, \widetilde{f} * \hat{\varphi}\rangle \\ &= \langle \Delta\Phi * f, \hat{\varphi}\rangle\,.\end{aligned}$$

Example 3.25. *On $\mathbb{R}^2$, $\Delta(e^{x_1}\cos x_2) = 0$. The function $e^{x_1}\cos x_2$ is not a tempered distribution because it grows too fast as $x_1 \to \infty$. As such, the Fourier transfor of $e^{x_1}\cos x_2$ is not defined.*

Using Fourier transform to convert PDE to linear algebraic equations only provides those solutions which do not grow too rapidly at ∞.

3.8.2 The Poisson integral formula on the half-space

Let $\Omega = \mathbb{R}^n \times \mathbb{R}_+ =$, and consider the Dirichlet problem

$$\begin{aligned}\Big[\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}\Big]u = \Big[\frac{\partial^2}{\partial t^2} + \Delta\Big]u &= 0 \quad \text{in } \ \Omega \times (0,\infty)\,, \\ u(x,0) = f(x)(\in \mathcal{S}(\mathbb{R}^n)) &\quad\ \text{on } \ \partial\Omega \times (0,\infty)\,.\end{aligned}$$

(Note that for any constant c, $c\,t$ is always a solution as it is harmonic and vanishes at the boundary $t = 0$.) For uniqueness, we insist that u be bounded. This in turn means u is in $\mathcal{S}'$ and hence we may use the Fourier transform. Applying the Fourier transform (in the x variable) $\mathcal{F}_x$, we see that

$$\frac{\partial^2}{\partial t^2}\mathcal{F}_x u(\xi,t) - |\xi|^2\mathcal{F}_x u(\xi,t) = 0\,, \qquad \mathcal{F}_x u(\xi,0) = \hat{f}(\xi)\,.$$

Therefore, $\mathcal{F}_x u(\xi,t) = C_1(\xi)e^{t|\xi|} + C_2(\xi)e^{-t|\xi|}$, and $C_1(\xi) = 0$ by the growth condition imposed on u. Then $\mathcal{F}_x u(\xi,t) = \hat{f}(\xi)e^{-t|\xi|}$ and hence using (3.5),

$$\begin{aligned} u(x,t) &= \mathcal{F}^{-1}(\hat{f}(\xi)e^{-t|\xi|})(x) = \Big[\mathcal{F}^{-1}(e^{-t|\xi|}) * f\Big](x) \\ &= \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \int_{\mathbb{R}^n} \frac{t f(y)}{(t^2+|x-y|^2)^{\frac{n+1}{2}}} dy\,. \end{aligned}$$

This is the *Poisson integral formula on the half-space.*

If f is bounded, i.e., $f \in L^\infty(\mathbb{R}^n)$, then the integral converges and $u \in L^\infty(\mathbb{R}^n \times \mathbb{R}_+)$. Therefore, $u \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}_+) \cap L^\infty(\mathbb{R}^n \times \mathbb{R}_+)$.

3.8.3 The Heat equation

Let $t \geq 0$ denote time, and x denote a point in space $\mathbb{R}^n$. The function $u(x,t)$ denotes the temperature at time t and position x, and $g \in \mathcal{S}(\mathbb{R}^n)$ denotes the initial temperature distribution. We wish to solve the *heat equation*

$$u_t(x,t) = \Delta u(x,t) \qquad \text{in} \quad \mathbb{R}^n \times (0,\infty)\,, \tag{3.8a}$$

$$u(x,0) = g(x) \qquad \text{on} \quad \mathbb{R}^n \times \{t=0\}\,. \tag{3.8b}$$

Taking the Fourier transform of (3.8), we find that

$$\begin{aligned} \partial_t \hat{u}(\xi,t) &= -|\xi|^2 \hat{u}(\xi,t)\,, \\ \hat{u}(\xi,0) &= \hat{g}(\xi)\,. \end{aligned}$$

Therefore, $\hat{u}(\xi,t) = \hat{g}(\xi)e^{-|\xi|^2 t}$ and hence

$$\begin{aligned} u(x,t) &= \mathcal{F}^{-1}\Big(\hat{g}(\xi)e^{-|\xi|^2 t}\Big)(x) = \Big[\mathcal{F}^{-1}\Big(e^{-|\xi|^2 t}\Big) * g\Big](x) \\ &= \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y)dy \quad (\equiv (H(\cdot,t) * g)(x))\,. \end{aligned} \tag{3.9}$$

Theorem 3.26. *If $g \in L^\infty(\mathbb{R}^n)$, then the solution u to (3.8) is in $\mathcal{C}^\infty(\mathbb{R}^n \times (0,\infty))$.*

Proof. The function $\dfrac{e^{-|x|^2/4t}}{(4\pi t)^{n/2}}$ is $\mathcal{C}^\infty(\mathbb{R}^n \times [\alpha,\infty))$ for all $\alpha > 0$. □

Remark 3.27. *The representation formula (3.9) shows that whenever g is bounded, continuous, and positive, the solution $u(x,t)$ to (3.8) is positive everywhere for $t > 0$.*

The representation formula (3.9) can also be used to prove the following

Theorem 3.28. *Assume that $g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Then u defined by (3.9) is continuous at $t = 0$, that is,*

$$\lim_{(x,t)\to(x_0,0^+)} u(x,t) = g(x_0) \qquad \forall\ x_0 \in \mathbb{R}^n\,.$$

In order to study the *Inhomogeneous heat equation*

$$u_t(x,t) - \Delta u(x,t) = f(x,t) \qquad \text{in} \quad \mathbb{R}^n \times (0,\infty)\,, \tag{3.10a}$$

$$u(x,0) = 0 \qquad \text{on} \quad \mathbb{R}^n \times \{t=0\}\,, \tag{3.10b}$$

we introduce the parameter $s > 0$, and consider the following problem for U:

$$\begin{aligned} U_t(x,t,s) &= \Delta U(x,t,s)\,, \\ U(x,s,s) &= f(x,s)\,. \end{aligned}$$

Then by (3.9),

$$U(x,t,s) = \int_{\mathbb{R}^n} H(x-y,t-s) f(y,s) dy\,.$$

We next invoke *Duhamel's principle* to find a solution $u(x,t)$ to (3.10):

$$u(x,t) = \int_0^t U(x,t,s) ds = \int_0^t \int_{\mathbb{R}^n} H(x-y,t-s) f(y,s) dy ds\,. \tag{3.11}$$

The principle of linear superposition then shows that the solution of the problem

$$\begin{aligned} u_t(x,t) - \Delta u(x,t) &= f(x,t) && \text{in} \quad \mathbb{R}^n \times (0,\infty)\,, \\ u(x,0) &= g(x) && \text{on} \quad \mathbb{R}^n \times \{t=0\}\,, \end{aligned}$$

is the sum of (3.9) and (3.11):

$$\begin{aligned} u(x,t) &= \int_0^t \int_{\mathbb{R}^n} H(x-y,t-s) f(y,s) dy ds + \int_{\mathbb{R}^n} H(x-y,t) g(y) dy \\ &= [H(\cdot,t) * g](x) + \int_0^t [H(\cdot,t-s) * f(\cdot,s)](x) ds\,. \end{aligned} \tag{3.12}$$

3.8.4 The Wave equation

For wave speed $c > 0$, and for $x \in \mathbb{R}^n$, $t \in \mathbb{R}$, consider the following second-order linear hyperbolic equation:

$$\begin{aligned} u_{tt}(x,t) &= c^2 \Delta u(x,t) && \text{in} \quad \mathbb{R}^n \times (0,\infty)\,, \\ u(x,0) &= f(x) && \text{on} \quad \mathbb{R}^n \times \{t=0\}\,, \\ u_t(x,0) &= g(x) && \text{on} \quad \mathbb{R}^n \times \{t=0\}\,. \end{aligned}$$

Taking the Fourier transform of (3.13), we find that

$$\begin{aligned} \hat{u}_{tt}(\xi,t) &= -c^2|\xi|^2\hat{u}(\xi,t) && \text{in} \quad \mathbb{R}^n\times(0,\infty)\,,\\ \hat{u}(\xi,0) &= \hat{f}(\xi) && \text{on} \quad \mathbb{R}^n\times\{t=0\}\,,\\ \hat{u}_t(\xi,0) &= \hat{g}(\xi) && \text{on} \quad \mathbb{R}^n\times\{t=0\}\,. \end{aligned}$$

The general solution of this second-order ordinary differential equations is given by

$$\hat{u}(\xi,t) = C_1(\xi)\cos c|\xi|t + C_2(\xi)\sin c|\xi|t\,.$$

Solving for C_1 and C_2 by using the initial conditions, we find that

$$\hat{u}(\xi,t) = \hat{f}(\xi)\cos c|\xi|t + \hat{g}(\xi)\frac{\sin c|\xi|t}{c|\xi|}\,.$$

Therefore,

$$\begin{aligned} u(x,t) &= \Big[\mathcal{F}^{-1}(\cos c|\xi|t) * f + \mathcal{F}^{-1}\Big(\frac{\sin c|\xi|t}{c|\xi|}\Big) * g\Big](x)\\ &= \frac{1}{c}\Big[\frac{d}{dt}\mathcal{F}^{-1}\Big(\frac{\sin c|\xi|t}{|\xi|}\Big) * f + \mathcal{F}^{-1}\Big(\frac{\sin c|\xi|t}{|\xi|}\Big) * g\Big](x)\,. \end{aligned}$$

For the case that $n=1$,

$$\int_{-m}^{m}\frac{\sin ct\lambda}{\lambda}e^{-ix\lambda}d\lambda = \int_{-m}^{m}\frac{e^{i(ct-x)\lambda}-e^{-i(ct+x)\lambda}}{2i\lambda}d\lambda\,.$$

By the Cauchy integral formula and the residue theorem, $\lim_{m\to\infty}\int_{-m}^{m}\frac{e^{iz}}{z}dz = i\pi$. Therefore, $\forall\ t>0$,

$$\lim_{m\to\infty}\frac{1}{\pi}\int_{-m}^{m}\frac{\sin ct\lambda}{\lambda}e^{-ix\lambda}d\lambda = \chi_{|x|<ct}(x) = \begin{cases} 1 & |x|<ct\\ 0 & |x|\ge ct \end{cases}\,.$$

Corollary 3.29. $\mathcal{F}^{-1}\Big(\frac{\sin c|\xi|t}{|\xi|}\Big)(x) = \sqrt{\frac{\pi}{2}}\ \chi_{|x|<ct}(x)$ *in* $\mathcal{S}'(\mathbb{R})$.

Proof. For all $\varphi\in\mathcal{S}(\mathbb{R})$,

$$\begin{aligned} \int_{\mathbb{R}}\mathcal{F}^{-1}\Big(\frac{\sin c|\xi|t}{|\xi|}\Big)(x)\varphi(x)dx &= \int_{\mathbb{R}}\frac{\sin c|\xi|t}{|\xi|}\mathcal{F}^{-1}(\varphi)(\xi)d\xi\\ &= \lim_{m\to\infty}\frac{1}{\sqrt{2\pi}}\int_{-m}^{m}\int_{\mathbb{R}}\frac{\sin c|\xi|t}{|\xi|}e^{ix\xi}\varphi(x)dxd\xi\,, \end{aligned}$$

and by Fubini's theorem together with the dominated convergence theorem, we see that

$$\lim_{m\to\infty}\int_{-m}^{m}\int_{\mathbb{R}}\frac{\sin c|\xi|t}{|\xi|}e^{ix\xi}\varphi(x)dxd\xi = \lim_{m\to\infty}\int_{\mathbb{R}}\int_{-m}^{m}\frac{\sin c|\xi|t}{|\xi|}e^{ix\xi}\varphi(x)d\xi dx$$
$$= \int_{\mathbb{R}}\lim_{m\to\infty}\int_{-m}^{m}\frac{\sin c|\xi|t}{|\xi|}e^{ix\xi}\varphi(x)d\xi dx = \pi\int_{\mathbb{R}}\chi_{|x|<ct}(x)\varphi(x)dx\,.$$

□

We have thus established *d'Alembert's formula* for the solution of the the 1-D wave equation:

$$u(x,t) = \frac{1}{c}\sqrt{\frac{\pi}{2}}\frac{1}{\sqrt{2\pi}}\Big[\frac{d}{dt}\int_{\mathbb{R}}f(x-y)\chi_{|y|<ct}(y)dy + \int_{\mathbb{R}}g(x-y)\chi_{|y|<ct}(y)dy\Big]$$
$$= \frac{1}{2c}\frac{d}{dt}\int_{-ct}^{ct}f(x-y)dy + \frac{1}{2c}\int_{-ct}^{ct}g(x-y)dy$$
$$= \frac{f(x-ct)+f(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct}g(y)dy\,.$$

We have just used the Fourier transform to find explicit solutions to the fundamental linear elliptic, parabolic, and hyperbolic equations. More generally, the Fourier transform is a powerful tool for the analysis of many other constant coefficient linear partial differential equations.

3.9 Exercises

Problem 3.1. **(a)** *For $f \in L^1(\mathbb{R})$, set $S_R f(x) = (2\pi)^{-\frac{1}{2}}\int_{-R}^{R}\hat{f}(\xi)e^{ix\xi}d\xi$. Show that*

$$S_R f(x) = K_R * f(x) = \int_{-\infty}^{\infty}K_R(x-y)f(y)dy$$

where

$$K_R(x) = (2\pi)^{-1}\int_{-R}^{R}e^{ix\xi}d\xi = \frac{\sin Rx}{\pi x}\,.$$

(b) *Show that if $f \in L^2(\mathbb{R})$, then $S_R f \to f$ in $L^2(\mathbb{R})$ as $R\to\infty$.*

Problem 3.2. *Show that for any $R \in (0,\infty)$, there exists $f \in L^1(\mathbb{R})$ such that $S_R f \notin L^1(\mathbb{R})$. (Hint. Note that $K_R \notin L^1(\mathbb{R})$.) For partial credit, explain why the result is interesting.*

Problem 3.3. *Assume* $w \in \mathcal{S}' \cap L^1_{loc}(\mathbb{R}^n)$ *and* $w(x) \geq 0$. *Show that if* $\hat{w} \in L^\infty(\mathbb{R}^n)$, *then* $w \in L^1(\mathbb{R}^n)$ *and*

$$\|\hat{w}\|_{L^\infty(\mathbb{R}^n)} = (2\pi)^{-n/2}\|w\|_{L^1(\mathbb{R}^n)}$$

(Hint. Consider $w_j(x) = \psi(\frac{x}{j})w(x)$ *with* $\psi \in C_0^\infty(\mathbb{R}^n)$ *and* $\psi(0) = 1$. *Use the fact that* $w_j \rightharpoonup w$ *in* $\mathcal{S}'$.*)*

Problem 3.4. *Consider the Poisson equation on* $\mathbb{R}^1$: $u_{xx} = f$.

(a) *Show that* $\varphi(x) = \dfrac{x+|x|}{2}$ *and* $\psi(x) = \dfrac{|x|}{2}$ *are both distributional solutions to* $u_{xx} = \delta_0$.

(b) *Let* f *be continuous with compact support in* $\mathbb{R}$. *Show that* $u(x) = \displaystyle\int_{\mathbb{R}} \varphi(x-y)f(y)dy$ *and* $v(x) = \displaystyle\int_{\mathbb{R}} \psi(x-y)f(y)dy$ *both solve the Poisson equation* $w_{xx}(x) = f(x)$ *(without relying upon distribution theory).*

Problem 3.5. *Let* $T \in \mathcal{S}'(\mathbb{R}^n)$ *and* $f \in S(\mathbb{R}^n)$. *Show that the Leibniz rule for distributional derivatives holds; that is, show that* $\dfrac{\partial}{\partial x_i}(fT) = f\dfrac{\partial T}{\partial x_i} + \dfrac{\partial f}{\partial x_i}T$ *in the sense of distribution.*

Problem 3.6. *Let* $f(x) = e^{-s|x|^2}$ *and* $g(x) = e^{-t|x|^2}$. *Find the Fourier transform of* f *(and* g*) and use the inversion formula to compute* $f * g$.

Problem 3.7. *Let* d_r *denote the map given by* $d_r f(x) = f(rx)$. *Show that*

$$\mathcal{F}(d_r f) = r^{-n} d_{1/r}\mathcal{F}(f) .$$

Problem 3.8. *Show that a function* $f \in L^2(\mathbb{R}^n)$ *is real if and only if* $\hat{f}(-\xi) = \overline{\hat{f}(\xi)}$.

Problem 3.9. *Find the Fourier transform of the function* $f(x) = xe^{tx^2}$ *for* $t < 0$.

Problem 3.10. *Find the Fourier transform of* $\mathbf{1}_{(-a,a)}$, *the characteristic (indicator) function of the set* $(-a, a)$.

Problem 3.11. *Let* $f(x) = \mathbf{1}_{(0,\infty)}(x)e^{-tx}$, *that is,*

$$f(x) = \begin{cases} e^{-tx} & \text{if } x > 0\,, \\ 0 & \text{if } x \leq 0\,. \end{cases}$$

Find the Fourier transform of f *for* $t > 0$.

Problem 3.12. *Find the Fourier transform of the function* $f(x) = x_1|x|^\alpha$ *, where* x_1 *is the first component of* x *and* $-n-2<\alpha<-2$ *.*
Hint*: Use the fact that for* $-n<\alpha<0$ *,*

$$\mathcal{F}(|x|^\alpha)(\xi) = \frac{\Gamma(\frac{n+\alpha}{2})}{\Gamma(-\frac{\alpha}{2})} 2^{\alpha+\frac{n}{2}} |\xi|^{-(\alpha+n)}$$

and $f(x) = \dfrac{1}{\alpha+2}\dfrac{\partial}{\partial x_1}|x|^{\alpha+2}$ *.*

Problem 3.13. *Let* $\alpha>0$ *be given. Show that the Fourier transform of the function*

$$f(x) = \frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}e^{-t}e^{-t|x|^2}dt$$

is positive.

Problem 3.14. *Let* $f \in L^1(\mathbb{R})$ *. Show that the anti-derivative of* f *can be written as the convolution of* f *and a function* $\varphi \in L^1_{loc}(\mathbb{R})$ *.*

Problem 3.15. *Let* f *be a continuous function with period* 2π *, and* $\hat{f}$ *be the Fourier transform of* f *. Show that*

$$\hat{f}(\xi) = \sum_{n=-\infty}^{\infty} (\sqrt{2\pi} f_n)\tau_{-n}\delta$$

in the sense of distribution, where f_n *is the Fourier coefficient defined by*

$$f_n = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}dx\,.$$

Problem 3.16. *Using Definition 3.15, compute the Fourier transform of the function/distribution*

$$R(x) = \begin{cases} x & \text{if } x \geq 0\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

by completing the following:

(1) Let H *be the Heaviside function. Show that* $\hat{H}(\xi) = p.v.\dfrac{1}{\sqrt{2\pi}i\xi} + C\delta(\xi)$ *for some constant* C *, where* $p.v.\dfrac{1}{\xi}$ *is defined as*

$$\left\langle p.v.\frac{1}{\xi}, \varphi\right\rangle = \lim_{\epsilon\to 0^+}\int_{\mathbb{R}\setminus[-\epsilon,\epsilon]}\frac{\varphi(\xi)}{\xi}d\xi = \lim_{\epsilon\to 0^+}\Big(\int_{-\infty}^{-\epsilon} + \int_\epsilon^\infty\Big)\frac{\varphi(\xi)}{\xi}d\xi\,.$$

Note that the integral above always exists as long as $\varphi \in \mathcal{S}(\mathbb{R})$ *.*

(2) Let $S(x) = H(x) - \dfrac{1}{2}$. *Then* S *is an odd function, and show that* $\hat{S}(\xi) = -\hat{S}(-\xi)$.

(3) Use (2) to determine the constant C *in (1).*

(4) By the definition of Fourier transform, show that $\langle \hat{R}, \varphi \rangle = -i \langle \hat{H}, \varphi' \rangle$, *and as a consequence*

$$\hat{R}(\xi) = i \frac{d}{d\xi} \hat{H}(\xi)\,.$$

4 The Sobolev Spaces $H^s(\mathbb{R}^n)$, $s \in \mathbb{R}$

4.1 $H^s(\mathbb{R}^n)$ via the Fourier Transform

The Fourier transform allows us to generalize the Hilbert spaces $H^k(\mathbb{R}^n)$ for $k \in \mathbb{Z}_+$ to $H^s(\mathbb{R}^n)$ for all $s \in \mathbb{R}$, and hence study functions which possess fractional derivatives (and anti-derivatives) which are square integrable.

Definition 4.1. *For any $s \in \mathbb{R}^n$, let $\langle \xi \rangle = \sqrt{1+|\xi|^2}$, and set*

$$\begin{aligned} H^s(\mathbb{R}^n) &= \{u \in \mathcal{S}'(\mathbb{R}^n) \mid \langle \xi \rangle^s \hat{u} \in L^2(\mathbb{R}^n)\} \\ &= \{u \in \mathcal{S}'(\mathbb{R}^n) \mid \Lambda^s u \in L^2(\mathbb{R}^n)\}\,, \end{aligned}$$

where $\Lambda^s u = \mathcal{F}^*(\langle \xi \rangle^s \hat{u})$.

The operator Λ^s can be thought of as a "differential operator" of order s, and according to Rellich's theorem, Λ^{-s} is a compact operator, yielding the isomorphism

$$H^s(\mathbb{R}^n) = \Lambda^{-s} L^2(\mathbb{R}^n)\,.$$

Definition 4.2. *The inner-product on $H^s(\mathbb{R}^n)$ is given by*

$$(u,v)_{H^s(\mathbb{R}^n)} = (\Lambda^s u, \Lambda^s v)_{L^2(\mathbb{R}^n)} \quad \forall u,v \in H^s(\mathbb{R}^n)\,.$$

and the norm on $H^s(\mathbb{R}^n)$ is

$$\|u\|^s_{H^s(\mathbb{R}^n)} = (u,u)_{H^s(\mathbb{R}^n)} \quad \forall u \in H^s(\mathbb{R}^n)\,.$$

The completeness of $H^s(\mathbb{R}^n)$ with respect to the $\|\cdot\|_{H^s(\mathbb{R}^n)})$ is induced by the completeness of $L^2(\mathbb{R}^n)$.

Theorem 4.3. *For $s \in \mathbb{R}$, $(H^s(\mathbb{R}^n), \|\cdot\|_{H^s(\mathbb{R}^n)})$ is a Hilbert space.*

Example 4.4 ($H^1(\mathbb{R}^n)$)**.** *The $H^1(\mathbb{R}^n)$ in Fourier representation is exactly the same as the that given by Definition 2.13:*

$$\begin{aligned} \|u\|^2_{H^1(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} \langle \xi \rangle^2 \|\hat{u}(\xi)\|^2 d\xi \\ &= \int_{\mathbb{R}^n} (1+|\xi|^2)\|\hat{u}(\xi)\|^2 d\xi \\ &= \int_{\mathbb{R}^n} (|u(x)|^2 + |Du(x)|^2) dx\,, \end{aligned}$$

the last equality following from the Plancheral theorem.

Example 4.5 ($H^{\frac{1}{2}}(\mathbb{R}^n)$)**.** *The $H^{\frac{1}{2}}(\mathbb{R}^n)$ can be viewed as interpolating between decay required for $\hat{u} \in L^2(\mathbb{R}^n)$ and $\hat{u} \in H^1(\mathbb{R}^n)$:*

$$H^{\frac{1}{2}}(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n) \mid \int_{\mathbb{R}^n} \sqrt{1+|\xi|^2}|\hat{u}(\xi)|^2 \, d\xi < \infty\} .$$

Example 4.6 ($H^{-1}(\mathbb{R}^n)$)**.** *The space $H^{-1}(\mathbb{R}^n)$ can be heuristically described as those distributions whose anti-derivative is in $L^2(\mathbb{R}^n)$; in terms of the Fourier representation, elements of $H^{-1}(\mathbb{R}^n)$ possess a transforms that can grow linearly at infinity:*

$$H^{-1}(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n) \mid \int_{\mathbb{R}^n} \frac{|\hat{u}(\xi)|^2}{1+|\xi|^2} \, d\xi < \infty\} .$$

For $T \in H^{-s}(\mathbb{R}^n)$ and $u \in H^s(\mathbb{R}^n)$, the duality pairing is given by

$$\langle T, u \rangle = (\Lambda^{-s} T, \Lambda^s u)_{L^2(\mathbb{R}^n)} ,$$

from which the following result follows.

Proposition 4.7. *For all $s \in \mathbb{R}$, $[H^s(\mathbb{R}^n)]' = H^{-s}(\mathbb{R}^n)$.*

The ability to define fractional-order Sobolev spaces $H^s(\mathbb{R}^n)$ allows us to refine the estimates of the trace of a function which we previously stated in Theorem 2.33. That result, based on the Gauss-Green theorem, stated that the trace operator was continuous from $H^1(\mathbb{R}^n_+)$ into $L^2(\mathbb{R}^{n-1})$. In fact, the trace operator is continuous from $H^1(\mathbb{R}^n_+)$ into $H^{\frac{1}{2}}(\mathbb{R}^{n-1})$.

To demonstrate the idea, we take $n = 2$. Given a continuous function $u : \mathbb{R}^2 \to \{x_1 = 0\}$, we define the operator

$$Tu = u(0, x_2) .$$

The trace theorem asserts that we can extend T to a continuous linear map from $H^1(\mathbb{R}^2)$ into $H^{\frac{1}{2}}(\mathbb{R})$ so that we only lose one-half of a derivative.

Theorem 4.8. *$T : H^1(\mathbb{R}^2) \to H^{\frac{1}{2}}(\mathbb{R})$, and there is a constant C such that*

$$\|Tu\|_{H^{\frac{1}{2}}(\mathbb{R})} \le C\|u\|_{H^1(\mathbb{R}^2)} .$$

Before we proceed with the proof, we state a very useful result.

Lemma 4.9. *Suppose that $u \in \mathcal{S}(\mathbb{R}^2)$ and define $f(x_2) = u(0, x_2)$. Then*

$$\hat{f}(\xi_2) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}_{\xi_1}} \hat{u}(\xi_1, \xi_2) d\xi_1 .$$

Proof. $\hat{f}(\xi_2) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{u}(\xi_1, \xi_2) d\xi_1$ if and only if $f(x_2) = \frac{1}{\sqrt{2\pi}} \mathcal{F}^* \int_{\mathbb{R}} \hat{u}(\xi_1, \xi_2) d\xi_1$, and

$$\frac{1}{\sqrt{2\pi}} \mathcal{F}^* \int_{\mathbb{R}} \hat{u}(\xi_1, \xi_2) d\xi_1 = \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \hat{u}(\xi_1, \xi_2) d\xi_1 e^{ix_2 \xi_2} d\xi_2 \,.$$

On the other hand,

$$u(x_1, x_2) = \mathcal{F}^*[\hat{u}(\xi_1, \xi_2)] = \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \hat{u}(\xi_1, \xi_2) e^{ix_1\xi_1 + ix_2\xi_2} d\xi_1 d\xi_2 \,,$$

so that

$$u(0, x_2) = \mathcal{F}^*[\hat{u}(\xi_1, \xi_2)] = \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \hat{u}(\xi_1, \xi_2) e^{ix_2\xi_2} d\xi_1 d\xi_2 \,.$$

□

Proof of Theorem 4.8. Suppose that $u \in \mathcal{S}(\mathbb{R}^2)$ and set $f(x_2) = u(0, x_1)$. According to Lemma 4.9,

$$\begin{aligned} \hat{f}(\xi_2) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}_{\xi_1}} \hat{u}(\xi_1, \xi_2) d\xi_1 = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}_{\xi_1}} \hat{u}(\xi_1, \xi_2) \langle \xi \rangle \, \langle \xi \rangle^{-1} d\xi_1 \\ &\le \frac{1}{\sqrt{2\pi}} \left(\int_{\mathbb{R}} |\hat{u}(\xi_1, \xi_2)|^2 \langle \xi \rangle^2 d\xi_1 \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} \langle \xi \rangle^{-2} d\xi_1 \right)^{\frac{1}{2}} , \end{aligned}$$

and hence

$$|f(\xi_2)|^2 \le C \int_{\mathbb{R}} |\hat{u}(\xi_1, \xi_2)|^2 \langle \xi \rangle^2 d\xi_1 \int_{\mathbb{R}} \langle \xi \rangle^{-2} d\xi_1 \,.$$

The key to this trace estimate is the explicit evaluation of the integral $\int_{\mathbb{R}} \langle \xi \rangle^{-2} d\xi_1$:

$$\int_{\mathbb{R}} \frac{1}{1 + \xi_1^2 + \xi_2^2} d\xi_1 = \left. \frac{\tan^{-1}\left(\frac{\xi_1}{\sqrt{1+\xi_2^2}} \right)}{\sqrt{1 + \xi_2^2}} \right|_{-\infty}^{+\infty} \le \pi (1 + \xi_2^2)^{-\frac{1}{2}} \,. \tag{4.1}$$

It follows that $\int_{\mathbb{R}} (1 + \xi_2^2)^{\frac{1}{2}} |\hat{f}(\xi_2)|^2 d\xi_2 \le C \int_{\mathbb{R}} |\hat{u}(\xi_1, \xi_2)|^2 \langle \xi \rangle^2 d\xi_1$, so that integration of this inequality over the set $\{\xi_2 \in \mathbb{R}\}$ yields the result. Using the density of $\mathcal{S}(\mathbb{R}^2)$ in $H^1(\mathbb{R}^2)$ completes the proof. □

The proof of the trace theorem in higher dimensions and for general $H^s(\mathbb{R}^n)$ spaces, $s > \frac{1}{2}$, replacing $H^1(\mathbb{R}^n)$ proceeds in a very similar fashion; the only difference is that the integral $\int_{\mathbb{R}} \langle \xi \rangle^{-2} d\xi_1$ is replaced by $\int_{\mathbb{R}^{n-1}} \langle \xi \rangle^{-2s} d\xi_1 \cdots d\xi_{n-1}$, and instead of obtaining an explicit anti-derivative of this integral, an upper bound is instead found. The result is the following general trace theorem.

Theorem 4.10 (The trace theorem for $H^s(\mathbb{R}^n)$)**.** *For $s > \frac{1}{2}$, the trace operator $T : H^s(\mathbb{R}^n) \to H^{s-\frac{1}{2}}(\mathbb{R}^{n-1})$ is continuous.*

We can extend this result to open, bounded, C^∞ domains $\Omega \subset \mathbb{R}^n$.

Definition 4.11. *Let $\partial\Omega$ denote a closed C^∞ manifold, and let $\{\omega_l\}_{l=1}^K$ denote an open covering of $\partial\Omega$, such that for each $l \in \{1, 2, ..., K\}$, there exist C^∞-class charts ϑ_l which satisfy*

$$\vartheta_l : B(0, r_l) \subset \mathbb{R}^{n-1} \to \omega_l \quad \text{is a } C^\infty \text{ diffeomorphism}.$$

Next, for each $1 \leq l \leq K$, let $0 \leq \varphi_l \in C_0^\infty(U_l)$ denote a partition of unity so that $\sum_{l=1}^L \varphi_l(x) = 1$ for all $x \in \partial\Omega$. For all real $s \geq 0$, we define

$$H^s(\partial\Omega) = \{u \in L^2(\partial\Omega) \ : \ \|u\|_{H^s(\partial\Omega)} < \infty\},$$

where for all $u \in H^s(\partial\Omega)$,

$$\|u\|^2_{H^s(\partial\Omega)} = \sum_{l=1}^K \|(\varphi_l u) \circ \vartheta_l\|^2_{H^s(\mathbb{R}^{n-1})}.$$

The space $(H^s(\partial\Omega), \|\cdot\|_{H^s(\partial\Omega)})$ is a Hilbert space by virtue of the completeness of $H^s(\mathbb{R}^{n-1})$; furthermore, any system of charts for $\partial\Omega$ with subordinate partition of unity will produce an equivalent norm.

Theorem 4.12 (The trace map on Ω)**.** *For $s > \frac{1}{2}$, the trace operator $T : \Omega \to \partial\Omega$ is continuous.*

Proof. Let $\{U_l\}_{l=1}^K$ denote an n-dimensional open cover of $\partial\Omega$ such that $U_l \cap \partial\Omega = \omega_l$. Define charts $\theta_l : V_l \to U_l$, as in (2.11) but with each chart being a C^∞ map, such that ϑ_l is equal to the restriction of θ_l to the $(n-1)$-dimensional ball $B(0, r_l) \subset \mathbb{R}^{n-1}$). Also, choose a partition of unity $0 \leq \zeta_l \in C_0^\infty(U_l)$ subordinate to the covering U_l such that $\varphi_l = \zeta_l|_{\omega_l}$.

Then by Theorem 4.10, for $s > \frac{1}{2}$,

$$\|u\|^2_{H^{s-\frac{1}{2}}(\partial\Omega)} = \sum_{l=1}^K \|(\varphi_l u) \circ \vartheta_l\|^2_{H^{s-\frac{1}{2}}(\mathbb{R}^{n-1})} \leq C \sum_{l=1}^K \|(\varphi_l u) \circ \vartheta_l\|^2_{H^s(\mathbb{R}^n)} \leq C\|u\|^2_{H^s(\Omega)}.$$

□

One may then ask if the trace operator T is onto; namely, given $f \in H^{s-\frac{1}{2}}(\mathbb{R}^{n-1})$ for $s > \frac{1}{2}$, does there exist a $u \in H^s(\mathbb{R}^n)$ such that $f = Tu$? By essentially reversing the order of the proof of Theorem 4.8, it is possible to answer this question in the affirmative. We first consider the case that $n = 2$ and $s = 1$.

Theorem 4.13. *$T : H^1(\mathbb{R}^2) \to H^{\frac{1}{2}}(\mathbb{R})$ is a surjection.*

Proof. With $\xi = (\xi_1, \xi_2)$, we define (one of many possible choices) the function u on $\mathbb{R}^2$ via its Fourier representation:

$$\hat{u}(\xi_1, \xi_2) = K\hat{f}(\xi_1)\frac{\langle \xi_1 \rangle}{\langle \xi \rangle^2}\,,$$

for a constant $K \neq 0$ to be determined shortly. To verify that $\|u\|_{H^1(\mathbb{R}^1)} \leq \|f\|_{H^{\frac{1}{2}}(\mathbb{R})}$, note that

$$\begin{aligned}
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\hat{u}(\xi_1,\xi_2)|^2 \langle \xi \rangle^2 d\xi_1 d\xi_2 &= K\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\hat{f}(\xi_1)|^2 \frac{\langle \xi_1 \rangle^2}{\langle \xi \rangle^2} d\xi_1 d\xi_2 \\
&= K\int_{-\infty}^{\infty} |\hat{f}(\xi_1)|^2 (1+\xi_1^2)\int_{-\infty}^{\infty} \frac{1}{1+\xi_1^2+\xi_2^2} d\xi_2\, d\xi_1 \\
&\leq C\|f\|^2_{H^{\frac{1}{2}}(\mathbb{R})}\,,
\end{aligned}$$

where we have used the estimate (4.1) for the inequality above.

It remains to prove that $u(x_1, 0) = f(x_1)$, but by Lemma 4.9, it suffices that

$$\int_{-\infty}^{\infty} \hat{u}(\xi_1,\xi_2) d\xi_2 = \sqrt{2\pi}\hat{f}(\xi_1)\,.$$

Integrating $\hat{u}$, we find that

$$\int_{-\infty}^{\infty} \hat{u}(\xi_1,\xi_2) d\xi_2 = K\hat{f}(\xi_1)\sqrt{1+\xi_1^2}\int_{-\infty}^{\infty} \frac{1}{1+\xi_1^2+\xi_2^2} d\xi_2 \leq K\pi\hat{f}(\xi_1)$$

so setting $K = \sqrt{2\pi}/\pi$ completes the proof. □

A similar construction yields the general result.

Theorem 4.14. *For $s > \frac{1}{2}$, $T : H^s(\mathbb{R}^n) \to H^{s-\frac{1}{2}}(\mathbb{R}^{n-1})$ is a surjection.*

By using the system of charts employed for the proof of Theorem 4.12, we also have the surjectivity of the trace map on bounded domains.

Theorem 4.15. *For $s > \frac{1}{2}$, $T : H^s(\Omega) \to H^{s-\frac{1}{2}}(\partial\Omega)$ is a surjection.*

The Fourier representation provides a very easy proof of a simple version of the Sobolev embedding theorem.

Theorem 4.16. *For $s > n/2$, if $u \in H^s(\mathbb{R}^n)$, then u is continuous and*

$$\max |u(x)| \leq C\|u\|_{H^s(\mathbb{R}^n)}.$$

Proof. By Theorem 3.6, $u = \mathcal{F}^* \hat{u}$; thus according to Hölder's inequality and the Riemann-Lebesgue lemma (Theorem 3.22), it suffices to show that

$$\|\hat{u}\|_{L^1(\mathbb{R}^n)} \le C\|u\|_{H^s(\mathbb{R}^n)} .$$

But this follows from the Cauchy-Schwarz inequality since

$$\begin{aligned}\int_{\mathbb{R}^n} |\hat{u}(\xi)| d\xi &= \int_{\mathbb{R}^n} |\hat{u}(\xi)| \langle\xi\rangle^s \langle\xi\rangle^{-s} d\xi \\ &\le \left(\int_{\mathbb{R}^n} |\hat{u}(\xi)|^2 \langle\xi\rangle^{2s} d\xi\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} \langle\xi\rangle^{-2s} d\xi\right)^{\frac{1}{2}} \\ &\le C\|u\|_{H^s(\mathbb{R}^n)} ,\end{aligned}$$

the latter inequality holding whenever $s > n/2$. □

Hölder's inequality can be used to prove the following

Theorem 4.17 (Interpolation inequality)**.** *Let $0 < r < t < \infty$, and $s = \alpha r + (1-\alpha)t$ for some $\alpha \in (0,1)$. Then*

$$\|u\|_{H^s(\mathbb{R}^n)} \le C\|u\|^{\alpha}_{H^r(\mathbb{R}^n)} \|u\|^{1-\alpha}_{H^t(\mathbb{R}^n)} . \tag{4.2}$$

Example 4.18 (Euler equation on $\mathbb{T}^2$)**.** *On some time interval $[0,T]$ suppose that $u(x,t)$, $x \in \mathbb{T}^2, t \in [0,T]$, is a smooth solution of the Euler equations:*

$$\begin{aligned}\partial_t u + (u \cdot D)u + Dp = 0 \ \text{in } \mathbb{T}^2 \times (0,T] , \\ \operatorname{div} u = 0 \ \text{in } \mathbb{T}^2 \times (0,T] ,\end{aligned}$$

with smooth initial condition $u|_{t=0} = u_0$. Written in components, $u = (u^1, u^2)$ satisfies $u^i_t + u^i,_j j^j + p,_i = 0$ for $i = 1, 2$, where we are using the Einstein summation convention for summing repeated indices from 1 to 2 and where $u^i,_j = \partial u^i / \partial x_j$ and $p,_i = \partial p / \partial x_i$.

Computing the $L^2(\mathbb{T}^2)$ inner-product of the Euler equations with u yields the equality

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^2} |u(x,t)|^2 dx + \underbrace{\int_{\mathbb{T}^2} u^i,_j u^j u^i dx}_{\mathcal{I}_1} + \underbrace{\int_{\mathbb{T}^2} p,_i u^i dx}_{\mathcal{I}_2} = 0 .$$

Notice that

$$\mathcal{I}_1 = \frac{1}{2}\int_{\mathbb{T}^2} (|u|^2),_j u^j dx = \frac{1}{2}\int_{\mathbb{T}^2} |u|^2 \operatorname{div} u dx = 0 ,$$

the second equality arising from integration by parts with respect to $\partial/\partial x_j$. Integration by parts in the integral $\mathcal{I}_2$ shows that $\mathcal{I}_2 = 0$ as well, from which the conservation law $\frac{d}{dt}\|u(\cdot,t)\|^2_{L^2(\mathbb{T}^2)}$ follows.

To estimate the rate of change of higher-order Sobolev norms of u relies on the use of the Sobolev embedding theorem. In particular, we claim that on a short enough time interval $[0,T]$, we have the inequality

$$\frac{d}{dt}\|u(\cdot,t)\|^2_{H^3(\mathbb{T}^2)} \le C\|u(\cdot,t)\|^3_{H^3(\mathbb{T}^2)} \tag{4.3}$$

from which it follows that $\|u(\cdot,t)\|^2_{H^3(\mathbb{T}^2)} \le M$ for some constant $M<\infty$.

To prove (4.3), we compute the $H^3(\mathbb{T}^2)$ inner-product of the Euler equations with u:

$$\frac{1}{2}\frac{d}{dt}\|u(\cdot,t)\|^2_{H^3(\mathbb{T}^2)} + \sum_{|\alpha|\le 3}\int_{\mathbb{T}^2} D^\alpha u^i{,_j}\, u^j D^\alpha u^i dx + \sum_{|\alpha|\le 3}\int_{\mathbb{T}^2} D^\alpha p{,_i}\, D^\alpha u^i dx = 0\,.$$

The third integral vanishes by integration by parts and the fact that $D^\alpha \operatorname{div} u = 0$; thus, we focus on the nonlinearity, and in particular, on the highest-order derivatives $|\alpha|=3$, and use D^3 to denote all third-order partial derivatives, as well as the notation l.o.t. for lower-order terms. We see that

$$\int_{\mathbb{T}^2} D^3(u^i{,_j}\, u^j) D^3 u^i dx = \underbrace{\int_{\mathbb{T}^2} D^3 u^i{,_j}\ u^j\, D^3 u^i dx}_{\mathcal{K}_1} + \underbrace{\int_{\mathbb{T}^2} u^i{,_j}\ D^3 u^j\, D^3 u^i dx}_{\mathcal{K}_2} + \int_{\mathbb{T}^2} \text{l.o.t.}\, dx\,.$$

By definition of being lower-order terms, $\int_{\mathbb{T}^2}\text{l.o.t.}\,dx \le C\|u\|^3_{H^3(\mathbb{T}^2)}$, so it remains to estimate the integrals $\mathcal{K}_1$ and $\mathcal{K}_2$. But the integral $\mathcal{K}_1$ vanishes by the same argument that proved $\mathcal{I}_1 = 0$. On the other hand, the integral $\mathcal{K}_2$ is estimated by Hölder's inequality:

$$|\mathcal{K}_2| \le \|u^i{,_j}\|_{L^\infty(\mathbb{T}^2)}\, \|D^3 u^j\|_{H^3(\mathbb{T}^2)}\, \|D^3 u^i\|_{H^3(\mathbb{T}^2)}\,.$$

Thanks to the Sobolev embedding theorem, for $s=2$ (s needs only to be greater than 1),

$$\|u^i{,_j}\|_{L^\infty(\mathbb{T}^2)} \le C\|u^i{,_j}\|_{H^2(\mathbb{T}^2)} \le \|u\|_{H^3(\mathbb{T}^2)}\,,$$

from which it follows that $\mathcal{K}_2 \le C\|u\|^3_{H^3(\mathbb{T}^2)}$, and this proves the claim.

Note well, that it is the Sobolev embedding theorem that requires the use of the space $H^3(\mathbb{T}^2)$ for this analysis; for example, it would not have been possible to establish the inequality (4.3) with the $H^2(\mathbb{T}^2)$ norm replacing the $H^3(\mathbb{T}^2)$ norm.

4.2 Fractional-order Sobolev spaces via difference quotient norms

4.2.1 The case that $s > 0$

Lemma 4.19. *For $0<s<1$, $u \in H^s(\mathbb{R}^n)$ is equivalent to*

$$u \in L^2(\mathbb{R}^n)\,, \quad \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}} dxdy < \infty\,.$$

Proof. The Fourier transform shows that for $h \in \mathbb{R}^n$,

$$\int_{\mathbb{R}^n} |u(x+h) - u(x)|^2 dx = \int_{\mathbb{R}^n} |e^{ih\cdot\xi} - 1|^2 |\hat{u}(\xi)|^2 d\xi = \int_{\mathbb{R}^n} \sin^2 \frac{h\cdot\xi}{2} |\hat{u}(\xi)|^2 d\xi \,.$$

It follows that

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}} dxdy = \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\sin^2 \frac{h\cdot\xi}{2}}{|h|^{n+2s}} |\hat{u}(\xi)|^2 d\xi dh = \int_{\mathbb{R}^n} |\hat{u}(\xi)|^2 \Big[\int_{\mathbb{R}^n} \frac{\sin^2 \frac{h\cdot\xi}{2}}{|h|^{n+2s}} dh \Big] d\xi$$

$$\text{(letting } h = 2|\xi|^{-1}z\text{)} = 2^{-2s} \int_{\mathbb{R}^n} |\xi|^{2s} |\hat{u}(\xi)|^2 \Big[\int_{\mathbb{R}^n} \frac{\sin^2(z\cdot\frac{\xi}{|\xi|})}{|z|^{n+2s}} dz \Big] d\xi \,.$$

As the integral inside of the square brackets is rotationally invariant, it is independent of the direction of $\xi/|\xi|$; as such we set $\xi/|\xi| = e_1$ and let $z_1 = z\cdot e_1$ denote the first component of the vector z. It follows that

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}} dxdy = C \int_{\mathbb{R}^n} |\xi|^{2s} |\hat{u}(\xi)|^2 d\xi \,,$$

where $C = \int_{\mathbb{R}^n} \frac{\sin^2 z_1}{|z|^{n+2s}} dz < \infty$. □

Corollary 4.20. *For $0 < s < 1$,*

$$\|u\|_{H^s(\mathbb{R}^n)} = \|u\|_{L^2(\mathbb{R}^n)} + \Big[\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}} dxdy \Big]^{\frac{1}{2}}$$

is an equivalent norm on $H^s(\mathbb{R}^n)$.

For real $s \geq 0$, $u \in H^s(\mathbb{R}^n)$ if and only if $D^\alpha u \in L^2(\mathbb{R}^n)$ for all $|\alpha| \leq [s]$ (where $[s]$ denotes the greatest integer that is not bigger than s), and

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|D^\alpha u(x) - D^\alpha u(y)|^2}{|x-y|^{n+2(s-[s])}} dxdy < \infty$$

for all $|\alpha| = [s]$. Moreover, an equivalent norm on $H^s(\mathbb{R}^n)$ is given by

$$\|u\|_{H^s(\mathbb{R}^n)} = \Big[\sum_{|\alpha|\leq[s]} \|D^\alpha u\|^2_{L^2(\Omega(\mathbb{R}^n)} + \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|D^\alpha u(x) - D^\alpha u(y)|^2}{|x-y|^{n+2(s-[s])}} dxdy \Big]^{\frac{1}{2}} .$$

If $u \in H^k(\mathbb{R}^n)$, $k \in \mathbb{N}$, and $\varphi \in \mathcal{S}(\mathbb{R}^n)$, application of the product rule shows that $\varphi u \in H^k(\mathbb{R}^n)$. When $s \notin \mathbb{N}$, however, the product rule is not directly applicable and we must rely on other means to show that $\varphi u \in H^s(\mathbb{R}^n)$.

Lemma 4.21. *Suppose that $u \in H^s(\mathbb{R}^n)$ for some $s \geq 0$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Then $\varphi u \in H^s(\mathbb{R}^n)$.*

Proof. We first conisder the case that $0 \leq s < 1$.

By Corollary 4.19, since φu is clearly an $L^2(\mathbb{R}^n)$-function, it suffices to show that

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|(\varphi u)(x) - (\varphi u)(y)|^2}{|x-y|^{n+2s}} dxdy < \infty\,.$$

Since $|(\varphi u)(x) - (\varphi u)(y)| \leq |\varphi(x) - \varphi(y)||u(x)| + |u(x) - u(y)||\varphi(y)|$,

$$\begin{aligned}
&\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|(\varphi u)(x) - (\varphi u)(y)|^2}{|x-y|^{n+2s}} dxdy \\
&\quad \leq 2\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|\varphi(x) - \varphi(y)|^2|u(x)|^2 + |u(x) - u(y)|^2|\varphi(y)|^2}{|x-y|^{n+2s}} dxdy \\
&\quad \leq \underbrace{2\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|\varphi(x) - \varphi(y)|^2|u(x)|^2}{|x-y|^{n+2s}} dxdy}_{\mathcal{I}_1} + \underbrace{2\|\varphi\|_{L^\infty(\mathbb{R}^n)} \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} dxdy}_{\mathcal{I}_2}\,.
\end{aligned}$$

Since $u \in H^s(\mathbb{R}^n)$, $\mathcal{I}_2 < \infty$. On the other hand,

$$\mathcal{I}_1 = \Big[\int_{\mathbb{R}^n}\int_{|x-y|\leq 1} + \int_{\mathbb{R}^n}\int_{|x-y|\geq 1}\Big] \frac{|\varphi(x) - \varphi(y)|^2|u(x)|^2}{|x-y|^{n+2s}} dxdy\,.$$

For the integral over $|x-y| \leq 1$, since $\varphi \in \mathcal{S}(\mathbb{R}^n)$, $|\varphi(x) - \varphi(y)| \leq C|x-y|$ for some constant C. Therefore,

$$\begin{aligned}
\int_{\mathbb{R}^n}\int_{|x-y|\leq 1} \frac{|\varphi(x) - \varphi(y)|^2|u(x)|^2}{|x-y|^{n+2s}} dxdy &\leq C\int_{\mathbb{R}^n}\int_{|x-y|\leq 1} |x-y|^{2-n-2s}|u(x)|^2 dxdy \\
&\leq C\int_{|z|\leq 1} |z|^{2-n-2s}dz \int_{\mathbb{R}^n} |u(x)|^2 dx < \infty \qquad \text{if } s < 1\,.
\end{aligned}$$

For the remaining integral,

$$\begin{aligned}
\int_{\mathbb{R}^n}\int_{|x-y|\geq 1} \frac{|\varphi(x) - \varphi(y)|^2|u(x)|^2}{|x-y|^{n+2s}} dxdy &\leq 4\|\varphi\|^2_{L^\infty(\mathbb{R}^n)} \int_{\mathbb{R}^n}\int_{|x-y|\leq 1} |x-y|^{-n-2s}|u(x)|^2 dxdy \\
&\leq 4\|\varphi\|^2_{L^\infty(\mathbb{R}^n)} \int_{|z|\geq 1} |z|^{-n-2s}dz \int_{\mathbb{R}}^n |u(x)|^2 < \infty \qquad \text{if } s > 0\,.
\end{aligned}$$

The general case of $s \geq 0$ can be proved in a similar fashion, and we leave the details to the reader. $\square$

4.2.2 The case that $s < 0$

For $s < 0$, we define the space $H^s(\mathbb{R}^n)$ to be the dual space of $H^{-s}(\mathbb{R}^n)$ with the corresponding dual space norm (or operator norm) defined by

$$\|u\|_{H^s(\mathbb{R}^n)} = \sup_{v \in H^{-s}(\mathbb{R}^n)} \frac{\langle u, v \rangle}{\|v\|_{H^{-s}(\mathbb{R}^n)}} = \sup_{\|v\|_{H^{-s}(\mathbb{R}^n)}=1} \langle u, v \rangle \,. \tag{4.4}$$

The norm defined in (4.4) is equivalent to

$$\|u\|_{H^s(\mathbb{R}^n)} = \Big[\int_{\mathbb{R}^n} (1 + |\xi|^{2s}) |\hat{u}(\xi)|^2 d\xi \Big]^{\frac{1}{2}} \,.$$

5 Fractional-order Sobolev spaces on domains with boundary

5.1 The space $H^s(\mathbb{R}^n_+)$

Let $\mathbb{R}^n_+ = \mathbb{R}^{n-1} \times \mathbb{R}_+$ denote the upper half space of $\mathbb{R}^n$.

5.1.1 The case $s = k \in \mathbb{N}$

The space $H^k(\mathbb{R}^n_+)$ is the collection of all $L^2(\mathbb{R}^n_+)$-functions so that the α-th weak derivatives belong to $L^2(\mathbb{R}^n_+)$ for all $|\alpha| \le k$, that is,

$$H^k(\mathbb{R}^n_+) = \left\{ u \in L^2(\mathbb{R}^n_+) \;\middle|\; D^\alpha u \in L^2(\mathbb{R}^n_+) \;\forall\; |\alpha| \le k \right\}$$

with norm

$$\|u\|^2_{H^k(\mathbb{R}^n_+)} = \sum_{|\alpha| \le k} \|D^\alpha u\|^2_{L^2(\mathbb{R}^n_+)} \,. \tag{5.1}$$

Note that we are not able to directly use the Fourier transform to define the $H^k(\mathbb{R}^n_+)$.

Definition 5.1 (Extension operator E)**.** *Fix $N \in \mathbb{N}$. Let $(a_1, \cdots, a_N)$ solve*

$$\sum_{j=1}^N (-j)^\ell a_j = 1 \,, \qquad \ell = 0, \cdots, N-1 \,.$$

We denote by $E : C(\overline{\mathbb{R}^n_+}) \to C(\mathbb{R}^n)$ the function

$$(Eu)(x) = \begin{cases} u(x) & \text{if} \quad x_n \ge 0 \,, \\ \sum_{j=1}^N a_j u(x', -jx_n) & \text{if} \quad x_n < 0 \,. \end{cases} \tag{5.2}$$

Note that the the coefficients a_j solve a linear system of N equations for N unknowns which is always solvable since the determinant never vanishes.

Theorem 5.2 (Sobolev extension theorem)**.** *The operator E has a continuous extension to an operator $E : H^k(\mathbb{R}^n_+) \to H^k(\mathbb{R}^n)$, $k \le N-1$, with N defined in (5.2).*

Proof. We must show that all derivatives of u of order not bigger than $N-1$ are continuous at $x_n = 0$. We compute $D^\ell_{x_n} Eu$:

$$D^\ell_{x_n}(Eu)(x) = \begin{cases} D^\ell_{x_n} u(x) & \text{if} \quad x_n > 0 \,, \\ \sum_{j=1}^N (-j)^\ell a_j (D^\ell_{x_n} u)(x', -jx_n) & \text{if} \quad x_n < 0 \,. \end{cases}$$

By the definition of a_j, $\lim_{x_1\to 0^+} D^\ell_{x_1}(Eu)(x) = \lim_{x_1\to 0^-} D^\ell_{x_1}(Eu)(x)$. So $Eu \in H^k(\mathbb{R}^n)$. Finally, the continuity of E is concluded by the following inequality:

$$\|Eu\|_{H^k(\mathbb{R}^n)} \le C\|u\|_{H^k(\mathbb{R}^n_+)} .$$

□

Lemma 5.3. *For $k \in \mathbb{N}$, each $u \in H^k(\mathbb{R}^n_+)$ is the restriction of some $w \in H^k(\mathbb{R}^n)$ to $\mathbb{R}^n_+$, that is, $u = w|_{\mathbb{R}^n_+}$.*

Proof. We define the restriction map $\varrho : H^k(\mathbb{R}^n) \to H^k(\mathbb{R}^n_+)$. By Theorem 5.2, the restriction map is onto, since $\varrho E = \mathrm{Id}$ on $H^k(\mathbb{R}^n_+)$. □

5.1.2 The case $s \notin \mathbb{N}$

Next, suppose that $N-2 < s < N-1$ for some $N \in \mathbb{N}$ given in (5.2), and let E continue to denote the Sobolev extension operator.

We define the space $H^s(\mathbb{R}^n_+)$ as the restriction of $H^s(\mathbb{R}^n)$ to $\mathbb{R}^n_+$ with norm

$$\|u\|_{H^s(\mathbb{R}^n_+)} \equiv \|Eu\|_{H^s(\mathbb{R}^n)} . \tag{5.3}$$

When $s = k \in \mathbb{N}$, it may not be immediately clear that the $H^s(\mathbb{R}^n_+)$-norm defined by (5.3) is equivalent to the $H^k(\mathbb{R}^n_+)$-norm defined by (5.1). Let $\|\cdot\|_1$ be the norm defined by (5.1) and $\|\cdot\|_2$ be the norm defined by (5.3). It is clear that $\|u\|_1 \le \|u\|_2$, and by the continuity of E, $\|u\|_2 \le C\|u\|_1$; therefore, $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent if $s \in \mathbb{N}$.

5.2 The Sobolev space $H^s(\Omega)$

We can now define the Sobolev spaces $H^s(\Omega)$ for any open and bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary $\partial\Omega$.

Definition 5.4 (Smoothness of the boundary)**.** *We say $\partial\Omega$ is C^k if for each point $x_0 \in \partial\Omega$ there exist $r > 0$ and a C^k-function $\gamma : \mathbb{R}^{n-1} \to \mathbb{R}$ such that - upon relabeling and reorienting the coordinates axes if necessary - we have*

$$\Omega \cap B(x_0, r) = \{x \in B(x_0, r) \mid x_n > \gamma(x_1, \cdots, x_{n-1})\} .$$

$\partial\Omega$ is C^∞ if $\partial\Omega$ is C^k for all $k \in \mathbb{N}$, and Ω is said to have smooth boundary if $\partial\Omega$ is C^∞.

Definition 5.5 (Partition of unity)**.** *Let X be a topological space. A partition of unity is a collection of continuous functions $\{\chi_j : X \to [0,1]\}$ such that $\sum_j \chi_i(x) = 1$ for all $x \in X$. A partition of unity is locally finite if each x in X is contained in an open set on which only a finite number of χ_j are non-zero. A partition of unity is subordinate to an open cover $\{\mathcal{U}_j\}$ of X if each χ_i is zero on the complement of $\mathcal{U}_j$.*

For a domain Ω with smooth boundary, we may assume that there exist $x_1, \cdots x_N \in \partial\Omega$, $r_1, \cdots r_N > 0$, $\gamma_j \in C^\infty$ such that, upon relabeling and reorienting the coordinates axes if necessary,

$$\Omega \cap \mathcal{U}_j = \{x \in \mathcal{U}_j \mid x_n > \gamma_j(x_1, \cdots, x_{n-1})\} \text{ where } \mathcal{U}_j = B(x_j, r_j)\,,$$

and $\Omega \subseteq \bigcup_{j=0}^N \mathcal{U}_j$, and $\{\chi_j\}_{j=0}^N$ is a partition of unity subordinate to the open cover $\{\mathcal{U}_j\}$ such that $\chi_j \in C_c^\infty(\mathcal{U}_j)$, and the function ψ_j defined by

$$\psi_j(x) = (x_1, \cdots, x_{n-1}, \gamma_j(x_1, \cdots, x_{n-1}) + x_n)\,.$$

is a diffeomorphism between a small neighborhood $\mathcal{V}_j$ of $\mathbb{R}^n$ and $\text{supp}(\chi_j)$.

Let $v_0 = \chi_0 u$ and $v_j = (\chi_j u) \circ \psi_j$. Then v_0 can be considered as a function defined on $\mathbb{R}^n$, and v_j can be considered as a function defined on $\mathbb{R}^n_+$. We then have the following definition.

Definition 5.6. *The space $H^s(\Omega)$ for $s > 0$ is the collection of all measurable functions u such that $\chi_0 u \in H^s(\mathbb{R}^n)$ and $(\chi_j u) \circ \psi_j \in H^s(\mathbb{R}^n_+)$. The $H^s(\Omega)$-norm is defined by*

$$\|u\|_{H^s(\Omega)} = \left[\|\chi_0 u\|^2_{H^s(\mathbb{R}^n)} + \sum_{j=1}^N \|(\chi_j u) \circ \psi_j\|^2_{H^s(\mathbb{R}^n_+)}\right]^{1/2}.$$

Theorem 5.7 (Extension)**.** *Let Ω be a bounded, smooth domain. For any open set V such that $\Omega \subset\subset U$, there exists a bounded linear operator $E : H^s(\Omega) \to H^s(\mathbb{R}^n)$ such that*

(i) $Eu = u$ *a.e. in* Ω *,*

(ii) Eu *has support within* V *,*

(iii) $\|Eu\|_{H^s(\mathbb{R}^n)} \le C\|u\|_{H^s(\Omega)}$ *, where the constant C depends only on s, Ω and V.*

Proof. Define

$$Eu = \chi_0 u + \sum_{j=1}^N \sqrt{\chi_j}\Big[E[(\sqrt{\chi_j}u) \circ \psi_j]\Big] \circ \psi_j^{-1}\,,$$

where $E : H^k(\mathbb{R}^n_+) \to H^k(\mathbb{R}^n)$ is the continuous extension defined by (5.2) for some $k \ge s$. One more constraint, $\text{supp}(\chi_j) \subseteq V$, must be imposed on χ_j for all j because of (ii), while this constraint is easily satisfied if we let $r_j \le \text{dist}(\Omega, \partial U)$ for all j. $\square$

6 The Sobolev Spaces $H^s(\mathbb{T}^n)$, $s \in \mathbb{R}$

6.1 The Fourier Series: Revisited

Definition 6.1. *For $u \in L^1(\mathbb{T}^n)$, define*

$$\mathcal{F}u(k) = \hat{u}_k = (2\pi)^{-n} \int_{\mathbb{T}_n} e^{-ik\cdot x} u(x) dx \,,$$

and

$$\mathcal{F}^* \hat{u}(x) = \sum_{k \in \mathbb{Z}^n} \hat{u}_k e^{ik\cdot x} \,.$$

Note that $\mathcal{F} : L^1(\mathbb{T}^n) \to l^\infty(\mathbb{Z}^n)$. If u is sufficiently smooth, then integration by parts yields

$$\mathcal{F}(D^\alpha u) = -(-i)^{|\alpha|} k^\alpha \hat{u}_k, \quad k^\alpha = k_1^{\alpha_1} \cdots k_n^{\alpha_n} \,.$$

Example 6.2. *Suppose that $u \in C^1(\mathbb{T}^n)$. Then for $j \in \{1, ..., n\}$,*

$$\begin{aligned} \mathcal{F}\left[\frac{\partial u}{\partial x_j}\right](k) &= (2\pi)^{-n} \int_{\mathbb{T}^n} \frac{\partial u}{\partial x_j} e^{-ik\cdot x} dx \\ &= -(2\pi)^{-n} \int_{\mathbb{T}^n} u(x) \left(-ik_j\right) e^{-ik\cdot x} dx \\ &= ik_j \hat{u}_k \,. \end{aligned}$$

Note that $\mathbb{T}^n$ is a closed manifold without boundary; alternatively, one may identify $\mathbb{T}^n$ with the $[0,1]^n$ with periodic boundary conditions, i.e., with opposite faces identified.

Definition 6.3. *Let $\mathfrak{s} = \mathcal{S}(\mathbb{Z}^n)$ denote the space of rapidly decreasing functions $\hat{u}$ on $\mathbb{Z}^n$ such that for each $N \in \mathbb{N}$,*

$$p_N(u) = \sup_{k \in \mathbb{Z}^n} \langle k \rangle^N |\hat{u}_k| < \infty \,,$$

where $\langle k \rangle = \sqrt{1 + |k|^2}$.

Then

$$\mathcal{F} : C^\infty(\mathbb{T}^n) \to \mathfrak{s} \,, \quad \mathcal{F}^* : \mathfrak{s} \to C^\infty(\mathbb{T}^n) \,,$$

and $\mathcal{F}^*\mathcal{F} = \text{Id}$ on $C^\infty(\mathbb{T}^n)$ and $\mathcal{F}\mathcal{F}^* = \text{Id}$ on $\mathfrak{s}$. These properties smoothly extend to the Hilbert space setting:

$$\begin{array}{ll} \mathcal{F} : L^2(\mathbb{T}^n) \to l^2 & \mathcal{F}^* : l^2 \to L^2(\mathbb{T}^n) \\ \mathcal{F}^*\mathcal{F} = \text{Id on } L^2(\mathbb{T}^n) & \mathcal{F}\mathcal{F}^* = \text{Id on } l^2 \,. \end{array}$$

Definition 6.4. *The inner-products on* $L^2(\mathbb{T}^n)$ *and* l^2 *are*

$$(u,v)_{L^2(\mathbb{T}^n)} = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{T}^n} u(x)\overline{v(x)}dx$$

and

$$(\hat{u},\hat{v})_{l^2} = \sum_{k\in\mathbb{Z}^n} \hat{u}_k \overline{\hat{v}_k}\,,$$

respectively.

Parseval's identity shows that $\|u\|_{L^2(\mathbb{T}^n)} = \|\hat{u}\|_{l^2}$.

Definition 6.5. *We set*

$$\mathcal{D}'(\mathbb{T}^n) = [C^\infty(\mathbb{T}^n)]' \quad \textit{and} \quad \mathfrak{s}' = [\mathfrak{s}]'\,.$$

The space $\mathcal{D}'(\mathbb{T}^n)$ *is termed the space of periodic distributions.*

In the same manner that we extended the Fourier transform from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$ by duality, we may produce a similar extension to the periodic distributions:

$$\begin{array}{ll} \mathcal{F}: \mathcal{D}'(\mathbb{T}^n) \to \mathfrak{s}' & \mathcal{F}^*: \mathfrak{s}' \to \mathcal{D}'(\mathbb{T}^n) \\ \mathcal{F}^*\mathcal{F} = \text{Id on } \mathcal{D}'(\mathbb{T}^n) & \mathcal{F}\mathcal{F}^* = \text{Id on } \mathfrak{s}'\,. \end{array}$$

Definition 6.6 (Sobolev spaces $H^s(\mathbb{T}^n)$)**.** *For all* $s \in \mathbb{R}$*, the Hilbert spaces* $H^s(\mathbb{T}^n)$ *are defined as follows:*

$$H^s(\mathbb{T}^n) = \{u \in \mathcal{D}'(\mathbb{T}^n) \mid \|u\|_{H^s(\mathbb{T}^n)} < \infty\}\,,$$

where the norm on $H^s(\mathbb{T}^n)$ *is defined as*

$$\|u\|^2_{H^s(\mathbb{T}^n)} = \sum_{k\in\mathbb{Z}^n} |\hat{u}_k|^2 \langle k\rangle^{2s}\,.$$

The space $(H^s(\mathbb{T}^n), \|\cdot\|_{H^s(\mathbb{T}^n)})$ is a Hilbert space, and we have that

$$H^{-s}(\mathbb{T}^n) = [H^s(\mathbb{T}^n)]'\,.$$

For any $s \in \mathbb{R}$, we define the operator Λ^s as follows: for $u \in \mathcal{D}'(\mathbb{T}^n)$,

$$\Lambda^s u(x) = \sum_{k\in\mathbb{Z}^n} |\hat{u}_k|^2 \langle k\rangle^s e^{ik\cdot x}\,.$$

It follows that

$$H^s(\mathbb{T}^n) = \Lambda^{-s}\, L^2(\mathbb{T}^n)\,,$$

and for $r, s \in \mathbb{R}$,

$$\Lambda^s : H^r(\mathbb{T}^n) \to H^{r-s}(\mathbb{T}^n) \text{ is an isomorphism}.$$

Notice then that for any $\epsilon > 0$,

$$\Lambda^{-\epsilon} : H^s(\mathbb{T}^n) \to H^s(\mathbb{T}^n) \text{ is a compact operator},$$

as it is an operator-norm limit of finite-rank operators. (In particular, the eigenvalues of $\Lambda^{-\epsilon}$ tend to zero in this limit.) Hence, the inclusion map $H^{s+\epsilon}(\mathbb{T}^n) \hookrightarrow H^s(\mathbb{T}^n)$ is compact, and we have the following

Theorem 6.7 (Rellich's theorem on $\mathbb{T}^n$)**.** *Suppose that a sequence u_j satisfies for $s \in \mathbb{R}$ and $\epsilon > 0$,*

$$\sup \|u_j\|_{H^{s+\epsilon}(\mathbb{T}^n)} \le M < \infty \quad \textit{for a constant } M \neq M(j)\,.$$

Then there exists a subsequence $u_{j_k} \to u$ in $H^s(\mathbb{T}^n)$.

6.2 The Poisson Integral Formula and the Laplace operator

For $f : \mathbb{S}^1 \to \mathbb{R}$, denote by $\text{PI}(f)(r,\theta)$ the harmonic function on the unit disk $D = \{x \in \mathbb{R}^2 \,:\, |x| < 1\}$ with trace f:

$$\begin{aligned} \Delta\,\text{PI}(f) &= 0 \ \text{ in } \ D \\ \text{PI}(f) &= f \ \text{ on } \ \partial D = \mathbb{S}^1\,. \end{aligned}$$

$\text{PI}(f)$ has an explicit representation via the Fourier series

$$\text{PI}(f)(r,\theta) = \sum_{k\in\mathbb{Z}} \hat{f}_k r^{|k|} e^{ik\theta} \quad r < 1, 0 \le \theta < 2\pi\,, \tag{6.1}$$

as well as the integral representation

$$\text{PI}(f)(r,\theta) = \frac{1-r^2}{2\pi} \int_{\mathbb{S}^1} \frac{f(\phi)}{r^2 - 2r\cos(\theta-\phi)+1} d\phi \quad r < 1, 0 \le \theta < 2\pi\,. \tag{6.2}$$

The dominated convergence theorem shows that if $f \in C^0(\mathbb{S}^1)$, then $\text{PI}(f) \in C^\infty(D) \cap C^0(\overline{D})$.

Theorem 6.8. PI *extends to a continuous map from $H^{k-\frac{1}{2}}(\mathbb{S}^1)$ to $H^k(D)$ for all $k \in \mathbb{Z}_+$.*

Proof. Define $u = \text{PI}(f)$.
Step 1. The case that $k = 0$**.** Assume that $f \in H^{-\frac{1}{2}}(\Gamma)$ so that

$$\sum_{k\in\mathbb{Z}} |\hat{f}_k|^2 \langle k\rangle^{-1} \le M_0 < \infty\,.$$

Since the functions $\{r^{|k|}e^{ik\theta} \ : \ k \in \mathbb{Z}\}$ are orthogonal with respect to the $L^2(D)$ inner-product,

$$\|u\|^2_{L^2(D)} = \int_0^{2\pi}\int_0^1 \left|\sum_{k\in\mathbb{Z}} \hat{f}_k r^{|k|} e^{ik\theta}\right|^2 r\,dr\,d\theta$$
$$\leq 2\pi \sum_{k\in\mathbb{Z}} |\hat{f}_k|^2 \int_0^1 r^{2|k|+1} dr = \pi \sum_{k\in\mathbb{Z}} |\hat{f}_k|^2 (1+|k|)^{-1} \leq \pi \|f\|^2_{H^{\frac{1}{2}}(\mathbb{S}^1)},$$

where we have used the monotone convergence theorem for the first inequality.
Step 2. The case that $k = 1$. Next, suppose that $f \in H^{\frac{1}{2}}(\Gamma)$ so that

$$\sum_{k\in\mathbb{Z}} |\hat{f}_k|^2 \langle k \rangle^1 \leq M_1 < \infty\,.$$

Since we have shown that $u \in L^2(D)$, we must now prove that $u_\theta = \partial_\theta u$ and $u_r = \partial_r u$ are both in $L^2(D)$. Notice that by definition of the Fourier transform and (6.1),

$$\frac{\partial}{\partial\theta}\,\mathrm{PI}(f) = \mathrm{PI}(f_\theta)\,. \tag{6.3}$$

By definition, $\partial_\theta : H^{\frac{1}{2}}(\mathbb{S}^1) \to H^{-\frac{1}{2}}(\mathbb{S}^1)$ continuously, so that for some constant C,

$$\|f_\theta\|_{H^{-\frac{1}{2}}(\mathbb{S}^1)} \leq C\|f\|_{H^{\frac{1}{2}}(\mathbb{S}^1)}\,.$$

It follows from the analysis of **Step 1** and (6.3) that (with $u = \mathrm{PI}(f)$),

$$\|u_\theta\|_{L^2(D)} \leq C\|f\|_{H^{\frac{1}{2}}(\mathbb{S}^1)}\,.$$

Next, using the identity (6.1) notice that $|ru_r| = |u_\theta|$. It follows that

$$\|ru_r\|_{L^2(D)} \leq C\|f\|_{H^{\frac{1}{2}}(\mathbb{S}^1)}\,. \tag{6.4}$$

By the interior regularity of $-\Delta$ proven in Theorem 7.1, $u_r(r,\theta)$ is smooth on $\{r < 1\}$; hence the bound (6.4) implies that, in fact,

$$\|u_r\|_{L^2(D)} \leq C\|f\|_{H^{\frac{1}{2}}(\mathbb{S}^1)}\,,$$

and hence

$$\|u\|_{H^1(D)} \leq C\|f\|_{H^{\frac{1}{2}}(\mathbb{S}^1)}\,,$$

Step 3. The case that $k \geq 2$. Since $f \in H^{k-\frac{1}{2}}(\mathbb{S}^1)$, it follows that

$$\|\partial_\theta^k f\|_{H^{-\frac{1}{2}}(\mathbb{S}^1)} \leq C\|f\|_{H^{k-\frac{1}{2}}(\mathbb{S}^1)}$$

and by repeated application of (6.3), we find that

$$\|u\|_{H^k(D)} \leq C\|f\|_{H^{k-\frac{1}{2}}(\mathbb{S}^1)}.$$

□

The Hölder spaces on $\overline{D}$ are defines as follows: if $u : D \to \mathbb{R}$ is bounded and continuous, we write

$$\|u\|_{C(\overline{D})} := \sup_{x \in D} |u(x)|.$$

For $0 < \alpha \leq 1$, the α^{th}-Hölder seminorm of u is

$$[u]_{C^{0,\alpha}(\overline{D})} := \sup_{x,y \in D, x \neq y} \left\{ \frac{|u(x) - u(y)|}{|x-y|^\alpha} \right\}$$

and the α^{th}-Hölder norm of u is

$$\|u\|_{C^{0,\alpha}(\overline{D})} = \|u\|_{C(\overline{D})} + [u]_{C^{0,\alpha}(\overline{D})}.$$

According to Morrey's inequality, if $f \in H^{3/2}(S^1)$ then for $0 < \alpha < 1$, $f \in C^{0,\alpha}(\mathbb{S}^1)$. Next, we use the result of Problem 6.3, together with Morrey's inequality (once again) and Theorem 2.30 to prove that $u \in C^{0,\alpha}(\overline{D})$. Let us explain this.

We first prove the following:

$$f \in H^{3/2}(\mathbb{S}^1) \text{ implies that } f \in H^{1/2+\alpha}(\mathbb{S}^1) \text{ for } \alpha \in (0,1) \text{ which implies that } f \in C^{0,\alpha}(\mathbb{S}^1),$$

the last assertion meaning that $|f(x+y) - f(x)| \leq C|y|^\alpha$.

We start with the identity

$$\begin{aligned} |f(x+y) - f(y)| &= \left| \sum_{k \in \mathbb{Z}} \hat{f}_k e^{ikx}(e^{iky} - 1) \right| \\ &= \left| \sum_{k \neq 0} \hat{f}_k e^{ikx}(e^{iky} - 1) \right| \\ &\leq \left(\sum_{k \neq 0} |\hat{f}_k|^2 \langle k \rangle^{1+2\alpha} \right)^{\frac{1}{2}} \left(\sum_{k \neq 0} |e^{iky} - 1|^2 \langle k \rangle^{-1-2\alpha} \right)^{\frac{1}{2}} \\ &= \|f\|_{H^{1/2+\alpha}} \left(\sum_{k \neq 0} |e^{iky} - 1|^2 \langle k \rangle^{-1-2\alpha} \right)^{\frac{1}{2}}. \end{aligned}$$

We consider $|y| \le \frac{1}{2}$ and break the sum into two parts:

$$\sum_{k\neq 0} |e^{iky}-1|^2 \langle k\rangle^{-1-2\alpha} = \sum_{|k|\le \frac{1}{|y|}} |e^{iky}-1|^2 \langle k\rangle^{-1-2\alpha} + \sum_{|k|\ge \frac{1}{|y|}} |e^{iky}-1|^2 \langle k\rangle^{-1-2\alpha}.$$

For the second sum, we use that $|e^{iky}-1|^2 \le 4$ and employ the integral test to see that $\int_{1/|y|}^\infty r^{-1-2\alpha}dr \le C|y|^{2\alpha}$. For the first sum, we Taylor expand about $y=0$: $e^{iky}-1 = iky + O(y^2)$. Once again, we employ the integral test:

$$\sum_{|k|\le \frac{1}{|y|}} |e^{iky}-1|^2 \langle k\rangle^{-1-2\alpha} \le |y|^2 + \int_1^{1/|y|} |y|^2 r^2 r^{-1-2\alpha} dr \le C(|y|^2 + |y|^{2\alpha})\,.$$

Since $|y| \le 1/2$, we see that

$$\sum_{k\neq 0} |e^{iky}-1|^2 \langle k\rangle^{-1-2\alpha} \le C|y|^\alpha$$

as $\alpha < 1$.

Next, according to Theorem 6.8, if $f \in H^{3/2}(\mathbb{S}^1)$, then u solves $-\Delta u = 0$ in D with $u = f$ on ∂D, and $\|u\|_{H^2(D)} \le C\|f\|_{H^{3/2}(\mathbb{S}^1)}$. By Theorem 2.30,

$$\|Du\|_{L^q(D)} \le C\sqrt{q}\|u\|_{H^2(D)} \quad \forall q \in [1,\infty).$$

Hence, by Morrey's inequality, we see that $u \in C^{0,1-2/q}(D)$, and thus in $C^{0,\alpha}(D)$ for $\alpha \in (0,1)$.

6.3 Exercises

Problem 6.1. *Let $D := B(0,1) \subset \mathbb{R}^2$ and let let u satisfy the Neumann problem*

$$\Delta u = 0 \text{ in } D\,, \tag{6.5a}$$

$$\frac{\partial u}{\partial r} = g \text{ in } \partial D := \mathbb{S}^1\,. \tag{6.5b}$$

If $u = PI(f) := \sum_{k\in\mathbb{Z}} \hat f_k r^{|k|} e^{ik\theta}$, show that for $f \in H^{3/2}(\mathbb{S}^1)$,

$$g = Nf, \tag{6.6}$$

which is the same as

$$\hat g_k = |k| \hat f_k\,.$$

N denotes the Dirichlet to Neumann map given by $Nf(\theta) = \sum_{k\in\mathbb{Z}} \hat f_k |k| e^{ik\theta}$ or $Nf = -i\frac{\partial}{\partial\theta} Hf = -iH\frac{\partial f}{\partial \theta}$, where H is the Hilbert transform, defined by $Hu(\theta) = \sum_{k\in\mathbb{Z}} (\operatorname{sgn} k)\hat g_k e^{ik\theta}$.

Problem 6.2. *Define the function* $K(\theta) = \sum_{k\neq 0} |k|^{-1} e^{ik\theta}$. *Show that* $K \in L^2(\mathbb{S}^1) \subset L^1(\mathbb{S}^1)$. *Next, show that if* $g \in L^2(\mathbb{S}^1)$ *and* $\int_{\mathbb{S}^1} g(\theta)d\theta = 0$, *a solution to (6.6) is given by* $f(\theta) = (2\pi)^{-1} \int_{\mathbb{S}^1} K(\theta - \phi)g(\phi)d\phi$.

Problem 6.3. *Consider the solution to the Neumann problem (6.5a) and (6.5b). Show that* $g \in H^{1/2}(\mathbb{S}^1)$ *implies that* $u \in H^2(D)$ *and that*

$$\|u\|^2_{H^2(D)} \le C \left(\|g\|^2_{H^{1/2}(\mathbb{S}^1)} + \|u\|^2_{L^2(D)} \right) .$$

7 Regularity of the Laplacian on Ω

We have studied the regularity properties of the Laplace operator on $D = B(0,1) \subset \mathbb{R}^2$ using the Poisson integral formula. These properties continue to hold on more general open, bounded, C^∞ subsets Ω of $\mathbb{R}^n$.

We revisit the Dirichlet problem

$$\Delta u = 0 \quad \text{in} \quad \Omega\,, \tag{7.1a}$$
$$u = f \quad \text{on} \quad \partial\Omega\,. \tag{7.1b}$$

Theorem 7.1. *For $k \in \mathbb{N}$, given $f \in H^{k-\frac{1}{2}}(\partial\Omega)$, there exists a unique solution $u \in H^k(\Omega)$ to (7.1) satisfying*

$$\|u\|_{H^k(\Omega)} \le C\|f\|_{H^{k-\frac{1}{2}}(\partial\Omega)}\,, \quad C = C(\Omega)\,.$$

Proof. **Step 1.** $k = 1$**.** We begin by converting (7.1) to a problem with homogeneous boundary conditions. Using the surjectivity of the trace operator provided by Theorem 4.15, there exists $F \in H^1(\Omega)$ such that $T(F) = f$ on $\partial\Omega$, and $\|F\|_{H^1(\Omega)} \le C\|f\|_{H^{\frac{1}{2}}(\partial\Omega)}$. Let $U = u - F$; then $U \in H^1(\Omega)$ and by linearity of the trace operator, $T(U) = 0$ on $\partial\Omega$. It follows from Theorem 2.35 that $U \in H^1_0(\Omega)$ and satisfies $-\Delta U = \Delta F$ in $H^1_0(\Omega)$; that is $\langle -\Delta U, v\rangle = \langle \Delta F, v\rangle$ for all $v \in H^1_0(\Omega)$.

According to Remark 2.45, $-\Delta : H^1_0(\Omega) \to H^{-1}(\Omega)$ is an isomorphism, so that $\Delta F \in H^{-1}(\Omega)$; therefore, by Theorem 2.44, there exists a unique weak solution $U \in H^1_0(\Omega)$, satisfying

$$\int_\Omega DU \cdot Dv\, dx = \langle \Delta F, v\rangle \quad \forall v \in H^1_0(\Omega)\,,$$

with

$$\|U\|_{H^1(\Omega)} \le C\|\Delta F\|_{H^{-1}(\Omega)}\,, \tag{7.2}$$

and hence

$$u = U + F \in H^1(\Omega) \quad \text{and} \quad \|u\|_{H^1(\Omega)} \le \|f\|_{H^{\frac{1}{2}}(\partial\Omega)}\,.$$

Step 2. $k = 2$**.** Next, suppose that $f \in H^{1.5}(\partial\Omega)$. Again employing Theorem 4.15, we obtain $F \in H^2(\Omega)$ such that $T(F) = f$ and $\|F\|_{H^2(\Omega)} \le C\|f\|_{H^{1.5}(\partial\Omega)}$; thus, we see that $\Delta F \in L^2(\Omega)$ and that, in fact,

$$\int_\Omega DU \cdot Dv\, dx = \int_\Omega \Delta F\, v\, dx \quad \forall v \in H^1_0(\Omega)\,. \tag{7.3}$$

We first establish interior regularity. Choose *any* (nonempty) open sets $\Omega_1 \subset\subset \Omega_2 \subset\subset \Omega$ and let $\zeta \in C^\infty_0(\Omega_2)$ with $0 \le \zeta \le 1$ and $\zeta = 1$ on Ω_1. Let $\epsilon_0 = \min \operatorname{dist}(\operatorname{spt}(\zeta), \partial\Omega_2)/2$. For all $0 < \epsilon < \epsilon_0$, define $U^\epsilon(x) = \eta_\epsilon * U(x)$ for all $x \in \Omega_2$, and set

$$v = -\eta_\epsilon * (\zeta^2 U^\epsilon,_j),_j\,.$$

Then $v \in H^1_0(\Omega)$ and can be used as a test function in (7.3); thus,

$$-\int_\Omega U,_i \ \eta_\epsilon * (\zeta^2 U^\epsilon,_j),_{ji} \ dx = -\int_\Omega U,_i \ \eta_\epsilon * [\zeta^2 U^\epsilon,_{ij} + 2\zeta\zeta,_i U^\epsilon,_j],_j \ dx$$
$$= \int_{\Omega_2} \zeta^2 U^\epsilon,_{ij} U^\epsilon,_{ij} \ dx - 2\int_\Omega \eta_\epsilon * [\zeta\zeta,_i U^\epsilon,_j],_j \ U,_i \ dx \,,$$

and

$$\int_\Omega \Delta F\, v\, dx = -\int_{\Omega_2} \Delta F\, \eta_\epsilon * (\zeta^2 U^\epsilon,_j),_j \ dx = -\int_{\Omega_2} \Delta F\, \eta_\epsilon * [\zeta^2 U^\epsilon,_{jj} + 2\zeta\zeta,_j U^\epsilon,_j]\, dx \,.$$

By Young's inequality (Theorem 1.53),

$$\|\eta_\epsilon * [\zeta^2 U^\epsilon,_{jj} + 2\zeta\zeta,_j U^\epsilon,_j]\|_{L^2(\Omega_2)} \le \|\zeta^2 U^\epsilon,_{jj} + 2\zeta\zeta,_j U^\epsilon,_j\|_{L^2(\Omega_2)};$$

hence, by the Cauchy-Young inequality with δ, Lemma 1.52, for $\delta > 0$,

$$\int_\Omega \Delta F\, v\, dx \le \delta \|\zeta D^2 U^\epsilon\|^2_{L^2(\Omega_2)} + C_\delta [\|DU^\epsilon\|^2_{L^2(\Omega_2)} + \|\Delta F\|^2_{L^2(\Omega)}] \,.$$

Similarly,

$$2\int_\Omega \eta_\epsilon * [\zeta\zeta,_i U^\epsilon,_j],_j \ U,_i \ dx \le \delta \|\zeta D^2 U^\epsilon\|^2_{L^2(\Omega_2)} + C_\delta [\|DU^\epsilon\|^2_{L^2(\Omega_2)} + \|\Delta F\|^2_{L^2(\Omega)}] \,.$$

By choosing $\delta < 1$ and readjusting the constant C_δ, we see that

$$\|D^2 U^\epsilon\|^2_{L^2(\Omega_1)} \le \|\zeta D^2 U^\epsilon\|^2_{L^2(\Omega_2)} \le C_\delta [\|DU^\epsilon\|^2_{L^2(\Omega_2)} + \|\Delta F\|^2_{L^2(\Omega)}]$$
$$\le C_\delta \|\Delta F\|^2_{L^2(\Omega)} \,,$$

the last inequality following from (7.2), and Young's inequality.

Since the right-hand side does not depend on $\epsilon > 0$, there exists a subsequence

$$D^2 U^{\epsilon'} \rightharpoonup \mathcal{W} \ \text{ in } \ L^2(\Omega_1) \,.$$

By Theorem 2.18, $U^\epsilon \to U$ in $H^1(\Omega_1)$, so that $\mathcal{W} = D^2 U$ on Ω_1. As weak convergence is lower semi-continuous, $\|D^2 U\|_{L^2(\Omega_1)} \le C_\epsilon \|\Delta F\|_{L^2(\Omega)}$. As Ω_1 and Ω_2 are arbitrary, we have established that $U \in H^2_{\text{loc}}(\Omega)$ and that

$$\|U\|_{H^2_{\text{loc}}(\Omega)} \le C \|\Delta F\|_{L^2(\Omega)} \,.$$

For any $w \in H^1_0(\Omega)$, set $v = \zeta w$ in (7.3). Since $u \in H^2_{\text{loc}}(\Omega)$, we may integrate by parts to find that

$$\int_\Omega (-\Delta U - \Delta F)\, \zeta w\, dx = 0 \ \ \forall w \in H^1_0(\Omega) \,.$$

Since w is arbitrary, and the spt(ζ) can be chosen arbitrarily close to $\partial\Omega$, it follows that for all x in the interior of Ω, we have that

$$-\Delta U(x) = \Delta F(x) \quad \text{for almost every } x \in \Omega\,. \tag{7.4}$$

We proceed to establish the regularity of U all the way to the boundary $\partial\Omega$. Let $\{\mathcal{U}_l\}_{l=1}^K$ denote an open cover of Ω which intersects the boundary $\partial\Omega$, and let $\{\theta_l\}_{l=1}^K$ denote a collection of charts such that

$$\begin{aligned}
&\theta_l : B(0,r_l) \to \mathcal{U}_l \text{ is a } C^\infty \text{ diffeomorphism }\,,\\
&\det D\theta_l = 1\,,\\
&\theta_l(B(0,r_l)\cap\{x_n=0\}) \to \mathcal{U}_l \cap \partial\Omega\,,\\
&\theta_l(B(0,r_l)\cap\{x_n>0\}) \to \mathcal{U}_l \cap \Omega\,.
\end{aligned}$$

Let $0 \le \zeta_l \le 1$ in $C_0^\infty(\mathcal{U}_l)$ denote a partition of unity subordinate to the open covering $\mathcal{U}_l$, and define the horizontal convolution operator, smoothing functions defined on $\mathbb{R}^n$ in the first $1,...,n-1$ directions, as follows:

$$\rho_\epsilon *_h F(x_h,x_n) = \int_{\mathbb{R}^{n-1}} \rho_\epsilon(x_h-y_h)F(y_h,x_n)dy_h\,,$$

where $\rho_\epsilon(x_h) = \epsilon^{-(n-1)}\rho(x_h/\epsilon)$, ρ the standard mollifier on $\mathbb{R}^{n-1}$, and $x_h = (x_1,...,x_{n-1})$. Let α range from 1 to $n-1$, and substitute the test function

$$v = -\left(\rho_\epsilon *_h [(\zeta_l\circ\theta_l)^2\rho_\epsilon *_h (U\circ\theta_l),_\alpha],_\alpha\right)\circ\theta_l^{-1} \in H_0^1(\Omega)$$

into (7.3), and use the change of variables formula to obtain the identity

$$\int_{B_+(0,r_l)} A_i^k(U\circ\theta_l),_k\, A_i^j(v\circ\theta_l),_j\; dx = \int_{B_+(0,r_l)} (\Delta F)\circ\theta_l\, v\circ\theta_l\, dx\,, \tag{7.5}$$

where the C^∞ matrix $A(x) = [D\theta_l(x)]^{-1}$ and $B_+(0,r_l) = B(0,r_l)\cap\{x_n>0\}$. We define

$$U^l = U\circ\theta_l\,, \text{ and denote the horizontal convolution operator by } H_\epsilon = \rho_\epsilon *_h\,.$$

Then, with $\xi_l = \zeta_l\circ\theta_l$, we can rewrite the test function as

$$v\circ\theta_l = -H_\epsilon[\xi_l^2 H_\epsilon U^l,_\alpha],_\alpha\,.$$

Since differentiation commutes with convolution, we have that

$$(v\circ\theta_l),_j = -H_\epsilon(\xi_l^2 H_\epsilon U^l,_{j\alpha}),_\alpha - 2H_\epsilon(\xi_l\xi_l,_j\, H_\epsilon U^l,_\alpha),_\alpha\,,$$

and we can express the left-hand side of (7.5) as

$$\int_{B_+(0,r_l)} A_i^k (U\circ\theta_l),_k\, A_i^j (v\circ\theta_l),_j \; dx = \mathcal{I}_1 + \mathcal{I}_2\,,$$

where

$$\mathcal{I}_1 = -\int_{B_+(0,r_l)} A_i^j A_i^k U^l,_k\; H_\epsilon(\xi_l^2 H_\epsilon U^l,_{j\alpha}),_\alpha\; dx\,,$$

$$\mathcal{I}_2 = -2\int_{B_+(0,r_l)} A_i^j A_i^k U^l,_k\; H_\epsilon(\xi_l \xi_l,_j\, H_\epsilon U^l,_\alpha),_\alpha\; dx\,.$$

Next, we see that

$$\mathcal{I}_1 = \int_{B_+(0,r_l)} [H_\epsilon(A_i^j A_i^k U^l,_k)],_\alpha\; (\xi_l^2 H_\epsilon U^l,_{j\alpha})\, dx = \mathcal{I}_{1a} + \mathcal{I}_{1b}\,,$$

where

$$\mathcal{I}_{1a} = \int_{B_+(0,r_l)} (A_i^j A_i^k H_\epsilon U^l,_k),_\alpha\; \xi_l^2 H_\epsilon U^l,_{j\alpha}\; dx\,,$$

$$\mathcal{I}_{1b} = \int_{B_+(0,r_l)} ([H_\epsilon, A_i^j A_i^k] U^l,_k),_\alpha\; \xi_l^2 H_\epsilon U^l,_{j\alpha}\; dx\,,$$

and where

$$[H_\epsilon, A_i^j A_i^k] U^l,_k = H_\epsilon(A_i^j A_i^k U^l,_k) - A_i^j A_i^k\; H_\epsilon U^l,_k \tag{7.6}$$

denotes the commutator of the horizontal convolution operator and multiplication. The integral $\mathcal{I}_{1a}$ produces the positive sign-definite term which will allow us to build the global regularity of U, as well as an error term:

$$\mathcal{I}_{1a} = \int_{B_+(0,r_l)} [\xi_l^2 A_i^j A_i^k H_\epsilon U^l,_{k\alpha}\; H_\epsilon U^l,_{j\alpha} + (A_i^j A_i^k),_\alpha\, H_\epsilon U^l,_k\; \xi_l^2 H_\epsilon U^l,_{j\alpha}]\, dx\,;$$

thus, together with the right hand-side of (7.5), we see that

$$\int_{B_+(0,r_l)} \xi_l^2 A_i^j A_i^k H_\epsilon U^l,_{k\alpha}\; H_\epsilon U^l,_{j\alpha}\; dx \le \left| \int_{B_+(0,r_l)} (A_i^j A_i^k),_\alpha\, H_\epsilon U^l,_k\; \xi_l^2 H_\epsilon U^l,_{j\alpha}]\, dx \right|$$

$$+ |\mathcal{I}_{1b}| + |\mathcal{I}_2| + \left| \int_{B_+(0,r_l)} (\Delta F)\circ\theta_l\, v\circ\theta_l\, dx \right|\,.$$

Since each θ_l is a C^∞ diffeomorphism, it follows that the matrix $A\,A^T$ is positive definite: there exists $\lambda > 0$ such that

$$\lambda |Y|^2 \le A_i^j A_i^k Y_j Y_k \quad \forall Y \in \mathbb{R}^n\,.$$

It follows that

$$\lambda \int_{B_+(0,r_l)} \xi_l^2 |\bar\partial D H_\epsilon U^l|^2 \, dx \le \left| \int_{B_+(0,r_l)} (A_i^j A_i^k),_\alpha H_\epsilon U^l,_k \; \xi_l^2 H_\epsilon U^l,_{j\alpha}] \, dx \right| + |\mathcal{I}_{1b}| + |\mathcal{I}_2| + \left| \int_{B_+(0,r_l)} (\Delta F)\circ\theta_l \, v\circ\theta_l \, dx \right| ,$$

where $D = (\partial_{x_1}, ..., \partial_{x_n})$ and $\bar p = (\partial_{x_1}, ..., \partial_{x_{n-1}})$. Application of the Cauchy-Young inequality with $\delta > 0$ shows that

$$\left| \int_{B_+(0,r_l)} (A_i^j A_i^k),_\alpha H_\epsilon U^l,_k \; \xi_l^2 H_\epsilon U^l,_{j\alpha}] \, dx \right| + |\mathcal{I}_2| + \left| \int_{B_+(0,r_l)} (\Delta F)\circ\theta_l \, v\circ\theta_l \, dx \right| \le \delta \int_{B_+(0,r_l)} \xi_l^2 |\bar\partial D H_\epsilon U^l|^2 \, dx + C_\delta \|\Delta F\|^2_{L^2(\Omega)} .$$

It remains to establish such an upper bound for $|\mathcal{I}_{1b}|$.

To do so, we first establish a pointwise bound for (7.6): for $\mathcal{A}^{jk} = A_i^j A_i^k$,

$$[H_\epsilon, A_i^j A_i^k] U^l,_k (x) = \int_{B(x_h,\epsilon)} \rho_\epsilon(x_h - y_h)[\mathcal{A}^{jk}(y_h, x_n) - \mathcal{A}^{jk}(x_h, x_n)] U^l,_k (y_h, x_n) \, dy_h$$

By Morrey's inequality, $|[\mathcal{A}^{jk}(y_h, x_n) - \mathcal{A}^{jk}(x_h, x_n)]| \le C\epsilon \|\mathcal{A}\|_{W^{1,\infty}(B_+(0,r_l))}$. Since

$$\partial_{x_\alpha} \rho_\epsilon(x_h - y_h) = \frac{1}{\epsilon^2} \rho' \left(\frac{x - h - y_h}{\epsilon} \right) ,$$

we see that

$$\left| \partial_{x_\alpha} \left([H_\epsilon, A_i^j A_i^k] U^l,_k \right)(x) \right| \le C \int_{B(x_h,\epsilon)} \frac{1}{\epsilon} \rho' \left(\frac{x - h - y_h}{\epsilon} \right) |U^l,_k (y_h, x_n)| \, dy_h$$

and hence by Young's inquality,

$$\left\| \partial_{x_\alpha} \left([H_\epsilon, A_i^j A_i^k] U^l,_k \right) \right\|_{L^2(B_+(0,r_l))} \le C \|U\|_{H^1(\Omega)} \le C \|\Delta F\|_{L^2(\Omega)} .$$

It follows from the Cauchy-Young inequality with $\delta > 0$ that

$$|\mathcal{I}_{1b}| \le \delta \int_{B_+(0,r_l)} \xi_l^2 |\bar\partial D H_\epsilon U^l|^2 \, dx + C_\delta \|\Delta F\|^2_{L^2(\Omega)} .$$

By choosing $2\delta < \lambda$, we obtain the estimate

$$\int_{B_+(0,r_l)} \xi_l^2 |\bar\partial D H_\epsilon U^l|^2 \, dx \le C_\delta \|\Delta F\|^2_{L^2(\Omega)} .$$

Since the right hand-side is independent of ϵ, we find that

$$\int_{B_+(0,r_l)} \xi_l^2 |\bar\partial D U^l|^2 \, dx \le C_\delta \|\Delta F\|^2_{L^2(\Omega)} \,. \tag{7.7}$$

From (7.4), we know that $\Delta U(x) = \Delta F(x)$ for a.e. $x \in \mathcal{U}_l$. By the chain-rule this means that almost everywhere in $B_+(0, r_l)$,

$$-\mathcal{A}^{jk} U^l,_{kj} = \mathcal{A}^{jk},_j U^l,_k + \Delta F \circ \theta_l \,,$$

or equivalently,

$$-\mathcal{A}^{nn} U^l_{nn} = \mathcal{A}^{j\alpha} U^l,_{\alpha j} + \mathcal{A}^{\beta k} U^l,_{k\beta} + \mathcal{A}^{jk},_j U^l,_k + \Delta F \circ \theta_l \,.$$

Since $\mathcal{A}^{nn} > 0$, it follows from (7.7) that

$$\int_{B_+(0,r_l)} \xi_l^2 |D^2 U^l|^2 \, dx \le C_\delta \|\Delta F\|^2_{L^2(\Omega)} \,. \tag{7.8}$$

Summing over l from 1 to K and combining with our interior estimates, we have that

$$\|u\|_{H^2(\Omega)} \le C \|\Delta F\|_{L^2(\Omega)} \,.$$

Step 3. $k \ge 3$**.** At this stage, we have obtained a pointwise solution $U \in H^2(\Omega) \cap H^1_0(\Omega)$ to $\Delta U = \Delta F$ in Ω, and $\Delta F \in H^{k-1}$. We differentiate this equation r times until $D^r \Delta F \in L^2(\Omega)$, and then repeat Step 2. □

8 Inequalities for the normal and tangential decomposition of vector fields on $\partial\Omega$

8.1 The regularity of $\partial\Omega$

Definition 8.1. *A (bounded) domain $\Omega \subseteq \mathbb{R}^n$ is said to be of class C^k if $\partial\Omega$ is an $n-1$ dimensional C^k sub-manifold of $\mathbb{R}^n$, or equivalently,*

(1) there are open sets $\mathcal{O}_\ell \subset \mathbb{R}^n$ such that $\partial\Omega \subseteq \bigcup_{\ell=1}^{N} \mathcal{O}_\ell$;

(2) there are maps $\varphi_\ell : \mathcal{U}_\ell := \partial\Omega \cap \mathcal{O}_\ell \to \mathbb{R}^{n-1}$ so that $\psi_\ell := \varphi_\ell^{-1} : \varphi_\ell(\mathcal{U}_\ell) \to \mathbb{R}^n$ is a C^k injective immersion, that is, for each ℓ, ψ_ℓ is C^k, one-to-one, and for a given coordinate $(y^1, \cdots, y^{n-1})$ in $\varphi_\ell(\mathcal{U}_\ell)$, the set of vectors $\{\psi_{\ell,1}(y), \cdots, \psi_{\ell,n-1}(y)\}$ are linearly independent for all $y \in \varphi_\ell(\mathcal{U}_\ell)$;

(3) and each transition map $\varphi_{\ell_1} \circ \varphi_{\ell_2}^{-1} : \mathcal{U}_{\ell_1} \cap \mathcal{U}_{\ell_2} \to \mathbb{R}^{n-1}$ is $\mathcal{C}^r$ for some $r \geq k$.

A domain Ω is smooth if Ω is a $\mathcal{C}^k$-domain for all $k > 0$.

Given a local chart $(\mathcal{U}, \varphi)$ on $\partial\Omega$ at a point $y \in \partial\Omega$ with local coordinates y^α, $\alpha = 1, ..., n-1$, let $T_y\partial\Omega$ denote the tangent space to $\partial\Omega$ at the point y:

$$T_y\partial\Omega = \{v \in \mathbb{R}^n | v \cdot N(y) = 0\} .$$

Each such tangent space is diffeomorphic to $\mathbb{R}^{n-1}$ with $n-1$ linearly independent tangent vectors spanning the space. The natural basis of tangent vectors is given by the partial derivatives of ψ:

$$\left(\frac{\partial}{\partial y^\alpha}\right)_y := \psi_{,\alpha}(y) \quad \alpha = 1, ..., n-1 ,$$

which condition (2) in Definition 8.1, are linearly independent.

For a function f defined on $\partial\Omega$, the partial derivative of f with respect to y^α at y, denoted by $f_{,\alpha}(y)$, is defined as

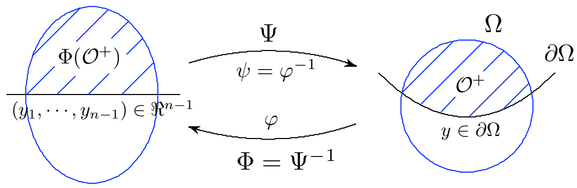
$$f_{,\alpha}(y) = \frac{\partial f}{\partial y^\alpha}(y) := \frac{\partial(f \circ \psi)}{\partial y^\alpha} . \tag{8.1}$$

Throughout, Greek indices run from 1 to $n-1$, while Latin indices run from 1 to n.

Having established a local coordinate system y^α, $\alpha = 1, ..., n-1$ on the boundary $\partial\Omega$, we extend the coordinate system to an n-dimensional neighborhood of $\partial\Omega$. For $y^n \in \mathbb{R}$, define the map $\Psi : \varphi(\mathcal{U}) \times \mathbb{R} \to \mathbb{R}^n$ by

$$\Psi(y^1, \cdots, y^{n-1}, y^n) = \psi(y^1, \cdots, y^{n-1}) + y^n N(\psi(y^1, \cdots, y^{n-1})) , \tag{8.2}$$

where N denotes the outward pointing unit normal on $\partial\Omega$. With the diameter of $\mathcal{O}$ taken sufficiently small, $\Psi : \mathcal{V} = \Phi(\mathcal{O}) \to \mathcal{O}$ is bijection mapping the *curvilinear cooridantes* $(y_1, ..., y_n)$ onto the standard Cartesian coordinates $(x_1, ..., x_n)$.



Let $p \in \mathcal{O}^+ = \mathcal{O} \cap \Omega$. By assumption there exists a unique $q = q(p) \in \partial\Omega$ so that $\text{dist}(p, q) = \text{dist}(p, \partial\Omega)$, and

$$\vec{pq} = \text{dist}(p, q) N(q) \,.$$

Suppose $p = \Psi(y^1, \cdots, y^n)$, then $q = \Psi(y^1, \cdots, y^{n-1}, 0)$. Given $y \in \mathcal{O}^+$, we define the tangent vectors

$$\left(\frac{\partial}{\partial y^i}\right)_y := \Psi,_i(y) \,.$$

Next, we define

$$n := N \circ \psi \,.$$

It follows that

$$\left(\frac{\partial}{\partial y^\alpha}\right)_y := \Big[\psi,_\alpha + y^n n,_\alpha\Big](y^1, \cdots, y^{n-1}) \,; \tag{8.3a}$$

$$\left(\frac{\partial}{\partial y^n}\right)_y := n(y^1, \cdots, y^{n-1}) \,. \tag{8.3b}$$

By adding an open set $\mathcal{O}_0 \subset\subset \Omega$, we may assume that $\Omega \subseteq \bigcup_{\ell=0}^{N} \mathcal{O}_\ell$. Let $\{\chi_\ell\}_{\ell=0}^N$ be a partition of unity (of $\overline{\Omega}$) subordinate to $\mathcal{O}_\ell$. We will further assume that $\text{spt}(\chi_\ell) \subset\subset \mathcal{O}_\ell$, $\mathcal{O}_\ell$ is smooth for all ℓ, and $\partial\mathcal{O}_\ell$ is transverse to $\{y_n = 0\}$ (and equivalently, $\partial\Phi(\mathcal{O}_\ell)$ is transverse to $\partial\Omega$).

Let f be a function defined on Ω, and $f_\ell = \chi_\ell f$. By the properties of $\{\chi_\ell\}_{\ell=0}^N$, $f = \sum_{\ell=0}^N (\chi_\ell f) = \sum_{\ell=0}^N f_\ell$. For $1 \le i \le n$, $f_\ell \circ \Psi_\ell$ is defined on $\Phi(\mathcal{O}_\ell)$, or in other words, $f_\ell \circ \Psi_\ell$ is a function of y. Let $A_\ell = (D_y \Psi_\ell)^{-1}$, by the chain rule,

$$\frac{\partial f_\ell}{\partial x_i} \circ \Psi_\ell = (A_\ell)_i^j \frac{\partial (f_\ell \circ \Psi_\ell)}{\partial y_j} = (A_\ell)_i^\alpha f_{\ell,\alpha} + (A_\ell)_i^n f_{\ell,n} \,. \tag{8.4}$$

Important notation: From now on, for a given function f, we use $f_{,i}$ to denote the derivative of f with respect to Euclidean/Cartesian coordinate x, while using $f_{,\alpha}$ and $f_{,n}$ to denote the derivative of f with respect to the curvilinear coordinate y.

8.2 Tangential and normal derivatives

8.2.1 Decomposing vectors on $\partial\Omega$ in 2-D

Suppose that $n = 2$. We begin with the definition of two important geometric quantities: in each chart $(\mathcal{U}, \varphi)$, let $g = \psi_{,1} \cdot \psi_{,1}$ and $b = n_{,1} \cdot \psi_{,1}$ denote the first and second fundamental forms, respectively. Then

$$n = (-\frac{\psi^2{}_{,1}}{\sqrt{g}}, \frac{\psi^1{}_{,1}}{\sqrt{g}}), \qquad \text{and} \qquad n_{,1} = g^{-1} b \psi_{,1} \ .$$

By the definition of Ψ,

$$D_y\Psi = \begin{bmatrix} \psi^1_{,1} + y_2 n^1_{,1} & -\frac{\psi^2_{,1}}{\sqrt{g}} \\ \psi^2_{,1} + y_2 n^2_{,1} & \frac{\psi^1_{,1}}{\sqrt{g}} \end{bmatrix} = \begin{bmatrix} \psi^1_{,1}(1 + \frac{y_2 b}{g}) & -\frac{\psi^2_{,1}}{\sqrt{g}} \\ \psi^2_{,1}(1 + \frac{y_2 b}{g}) & \frac{\psi^1_{,1}}{\sqrt{g}} \end{bmatrix};$$

hence, $\det(D_y\Psi) = \dfrac{g + y_2 b}{\sqrt{g}}$ and

$$A = \frac{\sqrt{g}}{g + y_2 b} \begin{bmatrix} \frac{\psi^1_{,1}}{\sqrt{g}} & \frac{\psi^2_{,1}}{\sqrt{g}} \\ -\psi^2_{,1}(1 + \frac{y_2 b}{g}) & \psi^1_{,1}(1 + \frac{y_2 b}{g}) \end{bmatrix}.$$

Consequently,

$$\tau^1_i := A^1_i = \frac{1}{g + y_2 b} \psi_{,1} \parallel \partial\Omega, \qquad \text{and} \qquad A^2_i = n_i \,.$$

and (8.4) implies that the gradient can be decomposed into *tangential* and *normal* derivatives:

$$\frac{\partial f}{\partial x_i} \circ \Psi = \tau^1_i f_{,1} + n_i f_{,n} \,,$$

where $f_{,1} = \partial f / \partial y_1$ and $f_{,n} = \partial f / \partial y_2$.

8.2.2 Decomposing vectors on $\partial\Omega$ in n-D

Let $e^\alpha = (0,\cdots,0,1,0,\cdots,0)^T$ be the α-th tangent vector on $\{y_n = 0\}$, and $e^n = (0,\cdots,0,-1)^T$ is the outward pointing normal to $\{y_n \geq 0\}$. Define $\tau^\alpha = A^T e^\alpha$ and $\nu = A^T e^n$ (where $A = (D_y\Psi)^{-1}$). By setting $\mathcal{G}_{\alpha\beta} = \Psi,_\alpha \cdot \Psi,_\beta$ and $\mathcal{G}^{\alpha\beta} = [\mathcal{G}_{\alpha\beta}]^{-1}$, we find that

$$\tau^\alpha = A^T e^\alpha = \mathcal{G}^{\alpha\beta}\Psi_{,\beta} \qquad \text{and} \qquad \nu = A^T e^n = n\,. \tag{8.5}$$

Note that τ^α and ν are also defined away from $y^n = 0$, and $\tau_\alpha \cdot \nu = 0$ for all $\alpha = 1,\cdots,n-1$.

By (8.4), for a given function f defined in $\mathcal{O}$,

$$\frac{\partial f}{\partial x^i} \circ \Psi = A_i^j \frac{\partial (f\circ\Psi)}{\partial y^j} = A_i^\alpha \frac{\partial (f\circ\Psi)}{\partial y^\alpha} + A_i^n \frac{\partial (f\circ\Psi)}{\partial y^n}\,.$$

Since $A_i^\alpha = \tau_i^\alpha$ and $A_i^n = n_i$, in $\Phi(\mathcal{O})$ we find that

$$(D_x f)\circ\Psi = \tau^\alpha \frac{\partial (f\circ\Psi)}{\partial y^\alpha} + n\frac{\partial (f\circ\Psi)}{\partial y^n} = \tau^\alpha f_{,\alpha} + n f_{,n}\,. \tag{8.6}$$

8.3 Some useful inequalities

The mean curvature H is defined as the trace of the second fundamental form $b_{\alpha\beta}$ so that in a local chart $(\mathcal{U},\psi)$,

$$H\circ\psi = \frac{g^{\alpha\beta}b_{\alpha\beta}}{n-1}\,, \qquad b_{\alpha\beta} = -\psi,_{\alpha\beta}\cdot n\,.$$

Lemma 8.2. *Suppose that $(\mathcal{O},\Psi)$ is a local chart in a neighborhood of $y \in \partial\Omega$ with coordinates (y^α, y^n). If $u \in H^s(\Omega)\cap H^1_0(\Omega)$ for $s > 2.5$, then on $\partial\Omega$ (or on $\{y_n\} = 0$ in $\Psi^{-1}(\mathcal{O})$),*

$$(u_{,ij}N_i - u_{,ii}N_j)\circ\psi = \tau_j^\alpha u_{,\alpha n} - (n-1)(H\circ\psi)n_j u_{,n}\,,$$

where H is the mean curvature of $\partial\Omega$.

Proof. By (8.5), for $y^n \geq 0$, $\tau^\alpha = \mathcal{G}^{\alpha\beta}\Psi_{,\beta}$. Therefore, by (8.6),

$$\begin{aligned} u_{,ij}\circ\Psi = {} & n_j(n_i u_{,nn} + \tau_i^\alpha u_{,\alpha n} + n_{i,n}u_{,n} + \tau^\alpha_{i,n}u_{,\alpha}) \\ & + \tau_j^\beta(\tau_i^\alpha u_{,\alpha\beta} + n_i u_{,\beta n} + \tau^\alpha_{i,\beta}u_{,\alpha} + n_{i,\beta}u_{,n})\,. \end{aligned}$$

Using this identity together with $\tau^\alpha \cdot n = 0$ and $u_{,\alpha} = 0$ if $y_n = 0$, we find that

$$\begin{aligned} \Big[u_{,ij}N_i - u_{,ii}N_j\Big]\circ\psi &= \tau_j^\beta(n_i u_{,\beta n} + \tau^\alpha_{i,\beta}u_{,\alpha} + n_{i,\beta}u_{,n})n_i - \tau_i^\beta(\tau_i^\alpha u_{,\alpha\beta} + \tau^\alpha_{i,\beta}u_{,\alpha} + n_{i,\beta}u_{,n})n_j \\ &= \tau_j^\alpha u_{,\alpha n} - n_j\tau_i^\beta n_{i,\beta}u_{,n}\,. \end{aligned}$$

The result then follows from the fact that

$$\tau_i^\beta n_{i,\beta}|_{y_n=0} = g^{\alpha\beta}\psi^i_{,\alpha} n_{i,\beta} = g^{\alpha\beta} b_{\alpha\beta} = (n-1)H\circ\psi\,. \qquad \square$$

Corollary 8.3. *Given $u \in H^s(\Omega)\cap H^1_0(\Omega)$ for $s>2.5$,*

$$\int_{\partial\Omega}[u_{,ij}N_i - u_{,ii}N_j]u_{,j}dS = (n-1)\int_{\partial\Omega} H\Big|\frac{\partial u}{\partial N}\Big|^2 dS\,.$$

Proof. By the definition of the surface integral,

$$\int_{\partial\Omega}[u_{,ij}N_i - u_{,ii}N_j]u_{,j}dS = \sum_{\ell=1}^N \int_{\partial\Omega}\chi_\ell[u_{,ij}N_i - u_{,ii}N_j]u_{,j}dS$$
$$= \sum_{\ell=1}^N \int_{\varphi(\mathcal{U}_\ell)}\Big[\chi_\ell[u_{,ij}N_i - u_{,ii}N_j]u_{,j}\Big]\circ\psi_\ell\sqrt{\det(g_\ell)}\;dy_1\cdots dy_{n-1}\,.$$

By Lemma 8.2,

$$\Big[u_{,ij}N_i - u_{,ii}N_j\Big]\circ\psi_\ell = \tau_j^\alpha u_{,\alpha n} - (n-1)(H\circ\psi)n_j u_{\ell,n}\,.$$

Therefore, by $u_{\ell,i}\circ\psi_\ell = n_i u_{\ell,n}$ on $\{y_n=0\}$,

$$\int_{\partial\Omega}[u_{,ij}N_i - u_{,ii}N_j]u_{,j}dS = -(n-1)\sum_{\ell=1}^N\int_{\varphi(\mathcal{U}_\ell)}[(\chi_\ell H)\circ\psi_\ell]u^2_{\ell,n}\sqrt{\det(g_\ell)}dy_1\cdots dy_{n-1}\,.$$

The corollary is immediately proved from this equality. $\square$

Corollary 8.4. *If $u\in H^2(\Omega)\cap H^1_0(\Omega)$, then*

$$\|D^2u\|^2_{L^2(\Omega)} = \|\Delta u\|^2_{L^2(\Omega)} + (n-1)\int_{\partial\Omega} H\Big|\frac{\partial u}{\partial N}\Big|^2 dS\,. \tag{8.7}$$

Proof. We first establish the identity is valid for all $u\in\mathcal{C}^\infty(\Omega)\cap H^1_0(\Omega)$; a density argument then completes the proof.

Let $u\in\mathcal{C}^\infty(\Omega)\cap H^1_0(\Omega)$. Integrating by parts, we find that

$$\|\Delta u\|^2_{L^2(\Omega)} = \int_\Omega |\Delta u(x)|^2dx = \int_\Omega u_{,ii}(x)u_{,jj}(x)dx$$
$$= \int_\Omega |D^2u(x)|^2dx - \int_{\partial\Omega}\Big[u_{,ij}(x)N_i - u_{,ii}(x)N_j\Big]u_{,j}(x)dS$$

and the conclusion follows from Corollary 8.3. $\square$

Corollary 8.5. *There is a constant $C = C(\Omega)$ so that*

$$\|u\|^2_{H^2(\Omega)} \le C\Big[\|Du\|^2_{L^2(\Omega)} + \|\Delta u\|^2_{L^2(\Omega)}\Big] \qquad \forall\ u \in H^2(\Omega)\cap H^1_0(\Omega)\,. \tag{8.8}$$

Proof. We only need to estimate $\displaystyle\int_{\partial\Omega} H\Big|\frac{\partial u}{\partial N}\Big|^2 dS$. Since $\Big|\dfrac{\partial u}{\partial N}\Big| = |Du\cdot N| \le |Du|$,

$$\int_{\partial\Omega} H\Big|\frac{\partial u}{\partial N}\Big|^2 dS \le C(\Omega)\|Du\|^2_{L^2(\partial\Omega)} \le C(\Omega)\|Du\|^2_{H^{0.25}(\partial\Omega)} \le \|Du\|^2_{H^{0.75}(\Omega)}\,.$$

By (4.2) and Young's inequality, we conclude that

$$\int_{\partial\Omega} H\Big|\frac{\partial u}{\partial N}\Big|^2 dS \le C_\delta\|Du\|^2_{L^2(\Omega)} + \delta\|Du\|^2_{H^1(\Omega)}\,.$$

The identity (8.8) then follows from (8.7) and taking $\delta > 0$ small enough. □

Corollary 8.6. *The norm $\|u\| := \|Du\|_{L^2(\Omega)} + \|\Delta u\|_{L^2(\Omega)}$ is an equivalent norm in $H^2(\Omega)\cap H^1_0(\Omega)$.*

Remark 8.7. *Similar to the proofs of Lemma 8.2 and Corollary 8.5, for every multi-index α,*

$$\Big|\int_{\partial\Omega}\Big[(D^\alpha u)_{,jj}N_i - (D^\alpha u)_{,ij}N_j\Big](D^\alpha u)_{,i}dS\Big| \le C\sum_{|\beta|=1}^{|\alpha|+1}\|D^{|\beta|}u\|^2_{L^2(\partial\Omega)}\,.$$

for all $u \in H^{|\alpha|+s}(\Omega)\cap H^1_0(\Omega)$. Therefore, one important conclusion of this inequality is that the norm $\|u\| := \|u\|_{H^{k+1}(\Omega)} + \|\Delta u\|_{H^k(\Omega)}$ is an equivalent norm in $H^{k+2}(\Omega)\cap H^1_0(\Omega)$, or more precisely, there are constants C_1 and C_2 so that for all $u \in H^{k+2}(\Omega)\cap H^1_0(\Omega)$,

$$C_1\|u\|^2_{H^{k+2}(\Omega)} \le \|u\|^2_{H^{k+1}(\Omega)} + \|\Delta u\|^2_{H^k(\Omega)} \le C_2\|u\|^2_{H^{k+2}(\Omega)}\,. \tag{8.9}$$

8.4 Elliptic estimates for vector fields

Proposition 8.8. *For an H^r domain Ω with $\Gamma = \partial\Omega$, $r \ge 3$, if $F \in L^2(\Omega;\mathbb{R}^3)$ with $\operatorname{curl} F \in H^{s-1}(\Omega;\mathbb{R}^3)$, $\operatorname{div} F \in H^{s-1}(\Omega)$, and $F\cdot N|_\Gamma \in H^{s-\frac12}(\Gamma)$ for $1 \le s \le r$, then there exists a constant $\bar C > 0$ depending only on Ω such that*

$$\begin{aligned}\|F\|_s &\le \bar C\left(\|F\|_0 + \|\operatorname{curl} F\|_{s-1} + \|\operatorname{div} F\|_{s-1} + |\bar\partial F\cdot N|_{s-\frac32}\right)\,,\\ \|F\|_s &\le \bar C\left(\|F\|_0 + \|\operatorname{curl} F\|_{s-1} + \|\operatorname{div} F\|_{s-1} + \textstyle\sum_{\alpha=1}^2|\bar\partial F\cdot T_\alpha|_{s-\frac32}\right)\,,\end{aligned} \tag{8.10}$$

where N denotes the outward unit-normal to Γ, and T_α, $\alpha = 1,2$, denotes the two tangent vectors to Γ.

Whenever, Ω is a C^∞-class domain, then (8.10) can be written as

$$\|F\|_s \le \bar{C}\left(\|F\|_0 + \|\operatorname{curl} F\|_{s-1} + \|\operatorname{div} F\|_{s-1} + |F\cdot N|_{s-\frac{1}{2}}\right)\,,$$
$$\|F\|_s \le \bar{C}\left(\|F\|_0 + \|\operatorname{curl} F\|_{s-1} + \|\operatorname{div} F\|_{s-1} + \textstyle\sum_{\alpha=1}^2 |F\cdot T_\alpha|_{s-\frac{1}{2}}\right)\,. \qquad (8.10')$$

9 The div-curl lemma

Lemma 9.1. *Suppose $v_k \rightharpoonup v$ and $w_k \rightharpoonup w$ both in $L^2(\Omega)$, and $\operatorname{div} v_k$ and $\operatorname{curl} w_k$ are compact in $H^{-1}(\Omega)$. Then $v_k \cdot w_k \to v\cdot w$ in $\mathcal{D}'(\Omega)$.*

Before proving Lemma 9.1, let us examine why this lemma should hold. Suppose $\operatorname{div} v_k$ and $\operatorname{curl} w_k$ both vanish; then $v_k = \operatorname{curl} u_k$ ($v = \operatorname{curl} u$) for some H^1-vector field u_k (u) and $w_k = Dp_k$ ($w = Dp$) for some H^1-scalar p_k (p). Therefore, for $\varphi \in \mathcal{D}(\Omega)$,

$$\int_\Omega v_k\cdot w_k \varphi dx = \int_\Omega \operatorname{curl} u_k \cdot Dp_k\varphi dx = -\langle \operatorname{curl} u_k, p_k D\varphi\rangle\,,$$

where we use the property that $\operatorname{div}\operatorname{curl} u_k = 0$ so that the derivative acting upon $\operatorname{curl} u_k$ vanishes when integrating by parts. Now since $\operatorname{curl} u_k$ is compact in $H^{-1}(\Omega)$, and p_k converges weakly in $H^1(\Omega)$, we find that

$$\lim_{k\to\infty}\int_\Omega v_k\cdot w_k\varphi dx = -\langle \operatorname{curl} u, pD\varphi\rangle = \int_\Omega \operatorname{curl} u\cdot Dp\varphi dx = \int_\Omega v\cdot w\varphi dx\,.$$

We will mimic this idea to prove Lemma 9.1.

Proof. Let $w_k \in H^2(\Omega)\cap H^1_0(\Omega)$ solve

$$\begin{aligned} -\Delta w_k &= v_k && \text{in} \quad \Omega\,,\\ w_k &= 0 && \text{on} \quad \partial\Omega\,. \end{aligned}$$

Then $v_k = \operatorname{curl}\operatorname{curl} w_k - D\operatorname{div} w_k$, and $\|w_k\|_{H^2(\Omega)} \le C\|v_k\|_{L^2(\Omega)}$. Moreover, let $\varphi\in\mathcal{D}(\Omega)$. Then

$$-\Delta(\varphi\operatorname{curl} w_k) = -\operatorname{curl} w_k\Delta\varphi - 2D\varphi\cdot D\operatorname{curl} w_k + \varphi\operatorname{curl} v_k\,.$$

We claim that the right-hand side, at least for a subsequence, converges strongly in $H^1(\Omega)'$. The convergence of the last term follows by the assumptions of the lemma, while the first term converges strongly in $L^2(\Omega)$ by Rellich's theorem. For the second term, by the definition of the dual space norm,

$$\begin{aligned} \|D\varphi\cdot D\operatorname{curl}(w_k - w)\|_{H^1(\Omega)'} &= \sup_{\|\psi\|_{H^1(\Omega)}=1} \langle D\varphi\cdot D\operatorname{curl}(w_k-w),\psi\rangle_{H^1(\Omega)}\\ &\le 2\|D\varphi\|_{W^{1,\infty}(\Omega)}\|\operatorname{curl}(w_k-w)\|_{L^2(\Omega)} \to 0 \quad \text{as } k\to\infty\,, \end{aligned}$$

where w denotes the limit of w_k. By elliptic regularity,

$$\|\varphi\, \mathrm{curl}(w_k - w)\|_{H^1(\Omega)} \to 0 \quad \text{as } k \to \infty\,.$$

Therefore,

$$\begin{aligned}\int_\Omega u_k \cdot v_k \varphi dx &= \int_\Omega u_k \cdot \mathrm{curl}\,\mathrm{curl}\, w_k \varphi dx - \int_\Omega u_k \cdot D\, \mathrm{div}\, w_k, \varphi dx \\ &= \int_\Omega u_k \cdot \mathrm{curl}(\varphi\, \mathrm{curl}\, w_k) dx - \int_\Omega u_k \cdot (D\varphi \times \mathrm{curl}\, w_k) dx \\ &\quad + \int_\Omega \mathrm{div}\, u_k\, \mathrm{div}\, w_k \varphi dx + \int_\Omega u_k \cdot D\varphi\, \mathrm{div}\, u_k dx\,.\end{aligned}$$

It is easy to see that the right-hand side converges to $\int_\Omega u \cdot v\varphi dx$. □

Example 9.2. *Consider the 1-D Burgers equation $u_t + uu_x = 0$ in $\mathbb{R}$. Suppose u^ϵ is the solution to the viscous Burgers equation*

$$u_t^\epsilon + u^\epsilon u_x^\epsilon = \epsilon u_{xx}^\epsilon \qquad \forall x \in \mathbb{R}\,, t > 0\,. \tag{9.1}$$

We want to show that u^ϵ converges to u in some appropriate topology. We consider the two dimensional space $\mathbb{R}^+ \times \mathbb{R}$ (treating the time axis t as another dimension). Then, (9.1) simply reads

$$\mathrm{div}_{t,x}(u^\epsilon, \frac{(u^\epsilon)^2}{2}) = \epsilon u_{xx}^\epsilon\,. \tag{9.2}$$

Multiplying (9.1) by u^ϵ, we obtain

$$\mathrm{curl}_{t,x}(-\frac{(u^\epsilon)^3}{3}, \frac{(u^\epsilon)^2}{2}) = \epsilon\Big[\frac{(u^\epsilon)^2}{2}\Big]_{xx} - \epsilon(u_x^\epsilon)^2\,. \tag{9.3}$$

Suppose that one knows that the right-hand side of (9.2) and (9.3) are compact in H^{-1}; then by the div-curl lemma,

$$-\frac{(u^\epsilon)^4}{12} \to -\frac{u_1 u_3}{3} + \frac{u_2^2}{4} \qquad \text{in} \quad \mathcal{D}'(\mathbb{R})\,,$$

where u_i is the weak limits of $(u^\epsilon)^i$. Therefore,

$$u_4 = 4u_1 u_3 - 3u_2^2\,. \tag{9.4}$$

Now consider the integral

$$\int_{\mathbb{R}} (u^\epsilon - u_1)^4 dx\,.$$

Binomial expansion gives us

$$\int_{\mathbb{R}} (u^\epsilon - u_1)^4 dx = \int_{\mathbb{R}} (u^\epsilon)^4 dx - 4\int_{\mathbb{R}} (u^\epsilon)^3 u_1 dx + 6\int_{\mathbb{R}} (u^\epsilon)^2 u_1^2 dx \\ - 4\int_{\mathbb{R}} u^\epsilon u_1^3 dx + \int_{\mathbb{R}} u_1^4 dx\,.$$

Passing ϵ *to* 0, *by (9.4) we find that*

$$\lim_{\epsilon\to 0} \int_{\mathbb{R}} (u^\epsilon - u_1)^4 \phi dx = \int_{\mathbb{R}} \left[u_4 - 4u_1u_3 + 6u_2u_1^2 - 4u_1^4 + u_1^4 \right] \phi dx \\ = -3\int_{\mathbb{R}} (u_2 - u_1^2)^2 \phi dx\,.$$

Note that the left-hand side is always non-negative, while the right-hand side is always non-positive. Therefore, $u_2 = u_1^2$ *or more precisely,*

$$w.\lim_{\epsilon\to 0} (u^\epsilon)^2 = \left(w.\lim_{\epsilon\to 0} u^\epsilon \right)^2\,.$$

9.1 Exercises

Problem 9.1. *Suppose that we have sequences* $\{f_k\}$ *bounded in* $W^{1,4}(D)$ *and* $\{w_k\}$ *bounded in* $W^{1,4}(D)$, *and let* $\{u_k\}$ *be the solution to*

$$-\Delta u_k = Df_k \cdot \operatorname{curl} w_k \text{ in } D\,, \\ u_k = 0 \text{ on } \partial D = \mathbb{S}^1\,.$$

Define the 2D gradient of f_k *and the 2D curl of the scalar-valued function* w_k *as the two-vectors*

$$\operatorname{curl} w_k = (-\frac{\partial w_k}{\partial x_2}, \frac{\partial w_k}{\partial x_1}) \\ Df_k = (\frac{\partial f_k}{\partial x_1}, \frac{\partial f_k}{\partial x_2})\,.$$

Suppose that $f_k \rightharpoonup f$ *in* $W^{1,4}(D)$ *and* $w_k \rightharpoonup w$ *in* $W^{1,4}(D)$.
(a) *Show that (up to a subsequence)* $u_k \rightharpoonup u$ *in* $H_0^1(D)$ *and* u *solves*

$$-\Delta u = Df \cdot \operatorname{curl} w \text{ in } D\,, \qquad (*) \\ u = 0 \text{ on } \partial D = \mathbb{S}^1\,.$$

with equality in (*) holding $H^{-1}(D)$. *(Hint. Prove that* $Df_k \cdot \operatorname{curl} w_k \rightharpoonup Df \cdot \operatorname{curl} w$ *in* $H^{-1}(D)$.*)*
(b) *Show that* u *is also in* $H^2(D)$.