Section 5.6

2.) a.) $\int_0^1 r\sqrt{1-r^2}\,dr$ (Let $u = 1-r^2 \to du = -2r\,dr$
$\to -\frac{1}{2}du = r\,dr$; $r: 0 \to 1$ so $u: 1 \to 0$)
$= \int_1^0 \frac{-1}{2}\sqrt{u}\,du = -\frac{1}{2}\cdot \frac{u^{3/2}}{3/2}\Big|_1^0 = -\frac{1}{3}(0^{3/2} - 1^{3/2})$
$= \left(-\frac{1}{3}\right)(-1) = 1/3$

4.) b.) $\int_{2\pi}^{3\pi} 3\cos^2 x \sin x\,dx$ (Let $u = \cos x \to$
$du = -\sin x\,dx \to -du = \sin x\,dx$;
$x: 2\pi \to 3\pi$ so $u: \cos 2\pi \to \cos 3\pi$ or
$u: 1 \to -1$)
$= \int_1^{-1} -3u^2\,du = -u^3\Big|_1^{-1} = -(-1)^3 - -(1)^3 = 2$

7.) b.) $\int_0^1 \frac{5r}{(4+r^2)^2}\,dr$ (Let $u = 4+r^2 \to$
$du = 2r\,dr \to$
$\frac{1}{2}du = r\,dr$; $r: 0 \to 1$ so $u: 4 \to 5$)
$= 5\left(\frac{1}{2}\right)\int_4^5 \frac{1}{u^2}\,du = \frac{5}{2}\int_4^5 u^{-2}\,du = \frac{5}{2}\cdot\frac{u^{-1}}{-1}\Big|_4^5$
$= -\frac{5}{2}\left(\frac{1}{5} - \frac{1}{4}\right) = -\frac{5}{2}\cdot\frac{-1}{20} = \frac{1}{8}$

12.) a.) $\int_0^{\frac{\pi}{6}} (1-\cos 3t)\sin 3t\,dt$ (Let $u = 1 - \cos 3t \to$
$du = -(-\sin 3t \cdot 3)\,dt = 3\sin 3t\,dt \to$
$\frac{1}{3}du = \sin 3t\,dt$; $t: 0 \to \frac{\pi}{6}$ so
$u: (1-\cos 0) \to (1 - \cos\frac{\pi}{2})$ or $u: 0 \to 1$)
$= \int_0^1 \frac{1}{3}\cdot u\,du = \frac{1}{3}\cdot\frac{u^2}{2}\Big|_0^1 = \frac{1}{6}(1-0) = 1/6$

16.) $\int_1^4 \frac{dy}{2\sqrt{y}(1+\sqrt{y})^2}$ (Let $u = 1+\sqrt{y} \to du = \frac{1}{2}y^{-1/2}\,dy$
$= \frac{1}{2\sqrt{y}}\,dy$; $y: 1 \to 4$ so $u: 2 \to 3$)
$= \int_2^3 \frac{1}{u^2}\,du = \frac{-1}{u}\Big|_2^3 = \frac{-1}{3} - \frac{-1}{2} = \frac{1}{6}$

22.) $\int_0^1 (y^3+6y^2-12y+9)^{-1/2}(y^2+4y-4)\,dy$

(Let $u = y^3+6y^2-12y+9 \rightarrow du = (3y^2+12y-12)\,dy$
$= 3(y^2+4y-4)\,dy \rightarrow \frac{1}{3}\,du = (y^2+4y-4)\,dy$;
$y: 0 \rightarrow 1$ so $u: 9 \rightarrow 4$)

$= \int_9^4 \frac{1}{3} u^{-1/2}\,du = \frac{1}{3}\cdot\frac{u^{1/2}}{1/2}\Big|_9^4 = \frac{2}{3}(2-3) = -\frac{2}{3}$

24.) $\int_{-1}^{-1/2} t^{-2}\sin^2\left(1+\frac{1}{t}\right)dt$ (Let $u = 1+\frac{1}{t} \rightarrow$
$du = \frac{-1}{t^2}\,dt \longrightarrow -du = t^{-2}\,dt$; $t: -1 \rightarrow -1/2$
so $u: 0 \rightarrow -1$)

$= -\int_0^{-1} \sin^2 u\,du$ (RECALL: $\cos 2\theta = 1 - 2\sin^2\theta$
$\rightarrow 2\sin^2\theta = 1-\cos 2\theta \rightarrow \sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$.)

$= -\int_0^{-1} \frac{1}{2}(1-\cos 2\theta)\,d\theta = -\frac{1}{2}\left(\theta - \frac{1}{2}\sin 2\theta\right)\Big|_0^{-1}$

$= -\frac{1}{2}\left(-1-\frac{1}{2}\sin(-2)\right) - -\frac{1}{2}\left(0-\frac{1}{2}\cancel{\sin 0}^{\,0}\right)$

$= \frac{1}{2} + \frac{1}{4}\cdot(-\sin 2) = \frac{1}{2} - \frac{\sin 2}{4}$

31.) $\int_2^4 \frac{dx}{x(\ln x)^2}$ (Let $u = \ln x \rightarrow du = \frac{1}{x}\,dx$;
$x: 2 \rightarrow 4$ so $u: \ln 2 \rightarrow \ln 4$)

$= \int_{\ln 2}^{\ln 4} \frac{1}{u^2}\,du = \frac{-1}{u}\Big|_{\ln 2}^{\ln 4} = \frac{-1}{\ln 4} - \frac{-1}{\ln 2}$

$= \frac{-1}{\ln 4} + \frac{2}{2\ln 2} = \frac{-1}{\ln 4} + \frac{2}{\ln 4} = \frac{1}{\ln 4}$

35.) $\int_0^{\frac{\pi}{3}} \tan^2\theta\cos\theta\,d\theta = \int_0^{\frac{\pi}{3}} \frac{\sin^2\theta}{\cos^2\theta}\cdot\cos\theta\,d\theta$

$= \int_0^{\frac{\pi}{3}} \frac{\sin^2\theta}{\cos\theta}\,d\theta = \int_0^{\frac{\pi}{3}} \frac{1-\cos^2\theta}{\cos\theta}\,d\theta$

$= \int_0^{\frac{\pi}{3}} \left(\frac{1}{\cos\theta} - \frac{\cos^2\theta}{\cos\theta}\right)d\theta = \int_0^{\frac{\pi}{3}} (\sec\theta - \cos\theta)\,d\theta$

$$= \left(\ln|\sec\theta + \tan\theta| - \sin\theta\right)\Big|_0^{\pi/3}$$

$$= \left(\ln\left|\sec\tfrac{\pi}{3} + \tan\tfrac{\pi}{3}\right| - \sin\tfrac{\pi}{3}\right)$$

$$\quad - \left(\ln|\sec 0 + \tan 0| - \sin 0\right)$$

$$= \left(\ln|2+\sqrt{3}| - \frac{\sqrt{3}}{2}\right) - \left(\ln|1+0| - 0\right)$$

$$= \ln\left(2+\sqrt{3}\right) - \frac{\sqrt{3}}{2}$$

37.) $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{2\cos\theta}{1+(\sin\theta)^2}\,d\theta$ (Let $u = \sin\theta \rightarrow du = \cos\theta\, d\theta$,

$\theta: -\frac{\pi}{2} \rightarrow \frac{\pi}{2}$, $u = -1 \rightarrow 1$)

$$= 2\int_{-1}^{1} \frac{1}{1+u^2}\,du = 2\arctan u\Big|_{-1}^{1}$$

$$= 2\arctan 1 - 2\arctan(-1)$$

$$= 2\cdot\left(\frac{\pi}{4}\right) - 2\left(-\frac{\pi}{4}\right) = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

39.) $\displaystyle\int_0^{\ln\sqrt{3}} \frac{e^x}{1+(e^x)^2}\,dx$ (Let $u = e^x \rightarrow du = e^x dx$,

$x: 0 \rightarrow \ln\sqrt{3}$, $u: 1 \rightarrow \sqrt{3}$)

$$= \int_1^{\sqrt{3}} \frac{1}{1+u^2}\,du = \arctan u\Big|_1^{\sqrt{3}}$$

$$= \arctan\sqrt{3} - \arctan 1$$

$$= \frac{\pi}{3} - \frac{\pi}{4} = \frac{4\pi}{12} - \frac{3\pi}{12} = \frac{\pi}{12}$$

40.) $\displaystyle\int_1^{e^{\pi/4}} \frac{4}{t(1+(\ln t)^2)}\,dt$ (Let $u = \ln t \rightarrow du = \frac{1}{t}\,dt$,

$t: 1 \rightarrow e^{\pi/4}$, $u: 0 \rightarrow \frac{\pi}{4}$

$$= 4\int_0^{\frac{\pi}{4}} \frac{1}{1+u^2}\,du = 4 \arctan u \Big|_0^{\pi/4}$$

$$= 4 \arctan \frac{\pi}{4} - \cancel{4 \arctan 0}^{\,0} = 4 \arctan \frac{\pi}{4}$$

41.) $$\int_0^1 \frac{4\,ds}{\sqrt{4-s^2}} = 4\int_0^1 \frac{ds}{\sqrt{4\left(1-\frac{s^2}{4}\right)}} = 4\cdot\left(\frac{1}{2}\right)\int_0^1 \frac{ds}{\sqrt{1-\left(\frac{s}{2}\right)^2}}$$

(Let $u = \frac{s}{2} \rightarrow du = \frac{1}{2}\,ds$; $s: 0 \rightarrow 1$ so $u: 0 \rightarrow \frac{1}{2}$)

$$= 4\int_0^{1/2} \frac{1}{\sqrt{1-u^2}}\,du = 4 \arcsin u \Big|_0^{1/2}$$

43.) $$\int_{\sqrt{2}}^{2} \frac{\sec^2(\operatorname{arcsec} x)}{x\sqrt{x^2-1}}\,dx$$ (Let $u = \operatorname{arcsec} x$ $\rightarrow du = \frac{1}{x\sqrt{x^2-1}}\,dx$, $x: \sqrt{2} \rightarrow 2$, $u: \frac{\pi}{4} \rightarrow \frac{\pi}{3}$)

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sec^2 u\,du = \tan u \Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}} = \tan\frac{\pi}{3} - \tan\frac{\pi}{4}$$

$$= \sqrt{3} - 1$$

45.) $$\int_{-1}^{-\sqrt{2}/2} \frac{dY}{Y\sqrt{(2Y)^2-1}}$$ (Let $u = 2Y \rightarrow du = 2\,dy$ $\rightarrow \frac{1}{2}\,du = dy$, $Y = \frac{1}{2}u$

($Y: -1 \rightarrow -\frac{\sqrt{2}}{2}$, $u: -2 \rightarrow -\sqrt{2}$)

$$= \frac{1}{2}\int_{-2}^{-\sqrt{2}} \frac{\cancel{\frac{1}{2}}\,du}{\cancel{\frac{1}{2}}u\sqrt{u^2-1}}$$

$$= \frac{1}{2}\int_{-2}^{-\sqrt{2}} \frac{du}{-|u|\cdot\sqrt{u^2-1}} = \frac{-1}{2}\operatorname{arcsec} u \Big|_{-2}^{-\sqrt{2}}$$

$$= \frac{-1}{2}\operatorname{arcsec}(-\sqrt{2}) - \frac{-1}{2}\operatorname{arcsec}(-2)$$

$$= -\frac{1}{2}\left(\frac{3}{4}\pi\right) + \frac{1}{2}\left(\frac{2}{3}\pi\right) = \frac{-9}{24}\pi + \frac{8}{24}\pi = \frac{-1}{24}\pi$$

47.) (By symmetry) Area $= 2\int_0^2 x\sqrt{4-x^2}\,dx$

(Let $u = 4 - x^2 \rightarrow du = -2x\,dx \rightarrow \frac{-1}{2}du = x\,dx$;
$x: 0 \rightarrow 2$ so $u: 4 \rightarrow 0$)

$= 2\cdot\left(\frac{-1}{2}\right)\int_4^0 \sqrt{u}\,du = -\frac{u^{3/2}}{3/2}\Big|_4^0 = -\frac{2}{3}\left(0^{3/2} - 4^{3/2}\right)$

$= 0 + \frac{2}{3}(8) = 16/3$

53.) (By symmetry) Area $= 2\int_0^2 [2x^2 - (x^4 - 2x^2)]\,dx$

$= 2\int_0^2 (4x^2 - x^4)\,dx = 2\left(\frac{4}{3}x^3 - \frac{x^5}{5}\right)\Big|_0^2$

$= 2\left(\frac{32}{3} - \frac{32}{5}\right) = 2\cdot\left(\frac{64}{15}\right) = \frac{128}{15}$

54.) Area $= \int_0^1 (y^2 - y^3)\,dy = \left(\frac{1}{3}y^3 - \frac{1}{4}y^4\right)\Big|_0^1$

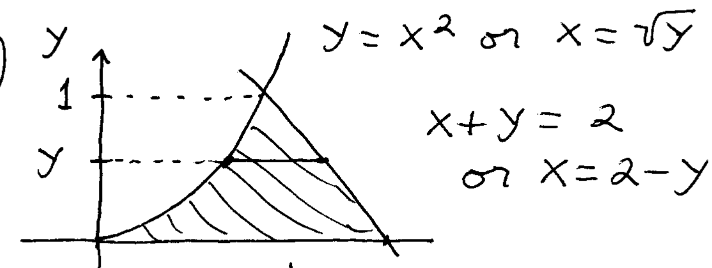
$= \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$

55.) Area $= \int_0^1 [(12y^2 - 12y^3) - (2y^2 - 2y)]\,dy$

$= \int_0^1 [-12y^3 + 10y^2 + 2y]\,dy = \left(-3y^4 + \frac{10}{3}y^3 + y^2\right)\Big|_0^1$

$= -3 + \frac{10}{3} + 1 = 4/3$

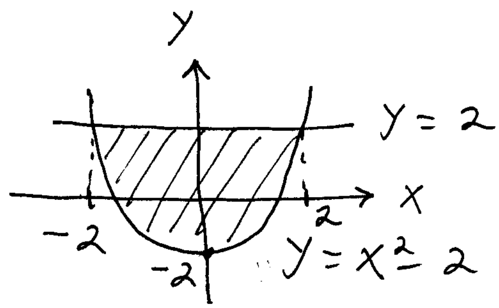
58.)



Area $= \int_0^1 [(2-y) - \sqrt{y}]\,dy$

$= \left(2y - \frac{y^2}{2} - \frac{y^{3/2}}{3/2}\right)\Big|_0^1 = 2 - \frac{1}{2} - \frac{2}{3} = \frac{5}{6}$

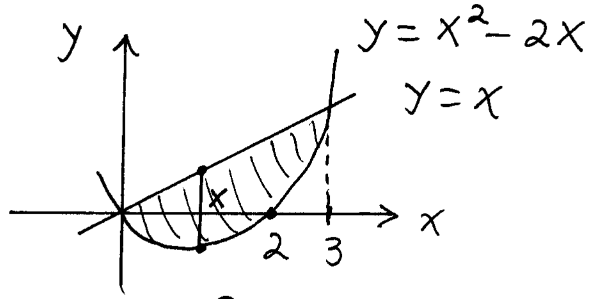
63.)



$x^2 - 2 = 2 \rightarrow x^2 = 4$
$\rightarrow x = \pm 2$

$$\text{Area} = \int_{-2}^{2} [2 - (x^2 - 2)]\,dx = \int_{-2}^{2} (4 - x^2)\,dx$$
$$= \left(4x - \frac{x^3}{3}\right)\Big|_{-2}^{2} = \left(8 - \frac{8}{3}\right) - \left(-8 + \frac{8}{3}\right)$$
$$= 16 - \frac{16}{3} = \frac{32}{3}$$

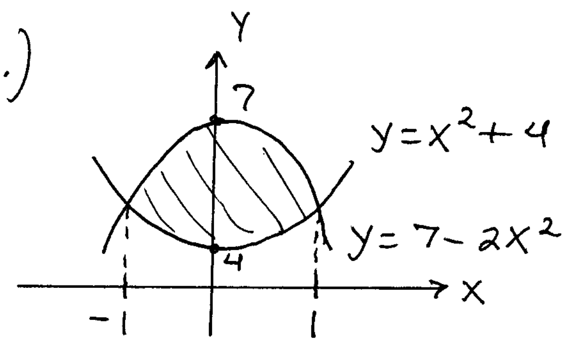
66.)



$x^2 - 2x = x \rightarrow$
$x^2 - 3x = 0 \rightarrow$
$x(x - 3) = 0 \rightarrow$
$x = 0$ or $x = 3$;

$$\text{Area} = \int_{0}^{3} (x - (x^2 - 2x))\,dx = \int_{0}^{3} (3x - x^2)\,dx$$
$$= \left(\frac{3}{2}x^2 - \frac{x^3}{3}\right)\Big|_{0}^{3} = \frac{27}{2} - 9 = \frac{9}{2}$$

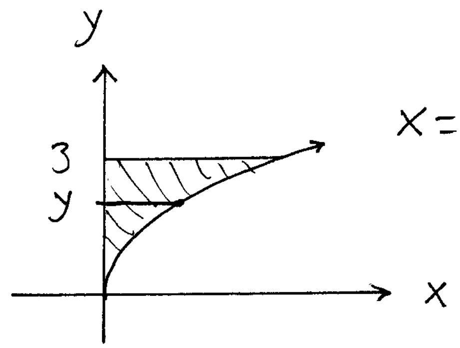
68.)



$x^2 + 4 = 7 - 2x^2 \rightarrow$
$3x^2 = 3 \rightarrow x^2 = 1 \rightarrow$
$x = \pm 1$

$$\text{Area} = \int_{-1}^{1} [(7 - 2x^2) - (x^2 + 4)]\,dx$$
$$= \int_{-1}^{1} [3 - 3x^2]\,dx = (3x - x^3)\Big|_{-1}^{1}$$
$$= (3 - 1) - (-3 + 1) = 2 - (-2) = 4$$

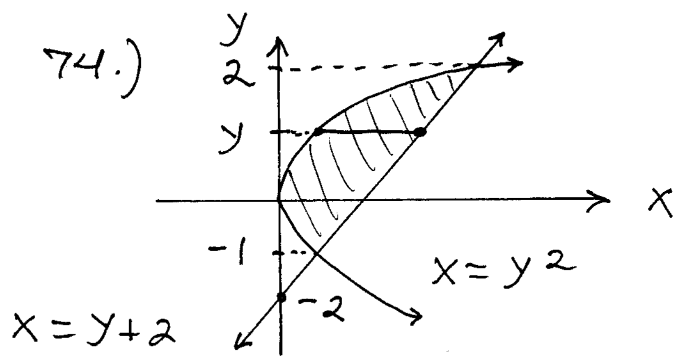
73.)



$x = 2y^2$ $\quad$ $\text{Area} = \int_0^3 2y^2\, dy$

$= \frac{2}{3} y^3 \Big|_0^3 = 18$

74.)



$y^2 = y + 2 \rightarrow$

$y^2 - y - 2 = 0 \rightarrow$

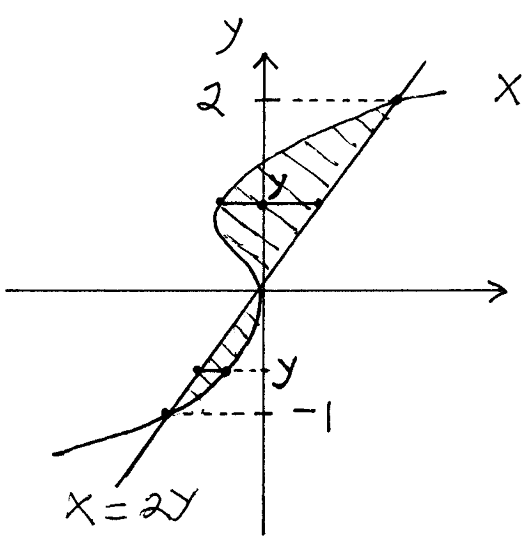
$(y-2)(y+1) = 0 \rightarrow$

$y = 2, \ y = -1$

$$\text{Area} = \int_{-1}^{2} [(y+2) - y^2]\, dy = \left(\tfrac{1}{2}y^2 + 2y - \tfrac{1}{3}y^3\right)\Big|_{-1}^{2}$$

$$= \left(2 + 4 - \tfrac{8}{3}\right) - \left(\tfrac{1}{2} - 2 + \tfrac{1}{3}\right) = 8 - 3 - \tfrac{1}{2} = \tfrac{9}{2}$$

80.)



$x = y^3 - y^2$

$y^3 - y^2 = 2y \rightarrow$

$y^3 - y^2 - 2y = 0 \rightarrow$

$y(y-2)(y+1) = 0 \rightarrow$

$y = 0, \ y = 2, \text{ or } y = -1$

$$\text{Area} = \int_{-1}^{0} [(y^3 - y^2) - 2y]\, dy$$

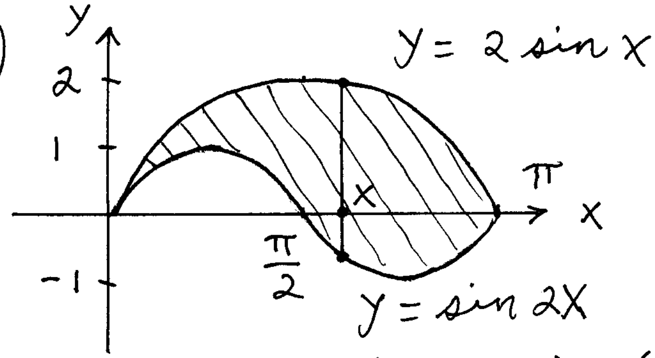
$$+ \int_0^2 [2y - (y^3 - y^2)]\, dy$$

$$= \left(\tfrac{1}{4}y^4 - \tfrac{1}{3}y^3 - y^2\right)\Big|_{-1}^{0}$$

$$+ \left(y^2 - \tfrac{1}{4}y^4 + \tfrac{1}{3}y^3\right)\Big|_0^2$$

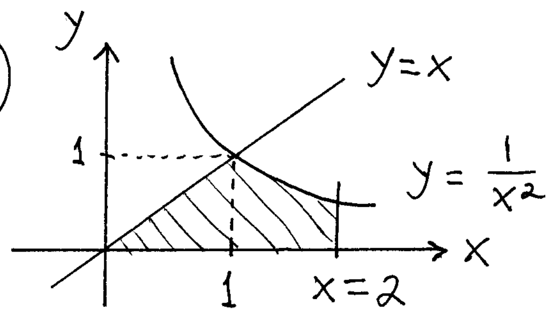
$$= (0) - \left(\tfrac{1}{4} + \tfrac{1}{3} - 1\right) + \left(4 - 4 + \tfrac{8}{3}\right) - (0)$$
$$= \frac{5}{12} + \frac{8}{3} = \frac{37}{12} .$$

85.)



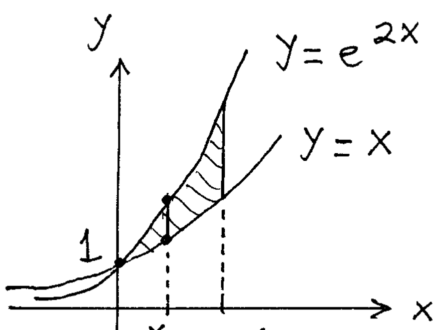
$$\text{Area} = \int_0^{\pi} (2\sin x - \sin 2x)\, dx$$
$$= \left(-2\cos x + \tfrac{1}{2}\cos 2x\right)\Big|_0^{\pi}$$
$$= \left(-2\cos\pi + \tfrac{1}{2}\cos 2\pi\right) - \left(-2\cos 0 + \tfrac{1}{2}\cos 0\right)$$
$$= +2 + \tfrac{1}{2} + 2 - \tfrac{1}{2} = 4$$

95.)



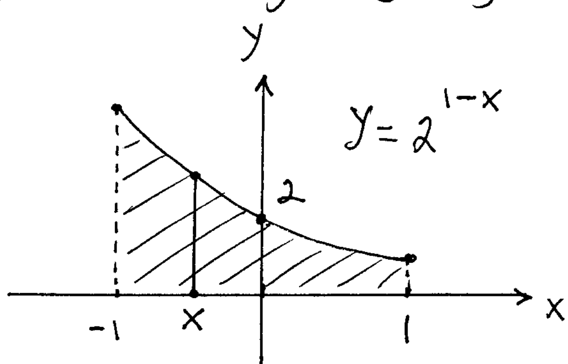
$$\text{Area} = \int_0^1 x\, dx + \int_1^2 \frac{1}{x^2}\, dx$$
$$= \frac{x^2}{2}\Big|_0^1 + \frac{-1}{x}\Big|_1^2$$
$$= \left(\tfrac{1}{2} - 0\right) + \left(\tfrac{-1}{2} - {}^{-}1\right) = 1$$

99.)



$$\text{Area} = \int_0^{\ln 3} (e^{2x} - e^x)\, dx$$
$$= \left(\tfrac{1}{2}e^{2x} - e^x\right)\Big|_0^{\ln 3}$$
$$= \left(\tfrac{1}{2}e^{2\ln 3} - e^{\ln 3}\right) - \left(\tfrac{1}{2}e^0 - e^0\right)$$
$$= \left(\tfrac{1}{2}e^{\ln 3^2} - 3\right) - \left(\tfrac{1}{2} - 1\right) = \tfrac{9}{2} - 3 + \tfrac{1}{2} = 2$$

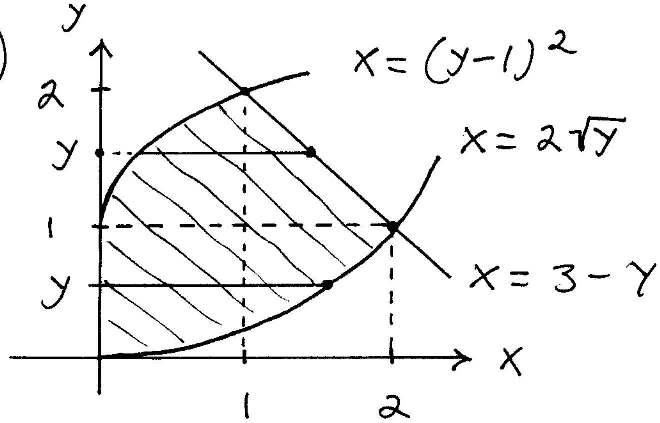
102.)



$$\text{Area} = \int_{-1}^{1} 2^{1-x}\, dx$$
$$= \frac{-1}{\ln 2}\, 2^{1-x}\Big|_{-1}^{1}$$
$$= \frac{-1}{\ln 2}\cdot 2^0 - \frac{-1}{\ln 2}\cdot 2^2$$

$= \frac{-1}{\ln 2} + \frac{4}{\ln 2} = \frac{3}{\ln 2}$

106.)

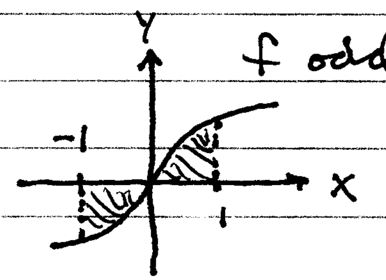


$$\text{Area} = \int_0^1 2\sqrt{y}\,dy + \int_1^2 \left[(3-y)-(y-1)^2\right] dy$$

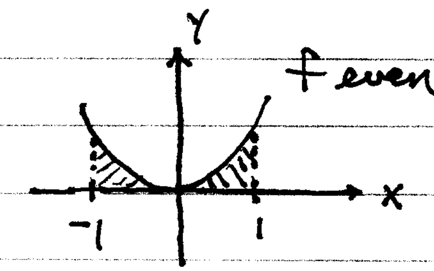
112.) $\int_0^1 f(1-x)\,dx$ (Let $u = 1-x \rightarrow$
$du = -dx \rightarrow -du = dx$,
$x: 0 \rightarrow 1$, $u: 1 \rightarrow 0$)

$= -\int_1^0 f(u)\,du = \int_0^1 f(u)\,du = \int_0^1 f(x)\,dx$

113.) $\int_0^1 f(x)\,dx = 3$



a.) f odd, so $f(-x) = -f(x)$,
then $\int_{-1}^0 f(x)\,dx = -3$



b.) f even, so $f(-x) = f(x)$,
then $\int_{-1}^0 f(x)\,dx = 3$

115.) $I = \int_0^a \frac{f(x)\,dx}{f(x)+f(a-x)}$ (Let $u = a - x \longrightarrow x = a-u$ and $du = -dx$; $x: 0 \to a$ so $u = a \to 0$)

$$= -\int_a^0 \frac{f(a-u)\,du}{f(a-u)+f(u)} = \int_0^a \frac{f(a-u)+f(u)-f(u)}{f(a-u)+f(u)}\,du$$

$$= \int_0^a \left(\frac{f(a-u)+f(u)}{f(a-u)+f(u)} - \frac{f(u)}{f(a-u)+f(u)} \right) du$$

$$= \int_0^a 1\,du - \int_0^a \frac{f(u)}{f(u)+f(a-u)}\,du$$

$$= u \Big|_0^a - I$$

$$= (a-0) - I$$

$$= a - I$$ so that

$$I = a - I \to 2I = a \to I = \frac{a}{2}$$

116.) $\int_1^Y \frac{1}{t}\,dt = \int_1^Y \frac{X}{Xt}\,dt$ (Let $u = Xt \to du = X\,dt$, $t: 1 \to Y$, $u: X \to XY$)

$$= \int_X^{XY} \frac{1}{u}\,du = \int_X^{XY} \frac{1}{t}\,dt$$