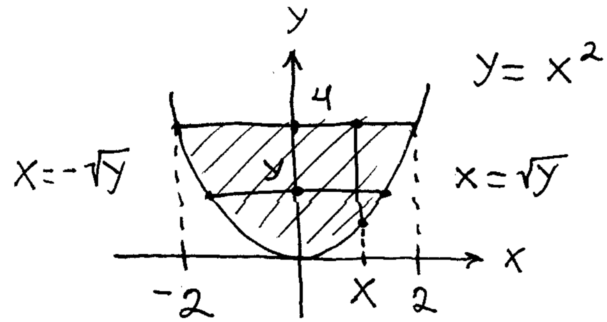
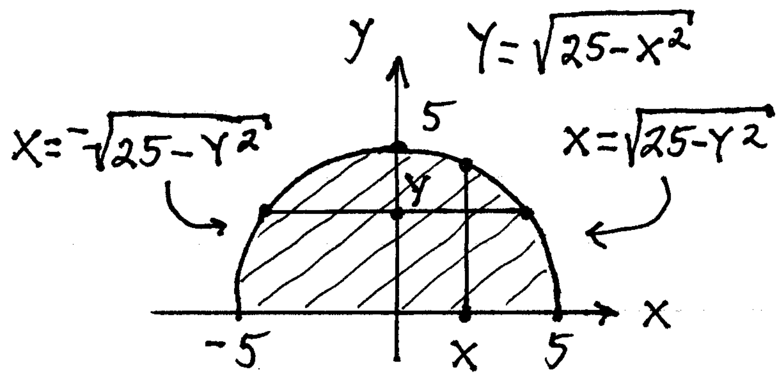
Section 6.6

1.) $$\bar{x} = \frac{\int_{-2}^{2} x\,(4-x^2)\,dx}{\int_{-2}^{2} (4-x^2)\,dx},$$



$$\bar{y} = \frac{\int_{0}^{4} y\,(\sqrt{y} - (-\sqrt{y}))\,dy}{\int_{0}^{4} (\sqrt{y} - (-\sqrt{y}))\,dy}$$

2.) $$\bar{x} = \frac{\int_{-5}^{5} x\sqrt{25-x^2}\,dx}{\int_{-5}^{5} \sqrt{25-x^2}\,dx},$$



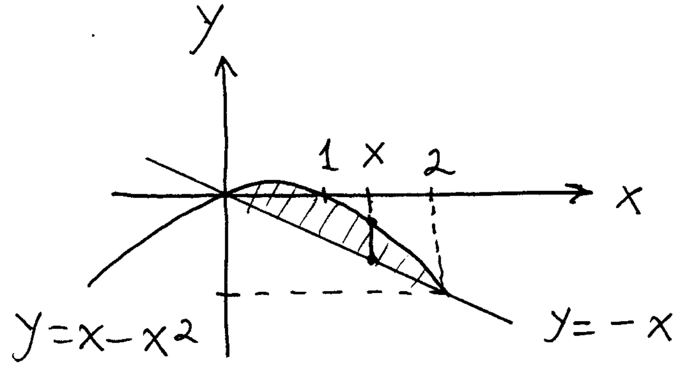
$$\bar{y} = \frac{\int_{0}^{5} y\,(\sqrt{25-y^2} - {}^{-}\sqrt{25-y^2})\,dy}{\int_{0}^{5} (\sqrt{25-y^2} - {}^{-}\sqrt{25-y^2})\,dy}$$

OR

$$\bar{y} = \frac{\frac{1}{2}\int_{-5}^{5} (\sqrt{25-x^2})^2 - (0)^2\,dx}{\int_{-5}^{5} \sqrt{25-x^2}\,dx}$$

3.) $x - x^2 = -x \rightarrow$

$0 = x^2 - 2x = x(x-2)$

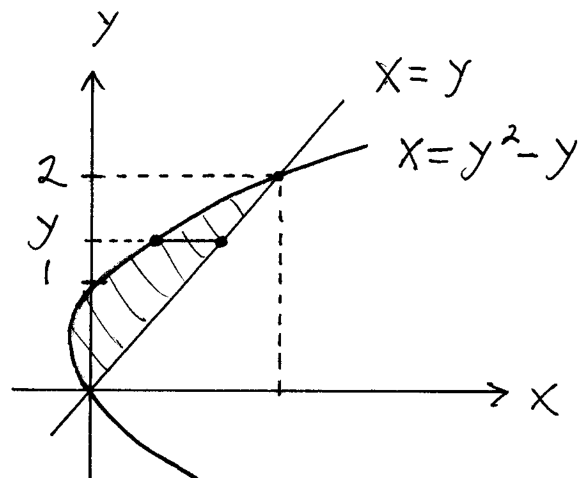


$$\bar{x} = \frac{\int_0^2 x\left((x-x^2)-(-x)\right)dx}{\int_0^2 \left((x-x^2)-(-x)\right)dx},$$

$$\bar{y} = \frac{\int_0^2 \frac{1}{2}\left((x-x^2)^2-(-x)^2\right)dx}{\int_0^2 \left((x-x^2)-(-x)\right)dx}$$

6.) $y^2 - y = y \rightarrow$

$y^2 - 2y = y(y-2) = 0$

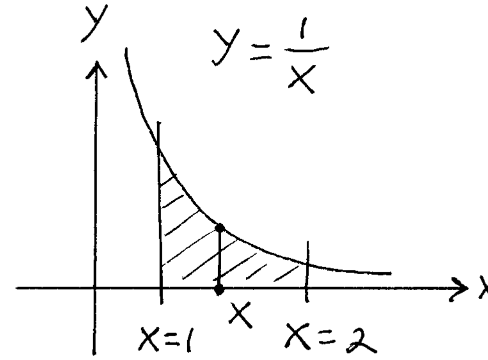


$$\bar{y} = \frac{\int_0^2 y\left(y-(y^2-y)\right)dy}{\int_0^2 \left(y-(y^2-y)\right)dy},$$

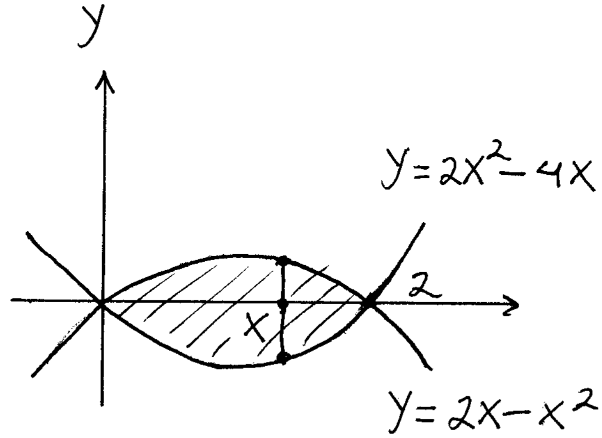
$$\bar{x} = \frac{\int_0^2 \frac{1}{2}\left((y)^2-(y^2-y)^2\right)dy}{\int_0^2 \left(y-(y^2-y)\right)dy}$$

9.) $\bar{x} = \int_1^2 x\left(\frac{1}{x}\right)dx \Big/ \int_1^2 \frac{1}{x}\,dx,$

$$\bar{y} = \frac{\int_1^2 \frac{1}{2}\left(\frac{1}{x}\right)^2 dx}{\int_1^2 \frac{1}{x}\,dx}$$



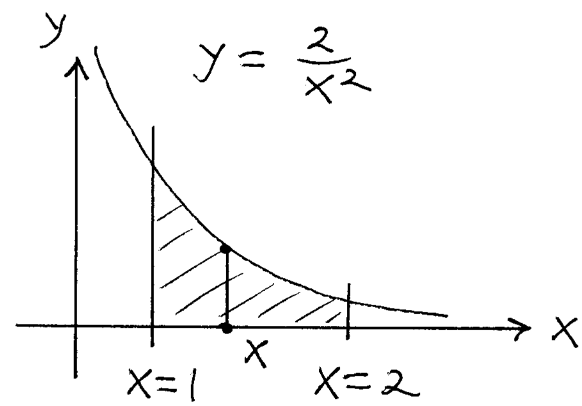
12.) $2x^2 - 4x = 2x - x^2 \rightarrow$
$3x^2 - 6x = 0 \rightarrow$
$3x(x-2) = 0$



$$\bar{x} = \frac{\int_0^2 x\left((2x-x^2)-(2x^2-4x)\right)dx}{\int_0^2 \left((2x-x^2)-(2x^2-4x)\right)dx},$$

$$\bar{y} = \frac{\int_0^2 \frac{1}{2}\left((2x-x^2)^2-(2x^2-4x)^2\right)dx}{\int_0^2 \left((2x-x^2)-(2x^2-4x)\right)dx}$$

15.) density $\delta(x,y) = x^2$

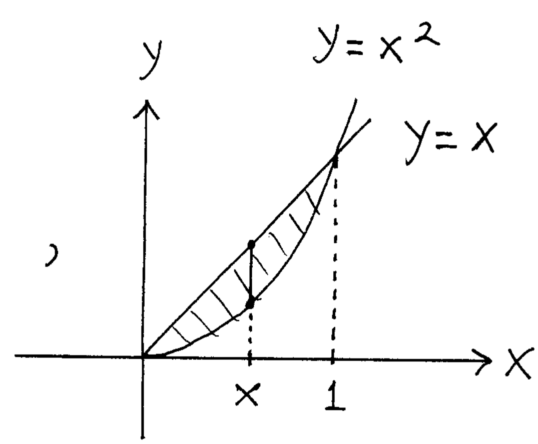


$$\bar{x} = \frac{\int_1^2 x\left(\frac{2}{x^2}\right)\cdot(x^2)\,dx}{\int_1^2 \left(\frac{2}{x^2}\right)\cdot(x^2)\,dx},$$

$$\bar{y} = \frac{\int_1^2 \frac{1}{2}\left(\frac{2}{x^2}\right)^2\cdot(x^2)\,dx}{\int_1^2 \left(\frac{2}{x^2}\right)(x^2)\,dx}$$

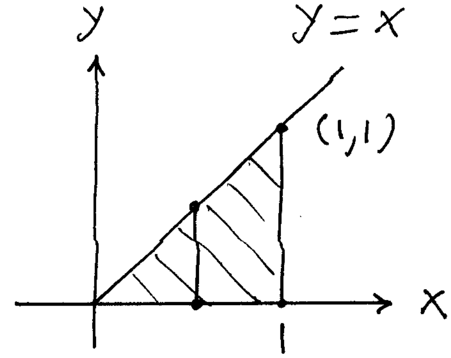
16.) $$\bar{x} = \frac{\int_0^1 x\cdot(x-x^2)\cdot(12x)\,dx}{\int_0^1 (x-x^2)\cdot(12x)\,dx},$$

$$\bar{y} = \frac{\int_0^1 \frac{1}{2}\left((x)^2-(x^2)^2\right)\cdot(12x)\,dx}{\int_0^1 (x-x^2)\cdot(12x)\,dx}$$



density
$\delta(x,y) = 12x$

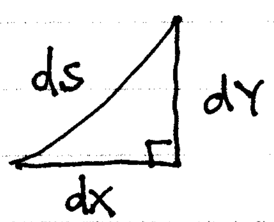
21.) $\bar{x} = \dfrac{\int_0^1 x(x)\,dx}{\int_0^1 x\,dx}$

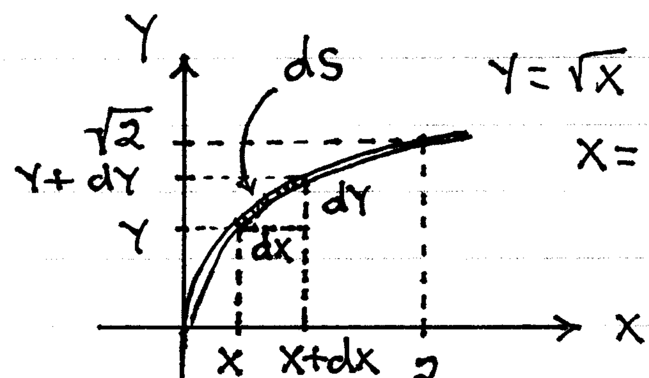


$= \dfrac{x^3/3\,\big|_0^1}{x^2/2\,\big|_0^1} = \dfrac{\frac{1}{3}}{\frac{1}{2}} = \dfrac{2}{3}$,

$\bar{y} = \dfrac{\int_0^1 \frac{1}{2}(x)^2\,dx}{\int_0^1 x\,dx} = \dfrac{\frac{1}{2}\cdot\frac{1}{3}x^3\big|_0^1}{1/2} = \dfrac{1}{3}$

25.)





$Y = \sqrt{X}$ or $X = Y^2$

$ds = \sqrt{(dx)^2 + (dY)^2}$

density $\delta(X) = K$

a.) Moment of small piece about $y=0$ is

$\approx$ (mass)(distance)

= (length)(density)(distance)

= (ds) (k) (Y)

$= K\cdot\sqrt{X}\cdot\sqrt{(dx)^2+(dY)^2}$

$= K\sqrt{X}\sqrt{\left(1+\dfrac{(dY)^2}{(dX)^2}\right)\cdot(dX)^2} = K\sqrt{X}\sqrt{1+\left(\dfrac{dY}{dX}\right)^2}\,dx$

$$= k\sqrt{x} \cdot \sqrt{1 + \left(\frac{1}{2\sqrt{x}}\right)^2}\, dx$$

$$= k\sqrt{x} \cdot \sqrt{1 + \frac{1}{4x}}\, dx$$

$$= k\sqrt{x} \cdot \sqrt{\frac{4x}{4x} + \frac{1}{4x}}\, dx$$

$$= k\sqrt{x} \cdot \frac{\sqrt{4x+1}}{\sqrt{4x}}\, dx$$

$= \frac{k}{2} \cdot \cancel{\frac{\sqrt{x}}{\sqrt{x}}} \sqrt{4x+1}\, dx$; so total moment of wire about $y=0$ is

$$M_{y=0} = \int_0^2 \frac{k}{2} \sqrt{4x+1}\, dx$$

$$= \frac{k}{\cancel{2}} \cdot \frac{1}{4} \cdot \frac{\cancel{2}}{3} (4x+1)^{3/2} \Big|_0^2$$

$$= \frac{1}{12} k \left(9^{3/2} - 1^{3/2}\right) = \frac{1}{12} k (27 - 1) = \frac{13}{6} k$$

b.) Moment of small piece about $x=0$ is

$\approx$ (mass)(distance)

$=$ (length)(density)(distance)

$= (ds)(k)(x)$

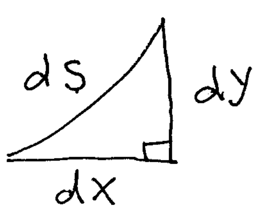
$= k y^2 \sqrt{(dx)^2 + (dy)^2}$

$$= kY^2\sqrt{\left(\frac{(dx)^2}{(dY)^2}+1\right)(dY)^2} = kY^2\sqrt{1+\left(\frac{dx}{dY}\right)^2}\ dY$$

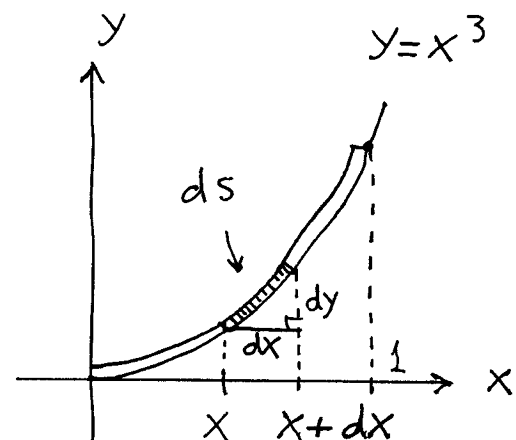
$$= ky^2\sqrt{1+(2Y)^2}\ dy$$

$$= ky^2\sqrt{1+4Y^2}\ dy\ ;$$ so total moment of wire about $x=0$ is

$$M_{x=0} = \int_0^{\sqrt{2}} ky^2\sqrt{1+4Y^2}\ dy$$

26.) 

$$ds = \sqrt{(dx)^2+(dy)^2}$$



density
$\delta(x) = k$

Moment of small piece about $y=0$ is

$\approx$ (mass) (distance)

$=$ (length)(density) (distance)

$= (ds)(k)(x^3)$

$= kx^3 \sqrt{(dx)^2 + (dy)^2}$

$= k \cdot x^3 \cdot \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \cdot dx$; so

total moment of wire about $y=0$ is

$$M_{y=0} = \int_0^1 k x^3 \cdot \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$

$$= \int_0^1 k x^3 \cdot \sqrt{1 + (3x^2)^2} \, dx$$

$$= \int_0^1 k x^3 \cdot \sqrt{1 + 9x^4} \, dx$$

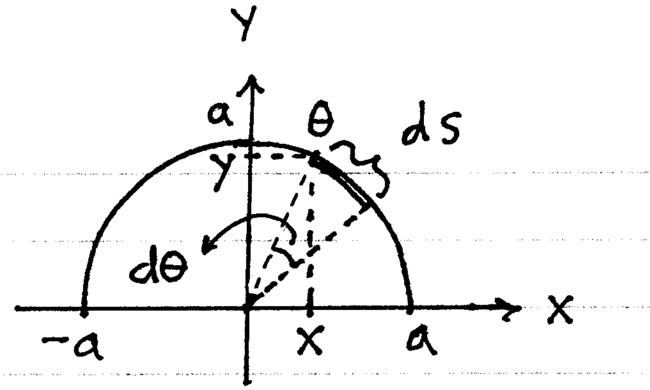
$$= k \cdot \frac{1}{36} \left. \frac{(1+9x^4)^{3/2}}{3/2} \right|_0^1$$

$$= \frac{k}{54} \left(10^{3/2} - 1^{3/2}\right)$$

$$= \frac{k}{54} \left(10^{3/2} - 1\right)$$

27.) By symmetry of shape and density, it follows that

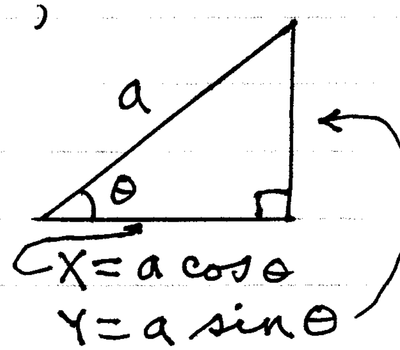
$$\bar{X} = 0 \;;$$



density $\delta = k \sin\theta$

$Y = \bar{Y}$

$ds = (\text{radius})(\text{angle}) = a \cdot d\theta \;;$



moment of small piece about $Y = \bar{Y}$ is

$= (\text{mass})(\text{distance})$

$= (\text{density})(\text{length})(\text{distance})$

$= (k \sin\theta)(ds)(Y - \bar{Y})$

$= k \sin\theta \, (Y - \bar{Y}) \, ds \;;$ so total moment about $Y = \bar{Y}$ is $M_{Y=\bar{Y}} = \int k \sin\theta \, (Y - \bar{Y}) \, ds \;;$

if $\bar{Y}$ is center of mass, then

$M_{Y=\bar{Y}} = \int k \sin\theta \, (Y - \bar{Y}) \, ds = 0 \rightarrow$

$\int (kY \sin\theta) \, ds - \int k\bar{Y} \sin\theta \, ds = 0 \rightarrow$

$\cancel{k} \int Y \sin\theta \, ds = \cancel{k} \bar{Y} \int \sin\theta \, ds \rightarrow$

$$\bar{Y} = \frac{\int Y \sin\theta \, ds}{\int \sin\theta \, ds}$$

$$= \frac{\int_0^\pi (a \sin\theta) \sin\theta \cdot a \, d\theta}{\int_0^\pi \sin\theta \cdot a \, d\theta}$$

$$= \frac{a^2 \int_0^\pi \sin^2\theta \, d\theta}{-a \cos\theta \Big|_0^\pi}$$

$$= \frac{a^2 \int_0^\pi \frac{1}{2}(1 - \cos 2\theta) \, d\theta}{-a \cancel{\cos}\overset{-1}{\pi} - -a \cancel{\cos}\overset{1}{0}}$$

$$= \frac{a^2 \, \frac{1}{2}\left(\theta - \frac{1}{2}\sin 2\theta\right) \Big|_0^\pi}{2a}$$

$$= \frac{\frac{1}{2} a^2 \left[\left(\pi - \frac{1}{2}\overset{0}{\cancel{\sin 2\pi}}\right) - \left(0 - \frac{1}{2}\overset{0}{\cancel{\sin 0}}\right)\right]}{2a}$$

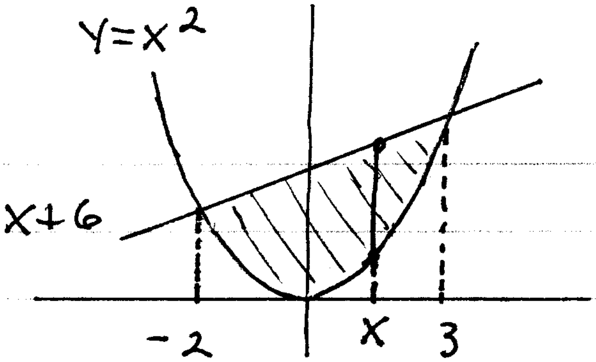
$$= \frac{1}{4} a \pi \; .$$

29.) $x^2 = x+6 \rightarrow$

$x^2 - x - 6 = 0 \rightarrow$

$(x-3)(x+2) = 0 \rightarrow$

$x = 3, \; x = -2$



$$\bar{x} = \frac{\int_{-2}^{3} x\left[(x+6) - x^2\right] dx}{\int_{-2}^{3} \left[(x+6) - x^2\right] dx},$$

$$\bar{y} = \frac{\frac{1}{2}\int_{-2}^{3} \left[(x+6)^2 - (x^2)^2\right] dx}{\int_{-2}^{3} \left[(x+6) - x^2\right] dx}$$

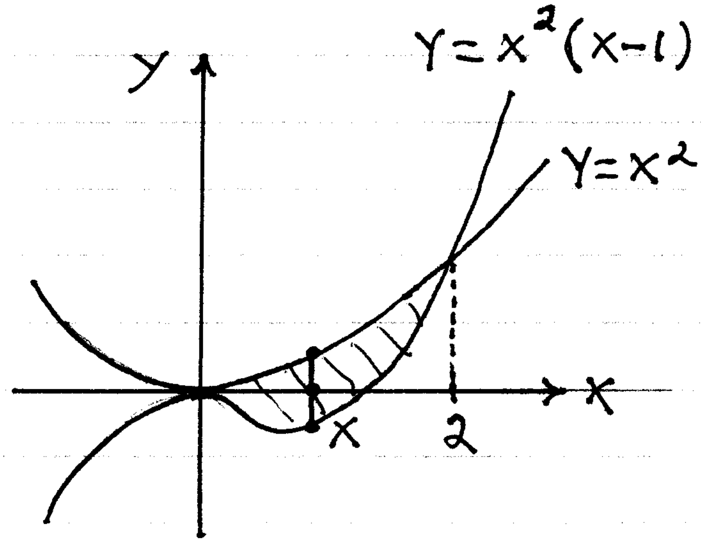
31.) $x^2(x-1) = x^2 \rightarrow$

$x^2(x-1) - x^2 = 0 \rightarrow$

$x^2\left[(x-1) - 1\right] = 0 \rightarrow$

$x^2(x-2) = 0 \rightarrow$

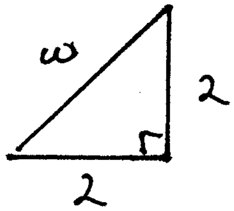
$x = 0, \; x = 2$

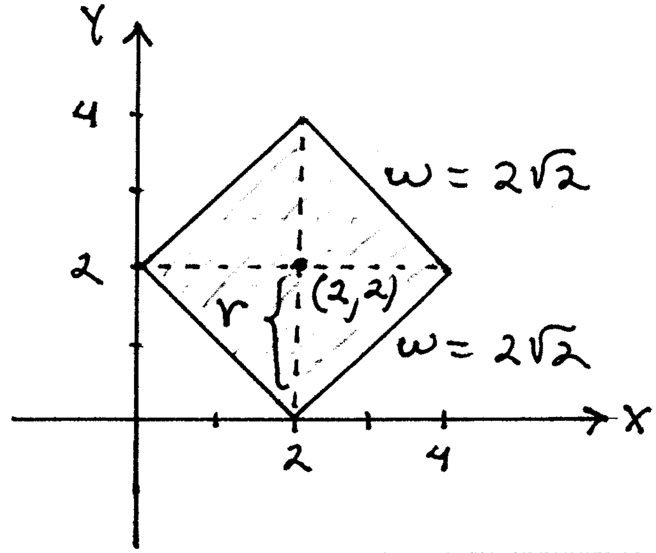


$$\bar{x} = \frac{\int_{0}^{2} x\left(x^2 - x^2(x-1)\right) dx}{\int_{0}^{2} \left(x^2 - x^2(x-1)\right) dx}$$

$$\bar{y} = \frac{\frac{1}{2}\int_{0}^{2} \left[(x^2)^2 - (x^2(x-1))^2\right] dx}{\int_{0}^{2} \left(x^2 - x^2(x-1)\right) dx}$$

35.)

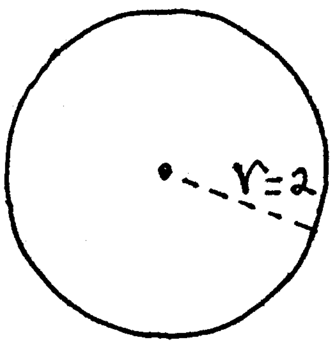




$w^2 = 2^2 + 2^2 = 8$

$\rightarrow w = \sqrt{8} = 2\sqrt{2}$

centroid is $\bar{x} = 2$, $\bar{Y} = 2$; then



$C = 2\pi r = 2\pi(2) = 4\pi$.

a.) Volume = (area) (distance)

$= (2\sqrt{2})^2 \cdot (4\pi)$

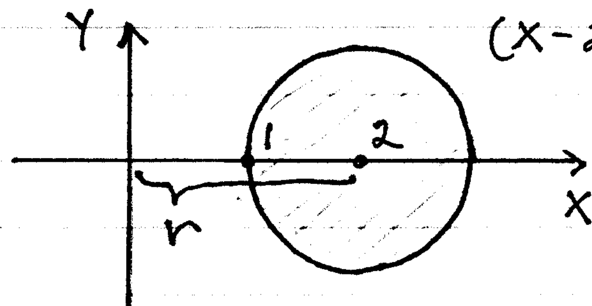
$= 32\pi$

b.) Surface area = (perimeter)(distance)

$= (4(2\sqrt{2}))(4\pi)$

$= 32\sqrt{2}\,\pi$

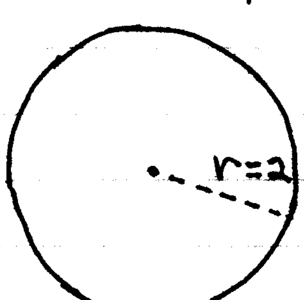
37.)



$(x-2)^2 + y^2 = 1$

centroid is $\bar{x} = 2$, $\bar{Y} = 0$; then

$C = 2\pi r = 2\pi(2) = 4\pi$



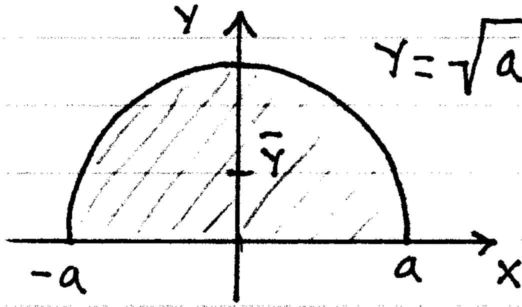
a.) Volume = (area) (distance)

$= (\pi(1)^2)(4\pi) = 4\pi^2$

b.) Surface area = (perimeter)(distance)

$= (2\pi(1))(4\pi) = 8\pi^2$

39.)



$Y = \sqrt{a^2 - x^2}$

By symmetry $\bar{X} = 0$;

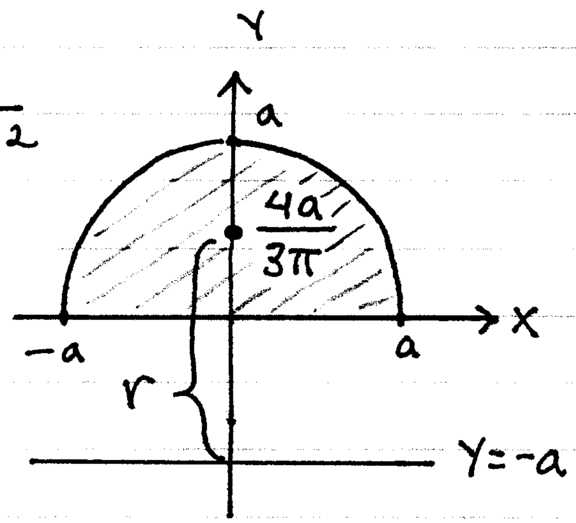
Surface area = (perimeter)(dist. $\bar{Y}$ travels) $\rightarrow$

$2\cancel{4\pi}a^{\cancel{2}} = \left(\tfrac{1}{\cancel{2}}\cancel{2\pi}(\cancel{a})\right)(\cancel{2}\pi\bar{Y}) \longrightarrow$

$\bar{Y} = \dfrac{2a}{\pi}$

$Y = \sqrt{a^2 - x^2}$

42.)



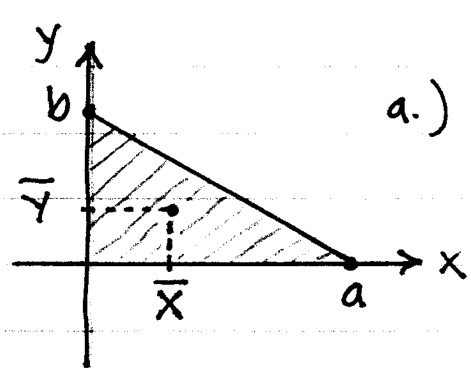
Volume = (area)(distance)

$= \left(\frac{1}{2}\pi(a)^2\right)\left(\frac{4a}{3\pi} + a\right)$

$= \left(\frac{1}{2}\pi a^2\right)\cdot a\left(\frac{4}{3\pi} + \frac{3\pi}{3\pi}\right)$

$= \frac{1}{2}\cancel{\pi}\, a^3 \cdot \frac{1}{3\cancel{\pi}}(4+3\pi) = \frac{1}{6}a^3(4+3\pi)$

45.)



a.) to find $\bar{X}$ rotate region about Y-axis to form cylinder:

Volume = (area)(distance) $\rightarrow$

$$\frac{1}{3}\pi a^2 b = \left(\frac{1}{\cancel{2}} ab\right)(\cancel{2}\pi \bar{x}) \rightarrow$$

$$\bar{x} = \frac{\frac{1}{3}\cancel{\pi}\, a^{\cancel{2}}\cancel{b}}{\cancel{a}\cancel{b}\cancel{\pi}} = \frac{1}{3}a$$

b.) to find $\bar{Y}$ rotate region about x-axis to form cylinder:

Volume = (area)(distance) $\rightarrow$

$$\frac{1}{3}\pi b^2 a = \left(\frac{1}{\cancel{2}} ab\right)(\cancel{2}\pi \bar{Y}) \rightarrow$$

$$\bar{Y} = \frac{\frac{1}{3}\cancel{\pi}\, b^{\cancel{2}}\cancel{a}}{\cancel{a}\cancel{b}\cancel{\pi}} = \frac{1}{3}b.$$