# MATH 21B PRACTICE MIDTERM I SPRING 2025: SOLUTION KEY

MAT 21B SPRING 2025

**Problem 1**. Find

$$\sum_{k=2}^{6}(2k+3)$$

*Solution.*

$$\begin{aligned}\sum_{k=2}^{6}(2k+3) &= \sum_{k=2}^{6}2k + \sum_{k=2}^{6}3 \\ &= 2\sum_{k=2}^{6}k + 3\cdot(6-2+1) \\ &= 2\left(\sum_{k=1}^{6}k - \sum_{k=1}^{1}k\right) + 3\cdot 5 \\ &= 2\left(\frac{6\cdot 7}{2} - \frac{1\cdot 2}{2}\right) + 15 \\ &= 2(21-1)+15 \\ &= 2\cdot 20 + 15 \\ &= 40 + 15 = \boxed{55}\end{aligned}$$

□

**Problem 2**. A car in an amusement park ride runs for twelve seconds along a straight track. The velocity of the car is recorded every three seconds and listed in this table:

| time in seconds | 0 | 3 | 6 | 9 | 12 |
|---|---|---|---|---|---|
| velocity in feet per second | 3 | 5 | 6 | 6 | 5 |

Estimate the distance that the car travels during these twelve seconds in two ways.

(a) ($L_4$): Use four equal intervals and the Left End rule.
(b) Use the average of the three estimates:
  (i) ($L_4$): Using four equal intervals and the Left End rule,
  (ii) ($R_4$): Using four equal intervals and the Right End rule, and
  (iii) ($M_2$): Using two equal intervals and the Midpoint rule.

*Solution.* The interval is from 0 to 12 seconds, and velocity is given every 3 seconds, so:

$$\Delta t = \frac{12-0}{4} = 3 \text{ seconds}$$

**(a) Left End Rule ($L_4$):**

$$L_4 = \Delta t\cdot[v(0)+v(3)+v(6)+v(9)] = 3(3+5+6+6) = 3\cdot 20 = \boxed{60 \text{ ft}}$$

**(b) Average of Three Estimates:**

(i) $L_4 = 60$ ft

(ii) Right End Rule ($R_4$):

$$R_4 = 3 \cdot [v(3) + v(6) + v(9) + v(12)] = 3(5 + 6 + 6 + 5) = 3 \cdot 22 = \boxed{66 \text{ ft}}$$

(iii) Midpoint Rule ($M_2$): Using midpoints of two intervals: [0, 6] and [6, 12]
Midpoint times: $t = 3, 9$

$$M_2 = 6 \cdot [v(3) + v(9)] = 6(5 + 6) = 6 \cdot 11 = \boxed{66 \text{ ft}}$$

**Average estimate:**

$$\text{Average} = \frac{L_4 + R_4 + M_2}{3} = \frac{60 + 66 + 66}{3} = \boxed{64 \text{ ft}}$$

□

**Problem 3.**

(a) Write a definite integral for the function $\operatorname{erf}(x)$ which is the area under the curve

$$\frac{2}{\sqrt{\pi}} e^{-t^2}$$

between $t = 0$ and $t = x$. (This function is popular in statistics).

(b) Find

$$\frac{d}{dx} \left[\operatorname{erf}(x^2)\right]\bigg|_{x=2}.$$

*Solution.* (a) The error function $\operatorname{erf}(x)$ is defined as:

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$$

(b) To find

$$\frac{d}{dx} \left[\operatorname{erf}(x^2)\right]\bigg|_{x=2},$$

we apply the chain rule. Let:

$$f(x) = \operatorname{erf}(x^2)$$

Then:

$$f'(x) = \frac{d}{dx}\left[\operatorname{erf}(x^2)\right] = \operatorname{erf}'(x^2) \cdot \frac{d}{dx}(x^2)$$

Recall that:

$$\frac{d}{dx}\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} e^{-x^2}$$

So:

$$f'(x) = \frac{2}{\sqrt{\pi}} e^{-(x^2)^2} \cdot 2x = \frac{4x}{\sqrt{\pi}} e^{-x^4}$$

Evaluate at $x = 2$:

$$f'(2) = \frac{4 \cdot 2}{\sqrt{\pi}} e^{-16} = \frac{8}{\sqrt{\pi}} e^{-16}$$

$$\boxed{\frac{d}{dx}\left[\operatorname{erf}(x^2)\right]\bigg|_{x=2} = \frac{8}{\sqrt{\pi}} e^{-16}}$$

□

**Problem 4**. Find the following as functions with a constant of integration:

(a) $\int \left(x^2 + \sqrt{x}\right)\, dx$
(b) $\int \cos^2(2x)\sin(2x)\, dx$

*Solution.* (a) Evaluate the integral:

$$\int \left(x^2 + \sqrt{x}\right)\, dx$$

Break the integral into two parts:

$$= \int x^2\, dx + \int x^{1/2}\, dx$$

Apply the power rule:

$$= \frac{x^3}{3} + \frac{2}{3}x^{3/2} + C$$

$$\boxed{\int (x^2 + \sqrt{x})\, dx = \frac{x^3}{3} + \frac{2}{3}x^{3/2} + C}$$

(b) Evaluate:

$$\int \cos^2(2x)\sin(2x)\, dx$$

Use substitution: let $u = \cos(2x)$, then $\frac{du}{dx} = -2\sin(2x) \Rightarrow \sin(2x)\, dx = -\frac{1}{2}\, du$

Rewriting the integral:

$$\int \cos^2(2x)\sin(2x)\, dx = -\frac{1}{2}\int u^2\, du = -\frac{1}{2}\cdot\frac{u^3}{3} + C = -\frac{1}{6}u^3 + C$$

Substituting back $u = \cos(2x)$:

$$\boxed{\int \cos^2(2x)\sin(2x)\, dx = -\frac{1}{6}\cos^3(2x) + C}$$

□

**Problem 5**. Compute the following numbers:

(a) $\int_0^1 \frac{x^3}{\sqrt{1-x^2}}\, dx$
(b) $\int_0^1 (1-x)(2x-x^2)^9\, dx$
(c) $\int_0^1 \sqrt{1-x^2}\, dx$
(d) $\int_4^5 f(x)\, dx$, given that:
  i. The average value of the function $f$ over the interval $[0, 5]$ is 3.
  ii. $\int_0^4 [2f(x) + x]\, dx = 10$

*Solution.* (a) Evaluate:

$$\int_0^1 \frac{x^3}{\sqrt{1-x^2}}\, dx$$

Let $u = 1 - x^2$, so $du = -2x\, dx \Rightarrow x\, dx = -\frac{1}{2}\, du$. Also, $x^2 = 1 - u \Rightarrow x^3 = x(1-u)$.

Thus,

$$x^3\, dx = (1-u)(x\, dx) = (1-u)\left(-\frac{1}{2}\, du\right)$$

Change of limits: $x = 0 \Rightarrow u = 1$, $x = 1 \Rightarrow u = 0$

So the integral becomes:

$$\int_0^1 \frac{x^3}{\sqrt{1-x^2}}\, dx = \frac{1}{2}\int_0^1 \frac{1-u}{\sqrt{u}}\, du = \frac{1}{2}\int_0^1 \left(u^{-1/2} - u^{1/2}\right)\, du$$

Evaluate:

$$\int_0^1 u^{-1/2}\,du = 2, \quad \int_0^1 u^{1/2}\,du = \boxed{\frac{2}{3}}$$

(b) Evaluate:

$$\int_0^1 (1-x)(2x-x^2)^9\,dx$$

Let $u = 2x - x^2$, then $du = (2-2x)\,dx = 2(1-x)\,dx$, so $(1-x)\,dx = \frac{1}{2}\,du$. When $x = 0$, $u = 0$, and when $x = 1$, $u = 1$. Then:

$$\int_0^1 (1-x)(2x-x^2)^9\,dx = \int_0^1 (2x-x^2)^9(1-x)\,dx = \int_0^1 u^9 \cdot \frac{1}{2}\,du = \frac{1}{2}\int_0^1 u^9\,du$$

$$= \frac{1}{2} \cdot \left[\frac{u^{10}}{10}\right]_0^1 = \frac{1}{2} \cdot \frac{1}{10} = \boxed{\frac{1}{20}}$$

(c) Evaluate:

$$\int_0^1 \sqrt{1-x^2}\,dx$$

This is a standard integral which represents one-quarter of the area of a unit circle (since $\sqrt{1-x^2}$ is the upper half of a circle of radius 1). Thus:

$$\int_0^1 \sqrt{1-x^2}\,dx = \boxed{\frac{\pi}{4}}$$

(d) From (i), the average value formula gives:

$$\frac{1}{5}\int_0^5 f(x)\,dx = 3 \quad \Rightarrow \quad \int_0^5 f(x)\,dx = 15$$

From (ii), we expand the integral:

$$\int_0^4 [2f(x) + x]\,dx = 2\int_0^4 f(x)\,dx + \int_0^4 x\,dx$$

Compute $\int_0^4 x\,dx$:

$$\int_0^4 x\,dx = \left[\frac{x^2}{2}\right]_0^4 = \frac{16}{2} = 8$$

So,

$$2\int_0^4 f(x)\,dx + 8 = 10 \quad \Rightarrow \quad 2\int_0^4 f(x)\,dx = 2 \quad \Rightarrow \quad \int_0^4 f(x)\,dx = 1$$

Now compute:

$$\int_4^5 f(x)\,dx = \int_0^5 f(x)\,dx - \int_0^4 f(x)\,dx = 15 - 1 = \boxed{14}$$

□