Section 8.3

1.) $\int \cos 2x \, dx = \frac{1}{2} \sin 2x + c$

2.) $\int_0^{\pi} 3 \sin\left(\frac{x}{3}\right) dx = 3 \cdot -3 \cos\left(\frac{x}{3}\right)\Big|_0^{\pi}$

$= -9 \cos \frac{\pi}{3} - -9 \cos 0$

$= -9\left(\frac{1}{2}\right) + 9(1) = 9/2$

3.) $\int \cos^3 x \cdot \sin x \, dx$ (Let $u = \cos x \xrightarrow{D}$
$du = -\sin x \, dx \rightarrow -du = \sin x \, dx$)

$= -\int u^3 \, du = -\frac{1}{4} u^4 + c = -\frac{1}{4} (\cos x)^4 + c$

5.) $\int \sin^3 x \, dx = \int \sin^2 x \cdot \sin x \, dx$

$= \int (1 - \cos^2 x) \sin x \, dx$ (Let $u = \cos x \xrightarrow{D}$
$du = -\sin x \, dx \rightarrow -du = \sin x \, dx$)

$= -\int (1 - u^2) \, du = -(u - \frac{1}{3} u^3) + c$

$= -\cos x + \frac{1}{3} (\cos x)^3 + c$

6.) $\int \cos^3 4x \, dx = \int \cos^2 4x \cdot \cos 4x \, dx$

$= \int (1 - \sin^2 4x) \cos 4x \, dx$ (Let $u = \sin 4x \xrightarrow{D}$
$du = 4 \cos 4x \, dx \rightarrow \frac{1}{4} du = \cos 4x \, dx$)

$= \frac{1}{4} \int (1 - u^2) \, du = \frac{1}{4} (u - \frac{1}{3} u^3) + c$

$= \frac{1}{4} \sin 4x - \frac{1}{12} (\sin 4x)^3 + c$

7.) $\int \sin^5 x\,dx = \int (\sin^2 x)^2 \cdot \sin x\,dx$

$= \int (1-\cos^2 x)^2 \sin x\,dx$ (Let $u = \cos x \xrightarrow{D}$ $du = -\sin x\,dx \rightarrow -du = \sin x\,dx$)

$= -\int (1-u^2)^2\,du = -\int (1-2u^2+u^4)\,du$

$= -\left(u - \frac{2}{3}u^3 + \frac{1}{5}u^5\right) + C$

$= -\cos x + \frac{2}{3}(\cos x)^3 - \frac{1}{5}(\cos x)^5 + C$

10.) $\int_0^{\frac{\pi}{6}} 3\cos^5(3x)\,dx = 3\int_0^{\frac{\pi}{6}} \cos^4(3x)\cos(3x)\,dx$

$= 3\int_0^{\frac{\pi}{6}} (\cos^2(3x))^2 \cdot \cos(3x)\,dx$

$= 3\int_0^{\frac{\pi}{6}} (1-\sin^2(3x))^2 \cdot \cos 3x\,dx$

(Let $u = \sin(3x) \rightarrow du = 3\cos(3x)\,dx$

$x: 0 \rightarrow \frac{\pi}{6}$ so $u: 0 \rightarrow 1$ $\quad \frac{1}{3}du = \cos(3x)\,dx$)

$= 3\int_0^1 \cdot \frac{1}{3}(1-u^2)^2\,du$

$= \int_0^1 (u^4 - 2u^2 + 1)\,du$

$= \left(\frac{u^5}{5} - \frac{2}{3}u^3 + u\right)\Big|_0^1$

$= \frac{1}{5} - \frac{2}{3} + 1 = \frac{8}{15}$

11.) $\int \sin^3 x \cos^3 x \, dx = \int \sin^3 x \cdot \cos^2 x \cdot \cos x \, dx$

$= \int \sin^3 x \, (1 - \sin^2 x) \cdot \cos x \, dx$

(Let $u = \sin x \xrightarrow{D} du = \cos x \, dx$)

$= \int u^3 (1 - u^2) \, du = \int (u^3 - u^5) \, du$

$= \frac{1}{4} u^4 - \frac{1}{6} u^6 + C = \frac{1}{4} (\sin x)^4 - \frac{1}{6} (\sin x)^6 + C$

14.) $\int_0^{\frac{\pi}{2}} \sin^2 x \, dx = \int_0^{\frac{\pi}{2}} \frac{1}{2} (1 - \cos 2x) \, dx$

$= \frac{1}{2} \left(x - \frac{1}{2} \sin 2x \right) \Big|_0^{\frac{\pi}{2}}$

$= \frac{1}{2} \left(\frac{\pi}{2} - \frac{1}{2} \cancelto{0}{\sin \pi} \right) - \frac{1}{2} \left(0 - \frac{1}{2} \cancelto{0}{\sin 0} \right) = \frac{\pi}{4}$

15.) $\int_0^{\frac{\pi}{2}} \sin^7 y \, dy = \int_0^{\frac{\pi}{2}} \sin^6 y \cdot \sin y \, dy$

$= \int_0^{\frac{\pi}{2}} (\sin^2 y)^3 \cdot \sin y \, dy$

$= \int_0^{\frac{\pi}{2}} (1 - \cos^2 y)^3 \cdot \sin y \, dy$

(Let $u = \cos y \rightarrow du = -\sin y \, dy \rightarrow$

$-du = \sin y \, dy$; $x : 0 \rightarrow \frac{\pi}{2}$ so $u : 1 \rightarrow 0$)

$= -\int_1^0 (1 - u^2)^3 \, du$

$= \int_0^1 (1 - 3u^2 + 3u^4 - u^6) \, du$

$= \left(u - u^3 + \frac{3}{5} u^5 - \frac{1}{7} u^7 \right) \Big|_0^1$

$= \cancel{1 - 1} + \frac{3}{5} - \frac{1}{7} = \frac{16}{35}$

17.) $\int_0^{\pi} 8\sin^4 x\,dx = 8\int_0^{\pi} (\sin^2 x)^2\,dx$

$= 8\int_0^{\pi} \left(\frac{1}{2}(1-\cos 2x)\right)^2 dx$

$= 8\cdot\frac{1}{4}\int_0^{\pi} (1-2\cos 2x+\cos^2 2x)\,dx$

$= 2\int_0^{\pi} \left(1-2\cos 2x+\frac{1}{2}(1+\cos 4x)\right) dx$

$= 2\int_0^{\pi} \left(\frac{3}{2}-2\cos 2x+\frac{1}{2}\cos 4x\right) dx$

$= 2\left(\frac{3}{2}x-\sin 2x+\frac{1}{8}\sin 4x\right)\Big|_0^{\pi}$

$= 2\left(\frac{3}{2}\pi-0+0\right)-2(0-0+0) = 3\pi$

19.) $\int 16\sin^2 x\cos^2 x\,dx$

$= 4\int 4\sin^2 x\cos^2 x\,dx = 4\int (2\sin x\cos x)^2 dx$

$= 4\int (\sin 2x)^2 dx = 4\int \sin^2 2x\,dx$

$= 4\int \frac{1}{2}(1-\cos 4x)\,dx = 2\left(x-\frac{1}{4}\sin 4x\right)+C$

20.) $\int_0^{\pi} 8\sin^4 y\cos^2 y\,dy$

$= 2\int_0^{\pi} \sin^2 y\cdot(2\sin y\cos y)^2\,dy$

$= \cancel{2}\int_0^{\pi} \frac{1}{\cancel{2}}(1-\cos 2y)\,(\sin 2y)^2\,dy$

$= \int_0^{\pi} (\sin^2 2y-\sin^2 2y\cdot\cos 2y)\,dy$

$= \int_0^{\pi} \left(\frac{1}{2}(1-\cos 4y)-\sin^2 2y\cdot\cos 2y\right) dy$

$$= \left\{ \frac{1}{2}\left(Y - \frac{1}{4}\sin 4Y\right) - \frac{1}{3}\cdot\frac{1}{2}\sin^3 2Y \right\} \Big|_0^{\pi}$$

$$= \left\{ \frac{1}{2}(\pi - 0) - \frac{1}{6}(0) \right\} - \{0\} \quad = \frac{\pi}{2}$$

22.) $\int_0^{\frac{\pi}{2}} \sin^2 2\theta \cos^3 2\theta \, d\theta$

$$= \int_0^{\frac{\pi}{2}} \sin^2 2\theta \cdot \cos^2 2\theta \cdot \cos 2\theta \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin^2 2\theta \cdot (1 - \sin^2 2\theta) \cos 2\theta \, d\theta$$

(Let $u = \sin 2\theta \rightarrow du = 2\cos 2\theta \, d\theta \rightarrow$
$\frac{1}{2} du = \cos 2\theta \, d\theta$; $\theta : 0 \rightarrow \frac{\pi}{2}$ so $u : 0 \rightarrow 0$)

$$= \frac{1}{2}\int_0^0 u^2(1-u^2)\, du = 0$$

24.) $\int_0^{\pi} \sqrt{1 - \cos 2x}\, dx = \int_0^{\pi} \sqrt{2 \cdot \underbrace{\tfrac{1}{2}(1 - \cos 2x)}_{\sin^2\theta}}\, dx$

$$= \sqrt{2}\int_0^{\pi} \sqrt{\sin^2 x}\, dx$$

$$= \sqrt{2}\int_0^{\pi} |\sin x|\, dx = \sqrt{2}\int_0^{\pi} \sin x \, dx$$

$$= \sqrt{2} \cdot -\cos x \Big|_0^{\pi} = -\sqrt{2}\cos\pi - -\sqrt{2}\cos 0$$

$$= -\sqrt{2}(-1) + \sqrt{2}(1) = 2\sqrt{2}$$

25.) $\int_0^{\pi} \sqrt{1 - \sin^2 t}\, dt = \int_0^{\pi} \sqrt{\cos^2 t}\, dt$

$$= \int_0^{\pi} |\cos t|\, dt = \int_0^{\frac{\pi}{2}} |\cos t|\, dt + \int_{\frac{\pi}{2}}^{\pi} |\cos t|\, dt$$

$$= \int_0^{\frac{\pi}{2}} \cos t \, dt + \int_{\frac{\pi}{2}}^{\pi} -\cos t \, dt$$

$$= \sin t \Big|_0^{\frac{\pi}{2}} - \sin t \Big|_{\frac{\pi}{2}}^{\pi}$$

$$= (\cancel{\sin}^{1}\tfrac{\pi}{2} - \cancel{\sin}^{0} 0) - (\cancel{\sin}^{0}\pi - \cancel{\sin}^{1}\tfrac{\pi}{2})$$

$$= 1 - (-1) = 2$$

27.) $$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\sin^2 x}{\sqrt{1-\cos x}}\,dx = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1-\cos^2 x}{\sqrt{1-\cos x}}\,dx$$

$$= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{(1-\cos x)(1+\cos x)}{\sqrt{1-\cos x}}\,dx$$

$$= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{(\sqrt{1-\cos x})^2(1+\cos x)}{\sqrt{1-\cos x}}\,dx$$

$$= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sqrt{1-\cos x}\,(\sqrt{1+\cos x})^2\,dx$$

$$= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (\sqrt{1-\cos x}\cdot\sqrt{1+\cos x})\cdot\sqrt{1+\cos x}\,dx$$

$$= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sqrt{(1-\cos x)(1+\cos x)}\cdot\sqrt{1+\cos x}\,dx$$

$$= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sqrt{1-\cos^2 x}\cdot\sqrt{1+\cos x}\,dx$$

$$= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sqrt{\sin^2 x}\,\sqrt{1+\cos x}\,dx$$

$$= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} |\sin x| \sqrt{1+\cos x}\, dx$$

$$= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin x\, (1+\cos x)^{1/2}\, dx \quad \left(\text{Let } u = 1+\cos x \xrightarrow{D} \cdots\right)$$

$$= \quad -\frac{2}{3}(1+\cos x)^{3/2} \Big|_{\frac{\pi}{3}}^{\frac{\pi}{2}}$$

$$= -\frac{2}{3}\left(1+\overset{0}{\cancel{\cos}\frac{\pi}{2}}\right)^{3/2} - -\frac{2}{3}\left(1+\overset{1/2}{\cancel{\cos}\frac{\pi}{3}}\right)^{3/2}$$

$$= -\frac{2}{3}(1)^{3/2} + \frac{2}{3}\left(\frac{3}{2}\right)^{3/2} = -\frac{2}{3} + \sqrt{\frac{3}{2}}$$

28.) $\displaystyle \int_0^{\frac{\pi}{6}} \sqrt{1+\sin x}\, dx = \int_0^{\frac{\pi}{6}} \sqrt{(1+\sin x)\frac{(1-\sin x)}{(1-\sin x)}}\, dx$

$$= \int_0^{\frac{\pi}{6}} \sqrt{\frac{1-\sin^2 x}{1-\sin x}}\, dx = \int_0^{\frac{\pi}{6}} \frac{\sqrt{\cos^2 x}}{\sqrt{1-\sin x}}\, dx$$

$$= \int_0^{\frac{\pi}{6}} \frac{|\cos x|}{\sqrt{1-\sin x}}\, dx = \int_0^{\frac{\pi}{6}} \cos x\, (1-\sin x)^{-1/2}\, dx$$

$$\left(\text{Let } u = 1-\sin x \xrightarrow{D} \cdots\right)$$

$$= \quad -2(1-\sin x)^{1/2} \Big|_0^{\frac{\pi}{6}}$$

$$= -2\sqrt{1-\underset{1/2}{\sin\frac{\pi}{6}}} - -\sqrt{1-\underset{0}{\sin 0}}$$

$$= 1 - 2\cdot\sqrt{\frac{1}{2}} = 1-\sqrt{2}$$

33.) $\displaystyle \int \sec^2 x \tan x\, dx = \int \sec x \cdot (\sec x \tan x)\, dx$

$$\left(\text{Let } u = \sec x \xrightarrow{D} du = \sec x \tan x\, dx\right)$$

$$= \int u\, du = \frac{1}{2}u^2 + C = \frac{1}{2}(\sec x)^2 + C$$

35.) $\int \sec^3 x \tan x \, dx = \int \sec^2 x \cdot \sec x \tan x \, dx$

(Let $u = \sec x \xrightarrow{D} du = \sec x \tan x \, dx$)

$= \int u^2 \, du = \frac{1}{3}u^3 + C = \frac{1}{3}(\sec x)^3 + C$

36.) $\int \sec^3 x \tan^3 x \, dx = \int \sec^2 x \cdot \tan^2 x \cdot \sec x \tan x \, dx$

$= \int \sec^2 x \cdot (\sec^2 x - 1) \cdot \sec x \tan x \, dx$

(Let $u = \sec x \xrightarrow{D} du = \sec x \tan x \, dx$)

$= \int u^2(u^2 - 1) \, du = \int (u^4 - u^2) \, du$

$= \frac{1}{5}u^5 - \frac{1}{3}u^3 + C = \frac{1}{5}(\sec x)^5 - \frac{1}{3}(\sec x)^3 + C$

39.) $\underline{\int \sec^3 x \, dx} = \int \sec x \cdot \sec^2 x \, dx$

(Let $u = \sec x$, $dv = \sec^2 x \, dx$

$\rightarrow du = \sec x \tan x \, dx$, $v = \tan x$)

$= \sec x \tan x - \int \sec x \cdot \tan^2 x \, dx$

$= \sec x \tan x - \int \sec x \cdot (\sec^2 x - 1) \, dx$

$= \sec x \tan x - \int (\sec^3 x - \sec x) \, dx$

$= \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx$

$= \sec x \tan x - \underline{\int \sec^3 x \, dx} + \ln|\sec x + \tan x| \rightarrow$

$2\int \sec^3 x \, dx = \sec x \tan x + \ln|\sec x + \tan x| + C \rightarrow$

$\underbrace{\int \sec^3 x \, dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C}$;

then $\int_{-\frac{\pi}{3}}^{0} 2\sec^3 x \, dx = 2\Big[\frac{1}{2}\sec x \tan x$

$+ \frac{1}{2}\ln|\sec x + \tan x|\Big]_{-\frac{\pi}{3}}^{0}$

$= \left[\sec x \tan x + \ln|\sec x + \tan x\right] \Big|_{-\frac{\pi}{3}}^{0}$

$= (\overset{1}{\cancel{\sec}}0 \overset{0}{\cancel{\tan}}0 + \ln|\overset{1}{\cancel{\sec}}0 + \overset{0}{\cancel{\tan}}0|)$

$- \left(\sec\left(-\frac{\pi}{3}\right)\tan\left(-\frac{\pi}{3}\right) + \ln\left|\sec\left(-\frac{\pi}{3}\right) + \tan\left(-\frac{\pi}{3}\right)\right|\right)$

$= -((2)(-\sqrt{3}) + \ln|(2) + (-\sqrt{3})|$

$= 2\sqrt{3} + \ln(2 - \sqrt{3})$

34.) $\underbrace{\int \sec x \cdot \tan^2 x \, dx} = \int \sec x \tan x \cdot \tan x \, dx$

(Let $u = \tan x$, $dv = \sec x \tan x \, dx$

$\to du = \sec^2 x \, dx$, $v = \sec x$)

$= \sec x \tan x - \int \sec x \cdot \sec^2 x \, dx$

$= \sec x \tan x - \int \sec x \cdot (1 + \tan^2 x) \, dx$

$= \sec x \tan x - \int (\sec x + \sec x \tan^2 x) \, dx$

$= \sec x \tan x - \int \sec x \, dx - \int \sec x \tan^2 x \, dx$

$= \sec x \tan x - \ln|\sec x + \tan x| - \underbrace{\int \sec x \tan^2 x \, dx}$;

then (TWIST)

$2 \int \sec x \tan^2 x \, dx = \sec x \tan x - \ln|\sec x + \tan x| + C$

$\to \int \sec x \tan^2 x \, dx = \frac{1}{2} \sec x \tan x - \frac{1}{2} \ln|\sec x + \tan x| + C$

41.) $\int \sec^4 \theta \, d\theta = \int \sec^2 \theta \sec^2 \theta \, d\theta$

$= \int (1 + \tan^2 \theta) \sec^2 \theta \, d\theta$ (Let $u = \tan \theta \to$

$du = \sec^2 \theta \, d\theta$)

$= \int (1 + u^2) \, du = u + \frac{1}{3} u^3 + C$

$= \tan \theta + \frac{1}{3} (\tan \theta)^3 + C$

44.) $\int \sec^6 x \, dx = \int \sec^4 x \cdot \sec^2 x \, dx$

$= \int (\sec^2 x)^2 \sec^2 x \, dx = \int (1+\tan^2 x)^2 \sec^2 x \, dx$

(Let $u = \tan x \xrightarrow{D} du = \sec^2 x \, dx$)

$= \int (1+u^2)^2 \, du = \int (u^4 + 2u^2 + 1) \, du$

$= \frac{1}{5}u^5 + \frac{2}{3}u^3 + u + C = \frac{1}{5}(\tan x)^5 + \frac{2}{3}(\tan x)^3 + \tan x + C$

46.) $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 6 \tan^4 x \, dx = 6 \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tan^2 x \cdot \tan^2 x \, dx$

$= 6 \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tan^2 x \cdot (\sec^2 x - 1) \, dx$

$= 6 \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (\tan^2 x \cdot \sec^2 x - \tan^2 x) \, dx$

$= 6 \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (\tan^2 x \cdot \sec^2 x - (\sec^2 x - 1)) \, dx$

$= 6 \left(\frac{1}{3} \tan^3 x - \tan x + x \right) \Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$

$= 6\left(\frac{1}{3}(1)^3 - (1) + \frac{\pi}{4}\right) - 6\left(\frac{1}{3}(-1)^3 - (-1) + -\frac{\pi}{4}\right)$

$= 6\left(\frac{\pi}{4} - \frac{2}{3}\right) - 6\left(\frac{2}{3} - \frac{\pi}{4}\right)$

$= \frac{3}{2}\pi - 4 - 4 + \frac{3}{2}\pi = 3\pi - 8$

47.) $\int \tan^5 x \, dx = \int \tan^3 x \cdot \tan^2 x \, dx$

$= \int \tan^3 x (\sec^2 x - 1) \, dx$

$= \int \tan^3 x \sec^2 x \, dx - \int \tan^3 x \, dx$

$$= \int \tan^3 x \sec^2 x \, dx - \int \tan x \cdot \tan^2 x \, dx$$

$$= \int \tan^3 x \sec^2 x \, dx - \int \tan x \cdot (\sec^2 x - 1) \, dx$$

$$= \int \tan^3 x \sec^2 x \, dx - \int \tan x \sec^2 x \, dx$$
$$+ \int \tan x \, dx$$

(Let $u = \tan x$
$\xrightarrow{D} du = \sec^2 x \, dx$)

$$= \int u^3 \, du - \int u \, du + \ln|\sec x| + c$$

$$= \frac{1}{4}u^4 - \frac{1}{2}u^2 + \ln|\sec x| + c$$

$$= \frac{1}{4}(\tan x)^4 - \frac{1}{2}(\tan x)^2 + \ln|\sec x| + c$$

51.) $\underline{\int \sin 3x \cos 2x \, dx}$ (Let $u = \sin 3x$, $dv = \cos 2x \, dx$
$\rightarrow du = 3\cos 3x$, $v = \frac{1}{2}\sin 2x$)

$$= \frac{1}{2}\sin 3x \sin 2x - \frac{3}{2}\int \cos 3x \sin 2x \, dx$$

(Let $u = \cos 3x$, $dv = \sin 2x \, dx$
$\rightarrow du = -3\sin 3x \, dx$, $v = \frac{-1}{2}\cos 2x$)

$$= \frac{1}{2}\sin 3x \sin 2x$$
$$- \frac{3}{2}\left[\frac{-1}{2}\cos 3x \cos 2x - \frac{3}{2}\int \sin 3x \cos 2x \, dx\right]$$

$$= \frac{1}{2}\sin 3x \sin 2x + \frac{3}{4}\cos 3x \cos 2x$$
$$+ \frac{9}{4}\underline{\int \sin 3x \cos 2x \, dx}$$; then (TWIST)

$$-\frac{5}{4}\int \sin 3x \cos 2x \, dx = \frac{1}{2}\sin 3x \sin 2x$$
$$+ \frac{3}{4}\cos 3x \cos 2x + C \longrightarrow$$

$$\int \sin 3x \cos 2x \, dx = \frac{-2}{5}\sin 3x \sin 2x$$
$$- \frac{3}{5}\cos 3x \cos 2x + C$$

54.) $\int_0^{\frac{\pi}{2}} \sin x \cos x \, dx$ (Let $u = \sin x \xrightarrow{D}$ $du = \cos x \, dx$)

$$= \int_{x=0}^{x=\pi/2} u \, du = \frac{1}{2}u^2 \Big|_{x=0}^{x=\frac{\pi}{2}} = \frac{1}{2}(\sin x)^2 \Big|_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2}(\overset{1}{\cancel{\sin}\tfrac{\pi}{2}})^2 - \frac{1}{2}(\overset{0}{\cancel{\sin}\, 0})^2 = 1/2$$

55.) $\underline{\int \cos 3x \cos 4x \, dx}$ (Let $u = \cos 3x$, $dv = \cos 4x \, dx$
$\rightarrow du = -3\sin 3x$, $v = \frac{1}{4}\sin 4x$)

$$= \frac{1}{4}\cos 3x \sin 4x - \frac{-3}{4}\int \sin 3x \sin 4x \, dx$$

(Let $u = \sin 3x$, $dv = \sin 4x \, dx$
$\rightarrow du = 3\cos 3x$, $v = \frac{-1}{4}\cos 4x$)

$$= \frac{1}{4}\cos 3x \sin 4x$$
$$+ \frac{3}{4}\left[\frac{-1}{4}\sin 3x \cos 4x - \frac{-3}{4}\int \cos 3x \cos 4x \, dx\right]$$
$$= \frac{1}{4}\cos 3x \sin 4x - \frac{3}{16}\sin 3x \cos 4x$$
$$+ \frac{9}{16}\underline{\int \cos 3x \cos 4x \, dx}$$; then (TWIST)

$\frac{7}{16}\int \cos 3x \cos 4x \, dx = \frac{1}{4}\cos 3x \sin 4x$

$- \frac{3}{16}\sin 3x \cos 4x + c \longrightarrow$

$\int \cos 3x \cos 4x \, dx = \frac{4}{7}\cos 3x \sin 4x$

$- \frac{3}{7}\sin 3x \cos 4x + c$

57.) $\int \sin^2\theta \cos 3\theta \, d\theta = \int \frac{1}{2}(1-\cos 2\theta)\cos 3\theta \, d\theta$

$= \frac{1}{2}\int \cos 3\theta \, d\theta - \frac{1}{2}\int \cos 2\theta \cos 3\theta \, d\theta$

$= \frac{1}{2}\cdot\frac{1}{3}\sin 3\theta - \frac{1}{2}\underbrace{\int \cos 2\theta \cos 3\theta \, d\theta}_{A}$;

$A = \int \cos 2\theta \cos 3\theta \, d\theta$ (Let $u = \cos 2\theta$, $dv = \cos 3\theta \, d\theta$

$\rightarrow du = -2\sin 2\theta$, $V = \frac{1}{3}\sin 3\theta$)

$= \frac{1}{3}\cos 2\theta \sin 3\theta - \frac{-2}{3}\int \sin 2\theta \sin 3\theta \, d\theta$

(Let $u = \sin 2\theta$, $dv = \sin 3\theta \, d\theta$

$\rightarrow du = 2\cos 2\theta$, $V = -\frac{1}{3}\cos 3\theta$)

$= \frac{1}{3}\cos 2\theta \sin 3\theta$

$+ \frac{2}{3}\left[\frac{-1}{3}\sin 2\theta \cos 3\theta - \frac{-2}{3}\int \cos 2\theta \cos 3\theta \, d\theta\right]$

$= \frac{1}{3}\cos 2\theta \sin 3\theta - \frac{2}{9}\sin 2\theta \cos 3\theta$

$+ \frac{4}{9}\underbrace{\int \cos 2\theta \cos 3\theta \, d\theta}_{A}$; then (TWIST)

$$\frac{5}{9}\int \cos 2\theta \cos 3\theta \, d\theta = \frac{1}{3}\cos 2\theta \sin 3\theta$$

$$-\frac{2}{9}\sin 2\theta \cos 3\theta + C \longrightarrow$$

$$A = \int \cos 2\theta \cos 3\theta \, d\theta = \frac{3}{5}\cos 2\theta \sin 3\theta$$

$$-\frac{2}{5}\sin 2\theta \cos 3\theta + C \text{ ; then}$$

$$\int \sin^2\theta \cos 3\theta \, d\theta = \frac{1}{6}\sin 3\theta$$

$$-\frac{1}{2}\left[\frac{3}{5}\cos 2\theta \sin 3\theta - \frac{2}{5}\sin 2\theta \cos 3\theta\right] + C$$

$$= \frac{1}{6}\sin 3\theta - \frac{3}{10}\cos 2\theta \sin 3\theta$$

$$+ \frac{1}{5}\sin 2\theta \cos 3\theta + C$$

51.) $\int \sin 3x \cos 2x \, dx$

$$= \int \frac{1}{2}\left(\sin(3x+2x) + \sin(3x-2x)\right) dx$$

$$= \frac{1}{2}\int (\sin 5x + \sin x) \, dx$$

$$= \frac{1}{2}\left(-\frac{1}{5}\cos 5x - \cos x\right) + C$$

55.) $\int \cos 4x \cos 3x \, dx$

$$= \int \frac{1}{2}\left(\cos(4x+3x) + \cos(4x-3x)\right) dx$$

$$= \frac{1}{2}\int (\cos 7x + \cos x) \, dx$$

$$= \frac{1}{2}\left(\frac{1}{7}\sin 7x + \sin x\right) + C$$

62.) $\int \sin 3\theta\,(\sin 2\theta \sin\theta)\, d\theta$

$= \int \sin 3\theta \cdot \frac{1}{2}\left[\cos(2\theta-\theta) - \cos(2\theta+\theta)\right] d\theta$

$= \frac{1}{2}\int \sin 3\theta\left[\cos\theta - \cos 3\theta\right] d\theta$

$= \frac{1}{2}\int\left[\sin 3\theta\cos\theta - \sin 3\theta \cos 3\theta\right] d\theta$

$= \frac{1}{2}\int\Big[\frac{1}{2}\left(\sin(3\theta+\theta) + \sin(3\theta-\theta)\right)$

$\qquad - \frac{1}{2}\left(\sin(3\theta+3\theta) + \sin(3\theta-3\theta)\right)\Big] d\theta$

$= \frac{1}{2}\cdot\frac{1}{2}\int\left[\sin 4\theta + \sin 2\theta - \sin 6\theta - \cancel{\sin}\overset{0}{0}\right] d\theta$

$= \frac{1}{4}\left(-\frac{1}{4}\cos 4\theta - \frac{1}{2}\cos 2\theta + \frac{1}{6}\cos 2\theta\right) + C$

63.) $\int \frac{\sec^3 x}{\tan x}\,dx = \int \frac{\sec^2 x \sec x}{\tan x}\,dx$

$= \int \frac{(1+\tan^2 x)}{\tan x}\sec x = \int\left[\frac{\sec x}{\tan x} + \sec x\tan x\right]dx$

$= \int \frac{\frac{1}{\cos x}}{\frac{\sin x}{\cos x}}\,dx + \sec x + C$

$= \int \frac{1}{\cancel{\cos x}}\cdot\frac{\cancel{\cos x}}{\sin x}\,dx + \sec x + C$

$= \int \csc x\,dx + \sec x + C$

$= \ln|\csc x - \cot x| + \sec x + C$

64.) $\int \frac{\sin^3 x}{\cos^4 x}\,dx = \int \frac{\sin x \cdot \sin^2 x}{\cos^4 x}\,dx$

$= \int \frac{\sin x \cdot (1-\cos^2 x)}{\cos^4 x}\,dx$ (Let $u = \cos x \xrightarrow{D}$ $du = -\sin x\,dx \rightarrow -du = \sin x\,dx$)

$= -\int \frac{1-u^2}{u^4}\,du = -\int (u^{-4} - u^{-2})\,du$

$= -\left(\frac{-1}{3}u^{-3} + u^{-1}\right) + C = \frac{1}{3}(\cos x)^{-3} - (\cos x)^{-1} + C$

65.) $\int \frac{\tan^2 x}{\csc x}\,dx = \int \frac{\left(\frac{\sin x}{\cos x}\right)^2}{1/\sin x}\,dx$

$= \int \frac{\sin^2 x}{\cos^2 x} \cdot \frac{\sin x}{1}\,dx = \int \sin x\,\frac{(1-\cos^2 x)}{\cos^2 x}\,dx$

$= \int \left[\frac{\sin x}{\cos^2 x} - \sin x\right] dx = \int \left[\frac{\sin x}{\cos x} \cdot \frac{1}{\cos x} - \sin x\right] dx$

$= \int (\sec x \tan x - \sin x)\,dx = \sec x + \cos x + C$

66.) $\int \frac{\cot x}{\cos^2 x}\,dx = \int \frac{\frac{\cos x}{\sin x}}{\frac{\cos^2 x}{1}}\,dx$

$= \int \frac{\cos x}{\sin x} \cdot \frac{1}{\cos^2 x}\,dx = \int \frac{1}{\sin x \cos x}\,dx$

$= \int \frac{2}{2\sin x \cos x}\,dx = \int \frac{2}{\sin 2x}\,dx$

$= 2\int \csc 2x\,dx = 2 \cdot \frac{1}{2}\ln|\csc 2x - \cot 2x| + C$

$= \ln|\csc 2x - \cot 2x| + C$

67.) $\int x \sin^2 x \, dx = \int x \cdot \frac{1}{2}(1-\cos 2x)\,dx$

$= \frac{1}{2}\int [x - x\cos 2x]\,dx$

$= \frac{1}{2}\cdot\frac{1}{2}x^2 - \frac{1}{2}\int x\cos 2x\,dx$

(~~Let~~ $u = x$, $dv = \cos 2x\,dx \rightarrow$
$du = 1\,dx$, $v = \frac{1}{2}\sin 2x$)

$= \frac{1}{4}x^2 - \frac{1}{2}\left[\frac{1}{2}x\sin 2x - \frac{1}{2}\int \sin 2x\,dx\right]$

$= \frac{1}{4}x^2 - \frac{1}{4}x\sin 2x + \frac{1}{4}\cdot\frac{-1}{2}\cos 2x + C$

$= \frac{1}{4}x^2 - \frac{1}{4}x\sin 2x - \frac{1}{8}\cos 2x + C$

69.) $Y = \ln(\sec x) \xrightarrow{D} Y' = \frac{1}{\sec x}\cdot \sec x\tan x = \tan x$,

$\text{Arc} = \int_0^{\frac{\pi}{4}} \sqrt{1+(Y')^2}\,dx = \int_0^{\frac{\pi}{4}}\sqrt{1+\tan^2 x}\,dx$

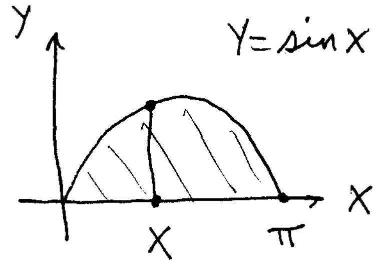
$= \int_0^{\frac{\pi}{4}}\sqrt{\sec^2 x}\,dx = \int_0^{\frac{\pi}{4}}|\sec x|\,dx$

$= \int_0^{\frac{\pi}{4}}\sec x\,dx = \ln|\sec x + \tan x|\Big|_0^{\frac{\pi}{4}}$

$= \ln\left|\sec\frac{\pi}{4} + \tan\frac{\pi}{4}\right| - \ln|\sec 0 + \tan 0|$

$= \ln|\sqrt{2}+1| - \cancel{\ln|1+0|}^{\,0} = \ln(\sqrt{2}+1)$

71.) (DISC METHOD)



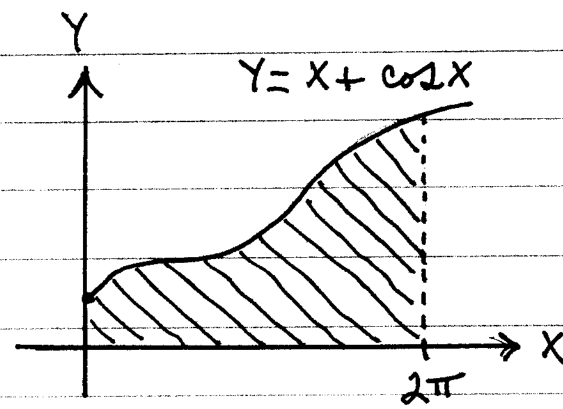
$$Vol = \pi \int_0^{\pi} (radius)^2\, dx$$

$$= \pi \int_0^{\pi} \sin^2 x\, dx$$

$$= \pi \int_0^{\pi} \frac{1}{2}(1 - \cos 2x)\, dx = \frac{\pi}{2}\left(x - \frac{1}{2}\sin 2x\right)\Big|_0^{\pi}$$

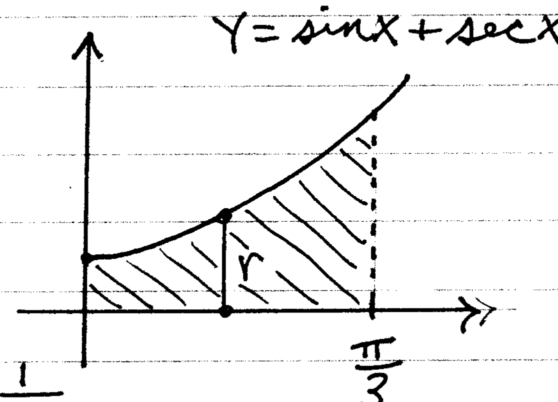
$$= \frac{\pi}{2}\left(\pi - \frac{1}{2}(0)\right) - \frac{\pi}{2}\left(0 - \frac{1}{2}(0)\right) = \frac{\pi^2}{2}$$

73.)



$$\overline{X} = \frac{\int_0^{2\pi} x(x + \cos x)\, dx}{\int_0^{2\pi} (x + \cos x)\, dx},$$

$$\overline{Y} = \frac{\frac{1}{2}\int_0^{2\pi} (x + \cos x)^2\, dx}{\int_0^{2\pi} (x + \cos x)\, dx}$$



74.) $$Vol = \pi \int_0^{\frac{\pi}{3}} (radius)^2\, dx$$

$$= \pi \int_0^{\frac{\pi}{3}} (\sin x + \sec x)^2\, dx$$

$$= \pi \int_0^{\frac{\pi}{3}} \left(\sin^2 x + 2 \sin x \overset{\frac{1}{\cos x}}{\sec x} + \sec^2 x\right) dx$$

$$= \pi \int_0^{\frac{\pi}{3}} \left[\frac{1}{2}(1 - \cos 2x) + 2 \tan x + \sec^2 x\right] dx$$

$$= \pi \left[\frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right) + 2 \ln|\sec x| + \tan x\right]\Big|_0^{\frac{\pi}{3}}$$

$$= \pi \left[\frac{1}{2}\left(\frac{\pi}{3}\right) - \frac{1}{4}\sin\left(\frac{2}{3}\pi\right) + 2\ln\left|\sec\left(\frac{\pi}{3}\right)\right| + \tan\left(\frac{\pi}{3}\right) \right]$$

$$- \pi \left[\frac{1}{2}(0) - \frac{1}{4}\underset{0}{\sin(0)} + 2\underset{0}{\ln}\left|\underset{1}{\sec(0)}\right| + \underset{0}{\tan(0)} \right]$$

$$= \pi \left[\frac{1}{6}\pi - \frac{1}{4} \cdot \frac{\sqrt{3}}{2} + 2\overset{0}{\overrightarrow{\ln 2}} + \sqrt{3} \right]$$

$$= \pi \left[\frac{1}{6}\pi + \frac{7}{8}\sqrt{3} + \ln 4 \right]$$