Section 7.2

1.) $2Y' + 3Y = e^{-x}$

<u>a.)</u> $Y = e^{-x} \xrightarrow{D} Y' = -e^{-x}$ then

$2Y' + 3Y = 2(-e^{-x}) + 3(e^{-x}) = e^{-x}$

<u>b.)</u> $Y = e^{-x} + e^{-\frac{3}{2}x} \xrightarrow{D} Y' = -e^{-x} - \frac{3}{2}e^{-\frac{3}{2}x}$ then

$2Y' + 3Y = 2(-e^{-x} - \frac{3}{2}e^{-\frac{3}{2}x}) + 3(e^{-x} + e^{-\frac{3}{2}x})$

$= -2e^{-x} - 3e^{-3/2x} + 3e^{-x} + 3e^{-3/2x}$

$= e^{-x}$

<u>c.)</u> $Y = e^{-x} + ce^{-\frac{3}{2}x} \xrightarrow{D} Y' = -e^{-x} - \frac{3}{2}ce^{-3/2x}$ then

$2Y' + 3Y = 2(-e^{-x} - \frac{3}{2}ce^{-3/2x}) + 3(e^{-x} + ce^{-3/2x})$

$= -2e^{-x} - 3ce^{-3/2x} + 3e^{-x} + 3ce^{-3/2x}$

$= e^{-x}$

2.) $Y' = Y^2$

<u>a.)</u> $Y = \frac{-1}{x} \xrightarrow{D} Y' = \frac{1}{x^2}$ then

$Y^2 = \left(\frac{-1}{x}\right)^2 = \frac{1}{x^2} = Y'$

<u>b.)</u> $Y = \frac{-1}{x+3} \xrightarrow{D} Y' = \frac{1}{(x+3)^2}$ then

$Y^2 = \left(\frac{-1}{x+3}\right)^2 = \frac{1}{(x+3)^2} = Y'$

<u>c.)</u> $Y = \frac{-1}{x+c} \xrightarrow{D} Y' = \frac{1}{(x+c)^2}$ then

$Y^2 = \left(\frac{-1}{x+c}\right)^2 = \frac{1}{(x+c)^2} = Y'$

3.) $x^2Y' + xY = e^x$

$$Y = \frac{1}{X} \cdot \int_1^X \frac{e^t}{t}\,dt \xrightarrow{D} \text{(product rule)}$$

$$Y' = \frac{1}{X} \cdot \frac{e^X}{X} + \left(\frac{-1}{X^2}\right) \cdot \int_1^X \frac{e^t}{t}\,dt$$

$$= \frac{1}{X^2}\left(e^X - \int_1^X \frac{e^t}{t}\,dt\right) \quad \text{then}$$

$$X^2 Y' + XY = \cancel{X^2} \cdot \frac{1}{\cancel{X^2}}\left(e^X - \int_1^X \frac{e^t}{t}\,dt\right) + \cancel{X} \cdot \frac{1}{\cancel{X}} \int_1^X \frac{e^t}{t}\,dt$$

$$= e^X - \int_1^X \frac{e^t}{t}\,dt + \int_1^X \frac{e^t}{t}\,dt = e^X$$

7.) $y = \dfrac{\cos X}{X} \longrightarrow X = \dfrac{\pi}{2}$ and $Y = \dfrac{\cos\frac{\pi}{2}}{\frac{\pi}{2}} = \dfrac{0}{\frac{\pi}{2}} = 0$

$$\xrightarrow{D} y' = \frac{X \cdot -\sin X - \cos X \cdot (1)}{X^2} \quad \text{then}$$

$$XY' + Y = X \cdot \frac{-X\sin X - \cos X}{X^2} + \frac{\cos X}{X}$$

$$= \frac{-X\sin X - \cos X}{X} + \frac{\cos X}{X}$$

$$= \frac{-\cancel{X}\sin X}{\cancel{X}} - \frac{\cos X}{X} + \frac{\cos X}{X} = -\sin X$$

10.) $\dfrac{dY}{dX} = X^2\sqrt{Y} \longrightarrow \displaystyle\int \frac{1}{\sqrt{Y}}\,dY = \int X^2\,dx \longrightarrow$

$$\int Y^{-1/2}\,dy = \frac{1}{3}X^3 + C \longrightarrow 2Y^{1/2} = \frac{1}{3}X^3 + C$$

11.) $\dfrac{dY}{dX} = e^{X-Y} = e^X e^{-Y} \longrightarrow \displaystyle\int \frac{1}{e^{-Y}}\,dY = \int e^X\,dx$

$\rightarrow \int e^{Y} dy = e^{X} + c \rightarrow e^{Y} = e^{X} + c$

12.) $\frac{dY}{dX} = 3X^2 e^{-Y} \longrightarrow \int \frac{1}{e^{-Y}} dY = \int 3X^2 dx \rightarrow$

$\int e^{Y} dy = X^3 + c \longrightarrow e^{Y} = X^3 + c$

13.) $\frac{dY}{dX} = \sqrt{Y} \cos^2 \sqrt{Y} \rightarrow \int \frac{1}{\sqrt{Y} \cos^2 \sqrt{Y}} dy = \int 1 \, dx \rightarrow$

$\int \frac{\sec^2 \sqrt{Y}}{\sqrt{Y}} dy = X + c$ (Let $u = \sqrt{Y} \xrightarrow{D}$

$du = \frac{1}{2\sqrt{Y}} dy \rightarrow 2\,du = \frac{1}{\sqrt{Y}} dy$) $\longrightarrow$

$2 \int \sec^2 u \, du = X + c \rightarrow 2 \tan u = X + c \rightarrow$

$2 \tan \sqrt{Y} = X + c$

16.) $\sec X \cdot \frac{dY}{dX} = e^{Y + \sin X} = e^{Y} e^{\sin X} \longrightarrow$

$\int \frac{1}{e^{Y}} dy = \int \frac{e^{\sin x}}{\sec X} dx \rightarrow \int e^{-Y} dy = \int \cos x \, e^{\sin x} dx$

(Let $u = \sin X \xrightarrow{D} du = \cos X \, dx$) $\longrightarrow$

$-e^{-Y} = \int e^{u} du \rightarrow -e^{-Y} = e^{u} + c \longrightarrow$

$-e^{-Y} = e^{\sin X} + c$

17.) $\frac{dY}{dx} = 2X\sqrt{1 - Y^2} \longrightarrow \int \frac{1}{\sqrt{1 - Y^2}} dy = \int 2X \, dx \rightarrow$

$\arcsin Y = X^2 + c$

19.) $Y^2 \frac{dY}{dx} = 3x^2Y^3 - 6x^2 = 3x^2(Y^3-2) \rightarrow$

$\int \frac{Y^2}{Y^3-2}\, dY = \int 3x^2\, dx$ (Let $u = Y^3 - 2 \xrightarrow{D}$ $du = 3Y^2 dy \rightarrow \frac{1}{3} du = Y^2 dy$)

$\frac{1}{3}\int \frac{1}{u}\, du = x^3 + c \rightarrow \frac{1}{3}\ln|u| = x^3 + c \rightarrow$

$\frac{1}{3}\ln|Y^3 - 2| = x^3 + c$

20.) $\frac{dY}{dX} = XY + 3X - 2Y - 6 = X(Y+3) - 2(Y+3) \rightarrow$

$\frac{dY}{dX} = (Y+3)(X-2) \rightarrow \int \frac{1}{Y+3}\, dy = \int (X-2)\, dx \rightarrow$

$\ln|Y+3| = \frac{1}{2}X^2 - 2X + c$

22.) $\frac{dY}{dX} = e^{X-Y} + e^X + e^{-Y} + 1 = e^X e^{-Y} + e^X + e^{-Y} + 1$

$\rightarrow \frac{dY}{dX} = e^X(e^{-Y}+1) + (e^{-Y}+1) = (e^{-Y}+1)(e^X+1) \rightarrow$

$\int \frac{1}{e^{-Y}+1}\, dy = \int (e^X+1)\, dx \rightarrow \int \frac{1}{\frac{1}{e^Y}+1}\, dy = e^X + x + c \rightarrow$

$\int \frac{1}{\frac{1}{e^Y}+1} \cdot \frac{e^Y}{e^Y}\, dy = e^X + x + c \rightarrow \int \frac{e^Y}{e^Y+1}\, dy = e^X + x + c$

(Let $u = e^Y + 1 \xrightarrow{D} du = e^Y dy$) $\rightarrow$

$\int \frac{1}{u}\, du = e^X + X + c \rightarrow \ln|u| = e^X + x + c \rightarrow$

$\ln|e^Y + 1| = e^X + x + c$

26.) assume $N = Ce^{kt}$, where N is mass of sugar (kg.) at time t (hrs.) :
$t = 0$ hr., $N = 1000$ kg. $\rightarrow$ $C = 1000$ $\rightarrow$
$N = 1000e^{kt}$; $t = 10$ hrs., $N = 800$ kg $\rightarrow$
$800 = 1000\, e^{10k}$ $\rightarrow$ $0.8 = e^{10k}$ $\rightarrow$
$\ln 0.8 = 10k$ $\rightarrow$ $k = \frac{1}{10} \ln 0.8$ $\rightarrow$
$\underline{\underline{N = 1000\, e^{(\frac{1}{10}\ln 0.8)\cdot t}}}$; if $t = 24$ hr., then
$N = 1000\, e^{(\frac{1}{10}\ln 0.8)(24)} \approx 585.35$ kg.

27.) $\frac{dL}{dx} = -k \cdot L$ $\rightarrow$ $L = Ce^{-kx}$:
$x = 0$ ft. $\rightarrow$ $L = Ce^{0} = C \cdot 1 = C$ so
C is intensity at the surface ;
$x = 18$ ft. $\rightarrow$ $L = \frac{1}{2}C$ so
$\frac{1}{2}\cancel{C} = \cancel{C}e^{-18k}$ $\rightarrow$ $\ln(1/2) = -18k$ $\rightarrow$
$k = \frac{-1}{18}\ln(1/2)$ so $\underline{\underline{L = Ce^{\frac{1}{18}\ln(1/2)\cdot x}}}$;
if $L = \frac{1}{10}C$, then $\frac{1}{10}\cancel{C} = \cancel{C}e^{\frac{1}{18}\ln(1/2)\cdot x}$ $\rightarrow$
$\ln(1/10) = \frac{1}{18}\ln(1/2) \cdot x$ $\rightarrow$
$x = \dfrac{\ln(1/10)}{\frac{1}{18}\ln(1/2)} \approx 59.8$ ft.

30.) assume $N = Ce^{kt}$, where N is the # of bacteria at time t (hrs.);

$t = 3$ hrs., $N = 10{,}000 \rightarrow \underline{10{,}000 = Ce^{3k}}$;

$t = 5$ hrs., $N = 40{,}000 \rightarrow \underline{40{,}000 = Ce^{5k}}$;

then $C = \dfrac{10{,}000}{e^{3k}}$ and $C = \dfrac{40{,}000}{e^{5k}} \rightarrow$

$$\frac{10{,}000}{e^{3k}} = \frac{40{,}000}{e^{5k}} \rightarrow \frac{e^{5k}}{e^{3k}} = 4 \rightarrow$$

$$e^{2k} = 4 \rightarrow 2k = \ln 4 \rightarrow \underline{k = \tfrac{1}{2}\ln 4}$$

so that $C = \dfrac{10{,}000}{e^{3/2 \ln 4}} = \dfrac{10{,}000}{e^{\ln 4^{3/2}}} = \dfrac{10{,}000}{8} = \underline{1250}$

so $N = 1250\, e^{\frac{1}{2}\ln 4 \cdot t} = 1250 \left(e^{\ln 4^{1/2}}\right)^t \rightarrow$

$\underline{\underline{N = 1250 \cdot 2^t}}$; if $t = 0$ hrs., $N = 1250$.

35.) This is analogous to the $A = Pe^{rt}$ formula for continuous compounding of interest, where r is the continuous annual growth rate. In this problem we are given the continuous annual decay rate of $r = -10\% = -0.1$;

$A = Pe^{rt} \rightarrow A = Pe^{-0.1t}$, where P is the initial amount; if $A = \frac{1}{5}P$, then $\frac{1}{5}\cancel{P} = \cancel{P}e^{-0.1t} \rightarrow$

$\ln(1/5) = -0.1t \rightarrow t = -10\ln(1/5) \approx 16.09$ yr.

37.) $N = Ce^{kt}$ and $t=0$, $N = 10\,g.$ $\rightarrow$

$N = 10e^{kt}$; if $t = 24{,}360$ yrs., then $N = 5g.$ $\rightarrow$

$5 = 10e^{24{,}360k} \rightarrow \frac{1}{2} = e^{24{,}360k} \rightarrow$

$\ln(1/2) = \ln e^{24{,}360k} = 24{,}360k \rightarrow k = \frac{\ln(1/2)}{24{,}360} \rightarrow$

$\boxed{N = 10e^{\frac{\ln(1/2)}{24{,}360}t}}$; if $N = 2g$ then

$2 = 10e^{\frac{\ln(1/2)}{24{,}360}t} \rightarrow \frac{1}{5} = e^{\frac{\ln(1/2)}{24{,}360}t} \rightarrow$

$\ln(1/5) = \frac{\ln(1/2)}{24{,}360}t \rightarrow t = \frac{24{,}360\ln(1/5)}{\ln(1/2)} \approx 56{,}562$ yrs.

38.) assume $N = Ce^{kt}$, where N is the amount of polonium after t days, and C is the initial amount ;

if $t = 139$ days, then $N = \frac{1}{2}C$ $\rightarrow$

$\frac{1}{2}\cancel{C} = \cancel{C}e^{139k} \rightarrow \ln(1/2) = 139k \rightarrow$

$k = \frac{1}{139}\ln(1/2) \rightarrow \underline{\underline{N = Ce^{\frac{1}{139}\ln(1/2)\cdot t}}}$;

if $N = 5\%$ of $C = 0.05C$, then

$0.05\cancel{C} = \cancel{C}\,e^{\frac{1}{139}\ln(1/2)\cdot t} \rightarrow$

$\ln(0.05) = \frac{1}{139}\ln(1/2)\cdot t \rightarrow$

$t = \frac{\ln(0.05)}{\frac{1}{139}\ln(1/2)} \approx 600.7$ days

Newton's Law of Cooling:
Let T_S be the (constant) surrounding temperature of an environment.
Let T_0 be the initial temperature of an object in this environment.
Then the temperature T of this object at time t is given by

$$T = T_S + (T_0 - T_S)e^{kt}.$$

41.) $T_S = 20°C$ and $t = 0$ min., $T = 90°C$ and $t = 10$ min., $T = 60°C$; then

$T = 20 + 70e^{kt}$ and $t = 10$ min., $T = 60°C$

$\rightarrow 60 = 20 + 70e^{10k} \rightarrow 40 = 70e^{10k} \rightarrow$

$\frac{4}{7} = e^{10k} \rightarrow \ln(4/7) = 10k \rightarrow$

$k = \frac{1}{10}\ln(4/7) \rightarrow \underline{\underline{T = 20 + 70e^{\frac{1}{10}\ln(4/7)\cdot t}}}$;

a.) If $T = 35°C$ then $35 = 20 + 70e^{\frac{1}{10}\ln(4/7)\cdot t}$

$\rightarrow 15 = 70e^{\frac{1}{10}\ln(4/7)\cdot t} \rightarrow \frac{3}{14} = e^{\frac{1}{10}\ln(4/7)\cdot t}$

$\rightarrow \ln(3/14) = \frac{1}{10}\ln(4/7)\cdot t$

$\rightarrow t = \dfrac{\ln(3/14)}{\frac{1}{10}\ln(4/7)} \approx 27.5$ minutes

(or 17.5 additional minutes)

b.) $T_S = -15°C$, $T_0 = 90°C$ then

$T = -15 + 105 e^{kt}$ (assume k is the same) $\longrightarrow$

$$\underline{\underline{T = -15 + 105 e^{\frac{1}{10}\ln(4/7)\cdot t}}} \quad ;$$

if $T = 35°C$, then $35 = -15 + 105 e^{\frac{1}{10}\ln(4/7)\cdot t}$,

$\rightarrow \quad 50 = 105 e^{\frac{1}{10}\ln(4/7)\cdot t}$

$\rightarrow \quad \frac{10}{21} = e^{\frac{1}{10}\ln(4/7)\cdot t}$

$\rightarrow \quad \ln(10/21) = \frac{1}{10}\ln(4/7)\cdot t$

$\rightarrow \quad t = \dfrac{\ln(10/21)}{\frac{1}{10}\ln(4/7)} \approx 13.26$ min.

43.) $T = T_S + (T_0 - T_S)e^{kt}$:

$t = 0$ min., $T = 46°C \rightarrow T_0 = 46$ so

$$T = T_S + (46 - T_S)e^{kt} \quad ;$$

$t = 10$ min., $T = 39°C \rightarrow$

$$\underline{39 = T_S + (46 - T_S)e^{10k}} \quad ;$$

$t = 20$ min., $T = 33°C \longrightarrow$

$$\underline{33 = T_S + (46 - T_S)e^{20k}} \quad ; \text{ then}$$

$e^{10k} = \dfrac{39 - T_S}{46 - T_S}$ and

$e^{20k} = (e^{10k})^2 = \dfrac{33 - T_S}{46 - T_S} \longrightarrow$

$$\left(\frac{39-T_S}{46-T_S}\right)^2 = \frac{33-T_S}{46-T_S} \rightarrow$$

$$\frac{T_S^2 - 78T_S + 1521}{T_S^2 - 92T_S + 2116} = \frac{33-T_S}{46-T_S} \rightarrow \cdots \rightarrow$$

$$124T_S^2 - 5109T_S + 69{,}966$$
$$= 125T_S^2 - 5152T_S + 69{,}828 \rightarrow$$

$$0 = T^2 - 43T - 138 \rightarrow$$

$$T = \frac{43 \pm \sqrt{(43)^2 + 4(138)}}{2}$$

$$= \frac{43 \pm 49}{2} \rightarrow \boxed{T = -3°C} \text{ or}$$

$T = 46°C$ (NO).

45.) assume $N = Ce^{kt}$, where N is the amount of carbon-14 present at time t; C is the initial amount; $t = 5730$ yrs, $N = \frac{1}{2}C \rightarrow$

$$\frac{1}{2}\cancel{C} = \cancel{C}e^{5730k} \rightarrow \ln(1/2) = 5730k \rightarrow$$

$$k = \frac{1}{5730}\ln(1/2) \rightarrow \underline{\underline{N = Ce^{\frac{1}{5730}\ln(1/2)\cdot t}}};$$

if $N = 44.5\%$ of $C = 0.445C$, then

$$0.445\cancel{C} = \cancel{C}e^{\frac{1}{5730}\ln(1/2)\cdot t} \rightarrow$$

$$\ln(0.445) = \frac{1}{5730}\ln(1/2)\cdot t \rightarrow$$

$$t = \frac{\ln(0.445)}{\frac{1}{5730}\ln(1/2)} \approx 6693 \text{ yrs.}$$

47.) $N = Ce^{kt}$ and $t = 5700$ yrs, $N = \frac{1}{2}C \rightarrow$

$\frac{1}{2}\cancel{C} = \cancel{C}e^{5700k} \rightarrow \ln(1/2) = 5700k \rightarrow$

$k = \frac{\ln(1/2)}{5700}$ so $\underline{N = Ce^{\frac{\ln(1/2)}{5700}t}}$; if

$t = 5000$ yrs. then $N = Ce^{\frac{\ln(1/2)}{5700}(5000)}$

$\approx 0.5444\,C = 54.44\%$ of original amount

48.) $N = Ce^{kt}$ and $k = \frac{\ln(1/2)}{5700}$ (SEE problem 45.) ; if $N = 0.995C$ then

$0.995\cancel{C} = \cancel{C}e^{kt} \rightarrow \ln(0.995) = kt \rightarrow$

$t = \frac{\ln(0.995)}{k} = \frac{\ln(0.995)}{\ln(1/2)/5700} \approx 41.2$ yrs.