MATH 21B MIDTERM 1: SOLUTION KEY

MAT 21B SPRING 2025

Problem 1. Find the number

$$\sum_{k=4}^{19}(k-3)$$

Solution.

$$\sum_{k=4}^{19}(k-3) = (4-3)+(5-3)+(6-3)+\cdots+(19-3)$$

$$= 1+2+3+\cdots+16$$

This is the sum of the first 16 natural numbers.

Using the formula:

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

Apply with $n = 16$:

$$\sum_{k=1}^{16} k = \frac{16\cdot 17}{2} = \frac{272}{2} = 136$$

Final Answer:

$$\sum_{k=4}^{19}(k-3) = \boxed{136}$$

□

Problem 2. Use four equal intervals and either the Left or Right End point rule to estimate the definite integral

$$\int_{10}^{30} f(x)\,dx$$

using **4 equal intervals** and either the **Left or Right Endpoint Rule**, given the following data:

x	0	5	10	15	20	25	30	35	40
$f(x)$	1.9	1.7	1.6	1.5	1.3	1.2	1.0	0.9	0.6

Solution. The interval is from $x = 10$ to $x = 30$, and we are using 4 equal subintervals:

$$\Delta x = \frac{30-10}{4} = 5$$

Use the Left Endpoint Rule:

The left endpoints are:

$$x = 10,\ 15,\ 20,\ 25$$

and corresponding $f(x)$ values are:

$$f(10) = 1.6, \quad f(15) = 1.5, \quad f(20) = 1.3, \quad f(25) = 1.2$$

Apply the Left Endpoint Rule:

$$\int_{10}^{30} f(x)\,dx \approx \Delta x\,[f(10) + f(15) + f(20) + f(25)]$$
$$= 5\,(1.6 + 1.5 + 1.3 + 1.2) = 5 \times 5.6 = 28$$

Final Answer Using Left Endpoint Rule:

$$\boxed{28}$$

Similarly, If we apply the right Endpoint Rule:

$$\int_{10}^{30} f(x)\,dx \approx \Delta x\,[f(15) + f(20) + f(25) + f(30)]$$
$$= 5\,(1.5 + 1.3 + 1.2 + 1.0) = 5 \times 5.0 = 25$$

Final Answer Using Right Endpoint Rule:

$$\boxed{25}$$

□

Problem 3. Find the area of the region bounded by the curves:

$$y = x^2 \quad \text{and} \quad y = \sqrt{x}$$

over the interval $x = 0$ to $x = 1$.

Solution. For $0 \leq x \leq 1$,

$$\sqrt{x} \geq x^2$$

So the area is given by:

$$\text{Area} = \int_0^1 \left(\sqrt{x} - x^2\right)\,dx = \int_0^1 \left(x^{1/2} - x^2\right)\,dx$$

Use the power rule:

$$= \left[\frac{2}{3}x^{3/2} - \frac{1}{3}x^3\right]_0^1$$

Evaluate

$$= \left(\frac{2}{3}(1) - \frac{1}{3}(1)\right) - \left(\frac{2}{3}(0) - \frac{1}{3}(0)\right) = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$$

Final Answer:

$$\boxed{\frac{1}{3}}$$

□

Problem 4.

- Evaluate the derivative:

$$\frac{d}{dx}\int_1^x \frac{\sqrt{t}+1}{\sqrt{t}}\,dt \quad \text{at } x=4$$

Solution. By the Fundamental Theorem of Calculus:

$$\frac{d}{dx}\int_1^x \frac{\sqrt{t}+1}{\sqrt{t}}\,dt = \frac{\sqrt{x}+1}{\sqrt{x}}$$

At $x=4$:

$$\frac{\sqrt{4}+1}{\sqrt{4}} = \boxed{\frac{3}{2}}$$

□

- Find the antiderivative:

$$\int \frac{\sqrt{x}+1}{\sqrt{x}}\,dx$$

Solution. Simplify the integrand:

$$\frac{\sqrt{x}+1}{\sqrt{x}} = \frac{\sqrt{x}}{\sqrt{x}} + \frac{1}{\sqrt{x}} = 1 + x^{-1/2}$$

Integrate term by term:

$$\int (1+x^{-1/2})\,dx = \int 1\,dx + \int x^{-1/2}\,dx$$

$$= x + \frac{x^{1/2}}{1/2} + C = x + 2\sqrt{x} + C$$

Final Answer:

$$\boxed{x + 2\sqrt{x} + C}$$

□

- Evaluate the definite integral:

$$\int_1^4 \frac{\sqrt{x}+1}{\sqrt{x}}\,dx$$

Solution. From the previous result:

$$\int \frac{\sqrt{x}+1}{\sqrt{x}}\,dx = x + 2\sqrt{x} + C$$

therefore:

$$\int_1^4 \frac{\sqrt{x}+1}{\sqrt{x}}\,dx = \left[x + 2\sqrt{x}\right]_1^4 = 4 + 2\sqrt{4} - 1 - 2\sqrt{1} = \boxed{5}$$

□

Problem 5.

- Find the antiderivative:

$$\int x\cos(x^2)\sin(x^2)\,dx$$

Solution. Let $u = x^2$

$$\Rightarrow du = 2x\,dx \quad \Rightarrow \quad x\,dx = \frac{1}{2}du$$

Now substitute into the integral:

$$\int x\cos(x^2)\sin(x^2)\,dx = \frac{1}{2}\int \cos(u)\sin(u)\,du$$

Use substitution again:
Let:

$$v = \sin(u) \quad \Rightarrow \quad dv = \cos(u)\,du$$

Now the integral becomes:

$$\frac{1}{2}\int v\,dv = \frac{1}{2}\cdot\frac{v^2}{2} = \frac{v^2}{4}$$

Substitute back:
Recall $v = \sin(u)$ and $u = x^2$, so:

$$\frac{v^2}{4} = \frac{\sin^2(x^2)}{4}$$

Final Answer:

$$\boxed{\frac{1}{4}\sin^2(x^2) + C}$$

□

- Evaluate the definite integral:

$$\int_0^{\sqrt{\pi}} x\cos(x^2)\sin(x^2)\,dx$$

Solution. Use the antiderivative from earlier:

$$\int x\cos(x^2)\sin(x^2)\,dx = \frac{1}{4}\sin^2(x^2) + C$$

Apply the limits:

$$\left[\frac{1}{4}\sin^2(x^2)\right]_0^{\sqrt{\pi}} = \frac{1}{4}\sin^2(\pi) - \frac{1}{4}\sin^2(0)$$

$$= \frac{1}{4}(0) - \frac{1}{4}(0) = 0$$

Final Answer:

$$\boxed{0}$$

□

Problem 6: Bonus points

Show that for all $x \geq 1$,

$$\int_0^x e^{-t^2} dt \leq \frac{x+1}{2}$$

Solution. We prove:

$$\int_0^x e^{-t^2} dt \leq \frac{x+1}{2}$$

Let's define the function:

$$F(x) = \frac{x+1}{2} - \int_0^x e^{-t^2} dt$$

Our goal is to show $F(x) \geq 0$ for $x \geq 1$.

Differentiate $F(x)$:

Using the Fundamental Theorem of Calculus:

$$F'(x) = \frac{d}{dx}\left(\frac{x+1}{2}\right) - \frac{d}{dx}\left(\int_0^x e^{-t^2} dt\right) = \frac{1}{2} - e^{-x^2}$$

Analyze the derivative for $x \geq 1$:

Since $x \geq 1 \Rightarrow x^2 \geq 1 \Rightarrow e^{-x^2} \leq e^{-1} < 0.5$, it follows that:

$$F'(x) = \frac{1}{2} - e^{-x^2} > 0$$

Therefore: $F(x)$ is **increasing** for $x \geq 1$.

Step 7: Evaluate at $x = 1$.

We compute:

$$F(1) = \frac{1+1}{2} - \int_0^1 e^{-t^2} dt = 1 - \int_0^1 e^{-t^2} dt$$

It is known that:

$$e^{-t^2} \leq 1 \text{ for all } t \text{ in } [0,1] \Rightarrow \int_0^1 e^{-t^2} dt \leq \int_0^1 1 dt \Rightarrow \int_0^1 e^{-t^2} dt \leq 1$$

Therefore $F(1) > 0$ and since $F(x)$ is increasing, it follows that:

$$F(x) > 0 \quad \text{for all } x \geq 1$$

Conclusion:

$$\int_0^x e^{-t^2} dt \leq \frac{x+1}{2} \quad \text{for all } x \geq 1$$

□