Section 8.8

1.) $\int_0^\infty \frac{1}{x^2+1}\,dx = \lim_{A\to\infty} \int_0^A \frac{1}{x^2+1}\,dx$

$= \lim_{A\to\infty} \arctan x \Big|_0^A$

$= \lim_{A\to\infty} (\arctan A - \cancelto{0}{\arctan 0}) = \frac{\pi}{2}$

4.) $\int_0^4 \frac{1}{\sqrt{4-x}}\,dx = \lim_{A\to 4^-} \int_0^A \frac{1}{\sqrt{4-x}}\,dx$

$= \lim_{A\to 4^-} \int_0^A (4-x)^{-1/2}\,dx$

$= \lim_{A\to 4^-} -2(4-x)^{1/2} \Big|_0^A$

$= \lim_{A\to 4^-} \{-2\sqrt{4-A} - {}^-2\sqrt{4}\} = 0+4 = 4$

6.) $\int_{-8}^1 \frac{1}{x^{1/3}}\,dx = \int_{-8}^0 \frac{1}{x^{1/3}}\,dx + \int_0^1 \frac{1}{x^{1/3}}\,dx$

$= C + D$:

$C = \lim_{A\to 0^-} \int_{-8}^A x^{-1/3}\,dx = \lim_{A\to 0^-} \frac{3}{2}x^{2/3} \Big|_{-8}^A$

$= \lim_{A\to 0^-} \left(\frac{3}{2}A^{2/3} - \frac{3}{2}(-8)^{2/3}\right) = 0 - \frac{3}{2}(4) = -6$;

$D = \lim_{A\to 0^+} \int_A^1 x^{-1/3}\,dx = \lim_{A\to 0^+} \frac{3}{2}x^{2/3} \Big|_A^1$

$= \lim_{A\to 0^+} \left(\frac{3}{2}(1)^{2/3} - \frac{3}{2}A^{2/3}\right) = \frac{3}{2} - 0 = \frac{3}{2}$;

then $\int_{-8}^1 \frac{1}{x^{1/3}}\,dx = C + D = -6 + \frac{3}{2} = \frac{-9}{2}$.

9.) $\int_{-\infty}^{-2} \frac{2}{x^2-1}\,dx = \int_{-\infty}^{-2} \frac{2}{(x-1)(x+1)}\,dx$

$= \int_{-\infty}^{-2} \left[\frac{A}{x-1} + \frac{B}{x+1}\right] dx = \int_{-\infty}^{-2} \left[\frac{1}{x-1} + \frac{-1}{x+1}\right] dx$

$= \lim_{A\to-\infty} \int_A^{-2} \left[\frac{1}{x-1} - \frac{1}{x+1}\right] dx$

$= \lim_{A\to-\infty} \left(\ln|x-1| - \ln|x+1|\right)\Big|_A^{-2}$

$= \lim_{A\to-\infty} \ln\left|\frac{x-1}{x+1}\right|\Big|_A^{-2}$

$= \lim_{A\to-\infty} \left\{\ln 3 - \ln\left|\frac{A-1}{A+1}\right|\right\}$

$= \ln 3 - \lim_{A\to-\infty} \ln\left|\frac{1-1/A}{1+1/A}\right|$

$= \ln 3 - \ln\left|\frac{1-0}{1+0}\right| = \ln 3 - \cancel{\ln 1}^{\,0} = \ln 3$

14.) $\int_{-\infty}^{\infty} \frac{x}{(x^2+4)^{3/2}}\,dx = \int_{-\infty}^{0} \frac{x}{(x^2+4)^{3/2}}\,dx$

$+ \int_0^{\infty} \frac{x}{(x^2+4)^{3/2}}\,dx = C + D$;

$C = \lim_{A\to-\infty} \int_A^0 x\cdot(x^2+4)^{-3/2}\,dx$

$= \lim_{A\to-\infty} \frac{1}{\cancel{2}}\cdot\frac{(x^2+4)^{-1/2}}{-1/\cancel{2}}\Big|_A^0$

$= \lim_{A\to-\infty} \left\{-(4)^{-1/2} - -(A^2+4)^{-1/2}\right\}$

$= \lim_{A\to-\infty} \left\{\frac{-1}{2} + \frac{1}{\sqrt{A^2+4}}\right\} = \frac{-1}{2} + 0 = \frac{-1}{2}$;

$$D = \lim_{A\to\infty} \int_0^A x(x^2+4)^{-3/2}\,dx$$

$$= \lim_{A\to\infty} \frac{-1}{\sqrt{x^2+4}}\Big|_0^A = \lim_{A\to\infty} \left\{ \frac{-1}{\sqrt{A^2+4}} - \frac{-1}{\sqrt{4}} \right\}$$

$$= 0 + \frac{1}{2} = 1/2 \text{ ; then}$$

$$\int_{-\infty}^{\infty} \frac{x}{(x^2+4)^{3/2}}\,dx = C + D = \frac{-1}{2} + \frac{1}{2} = 0$$

15.) $$\int_0^1 \frac{\theta+1}{\sqrt{\theta^2+2\theta}}\,d\theta = \lim_{A\to 0^+} \int_A^1 (\theta+1)(\theta^2+2\theta)^{-1/2}\,d\theta$$

$$= \lim_{A\to 0^+} (\theta^2+2\theta)^{1/2}\Big|_A^1 = \lim_{A\to 0^+} \left(\sqrt{3} - \sqrt{A^2+2A}\right)$$

$$= \sqrt{3} - 0 = \sqrt{3}$$

20.) $$\int_0^{\infty} \frac{16 \arctan x}{1+x^2}\,dx$$

$$= \lim_{A\to\infty} \int_0^A \frac{16 \arctan x}{1+x^2}\,dx$$

$$= \lim_{A\to\infty} 16 \cdot \frac{1}{2}(\arctan x)^2 \Big|_0^A$$

$$= \lim_{A\to\infty} \left\{ 8(\arctan x)^2 - 8(\arctan 0)^2 \right\}$$

$$= 8\left(\frac{\pi}{2}\right)^2 - 8(0)^2 = 2\pi^2$$

21.) $$\int_{-\infty}^{0} \theta e^{\theta}\,d\theta \quad \left(\text{Let } u = \theta,\ dv = e^{\theta}\,d\theta \atop du = d\theta,\ v = e^{\theta}\right)$$

$$= \lim_{A\to-\infty} \int_A^0 \theta e^{\theta}\,d\theta$$

$$= \lim_{A\to-\infty} \left\{ \theta e^{\theta}\Big|_A^0 - \int_A^0 e^{\theta}\, d\theta \right\}$$

$$= \lim_{A\to-\infty} \left\{ 0 - Ae^{A} - e^{\theta}\Big|_A^0 \right\}$$

$$= \lim_{A\to-\infty} \left\{ -Ae^{A} - (1-e^{A}) \right\}$$

"$\infty \cdot 0$" (indeterminate)

$$= \lim_{A\to-\infty} \left\{ \frac{-A}{e^{-A}} - 1 + e^{A} \right\}$$

"$\frac{\infty}{\infty}$" (L'Hopital)

$$= \lim_{A\to-\infty} \left\{ \frac{-1}{-e^{-A}} - 1 + e^{A} \right\}$$

$$= \lim_{A\to-\infty} \left\{ 2e^{A} - 1 \right\} = 2e^{-\infty} - 1$$

$$= 2 \cdot \frac{1}{e^{\infty}} - 1 = 2(0) - 1 = -1$$

25.) $\int_0^1 x \ln x \, dx = \lim_{A\to 0^+} \int_A^1 x \ln x \, dx$

(Let $u = \ln x$, $dv = x\,dx$
$du = \frac{1}{x}\,dx$, $v = \frac{1}{2}x^2$)

$$= \lim_{A\to 0^+} \left\{ \frac{1}{2}x^2 \ln x \Big|_A^1 - \frac{1}{2}\int_A^1 x\, dx \right\}$$

$$= \lim_{A\to 0^+} \left\{ \frac{1}{2}(1)^2 \overbrace{\ln 1}^{0} - \frac{1}{2}A^2 \ln A - \frac{1}{2}\cdot\frac{x^2}{2}\Big|_A^1 \right\}$$

$$= \lim_{A\to 0^+} \left\{ -\frac{1}{2}A^2 \ln A - \left(\frac{1}{4} - \frac{A^2}{4}\right) \right\}$$

"$0 \cdot \infty$" (indeterminate)

$$= \lim_{A\to 0^+} \left\{ \frac{-\ln A}{2/A^2} - \frac{1}{4} + \frac{A^2}{4} \right\}$$

$\leftarrow$ " $\frac{\infty}{\infty}$ " (L'Hopital)

$$= \lim_{A\to 0^+} \left\{ \frac{-1/A}{-4/A^3} - \frac{1}{4} + \frac{A^2}{4} \right\}$$

$$= \lim_{A\to 0^+} \left\{ \frac{1}{2}A^2 - \frac{1}{4} \right\} = 0 - \frac{1}{4} = -1/4$$

30.) $$\int_2^4 \frac{1}{t\sqrt{t^2-2^2}}\,dt = \lim_{A\to 2^+} \int_A^4 \frac{1}{t\sqrt{t^2-2^2}}\,dt$$

$$= \lim_{A\to 2^+} \frac{1}{2}\operatorname{arcsec}\left(\frac{t}{2}\right)\Big|_A^4$$

$$= \lim_{A\to 2^+} \left\{ \frac{1}{2}\operatorname{arcsec}(2) - \frac{1}{2}\operatorname{arcsec}\left(\frac{A}{2}\right) \right\}$$

$$= \frac{1}{2}\cdot\left(\frac{\pi}{3}\right) - \frac{1}{2}(0) = \frac{\pi}{6}$$

34.) $$\int_0^\infty \frac{1}{(x+1)(x^2+1)}\,dx = \int_0^\infty \left[\frac{A}{x+1} + \frac{Bx+C}{x^2+1}\right] dx$$

$\{ A(x^2+1) + (Bx+C)(x+1) = 1$

Let $x=-1$: $2A = 1 \to A = 1/2$

Let $x = i$: $(Bi+C)(i+1) = 1 \to$

$-B + Ci + Bi + C = 1 \to$

$(B+C)i + (C-B) = (0)i + (1) \to$

$\left.\begin{matrix} B+C=0 \\ C-B=1 \end{matrix}\right\rangle 2C = 1 \to C = 1/2$,

$B = -1/2$ $\}$

$$= \int_0^\infty \left[\frac{1/2}{x+1} + \frac{-1/2 x + 1/2}{x^2+1} \right] dx$$

$$= \lim_{A\to\infty} \left\{ \frac{1}{2}\ln|x+1| - \frac{1}{4}\ln|x^2+1| + \frac{1}{2}\arctan x \right\} \Big|_0^A$$

$$= \lim_{A\to\infty} \Big\{ \frac{1}{2}\ln|A+1| - \frac{1}{4}\ln|A^2+1| + \frac{1}{2}\arctan A$$
$$- \Big(\frac{1}{2}\overset{0}{\cancel{\ln 1}} - \frac{1}{4}\overset{0}{\cancel{\ln 1}} + \frac{1}{2}\overset{0}{\cancel{\arctan 0}} \Big) \Big\}$$

$$= \lim_{A\to\infty} \left\{ \frac{1}{2}\ln(A+1) - \frac{1}{2}\cdot\frac{1}{2}\ln(A^2+1) \right\} + \frac{1}{2}\left(\frac{\pi}{2}\right)$$

$$= \lim_{A\to\infty} \left\{ \frac{1}{2}\ln(A+1) - \frac{1}{2}\ln\sqrt{A^2+1} \right\} + \frac{\pi}{4}$$

$$= \lim_{A\to\infty} \frac{1}{2}\cdot\ln\left(\frac{A+1}{\sqrt{A^2+1}}\right) + \frac{\pi}{4}$$

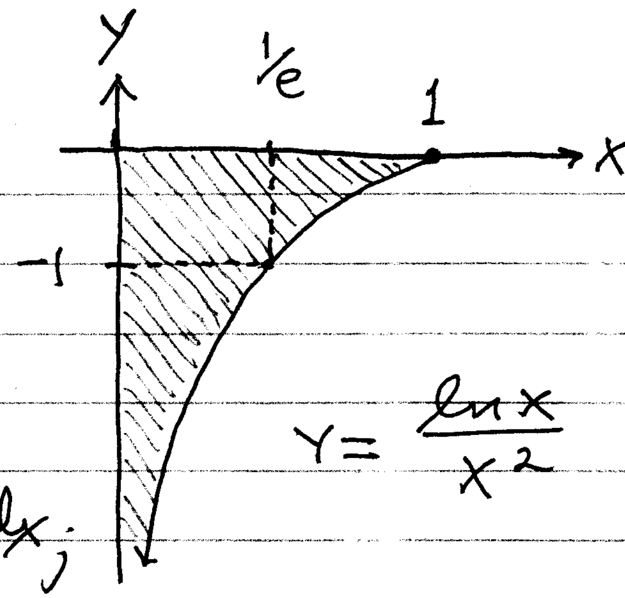
$$= \lim_{A\to\infty} \frac{1}{2}\ln\left(\frac{A+1}{\sqrt{A^2(1+1/A^2)}}\right) + \frac{\pi}{4}$$

$$= \lim_{A\to\infty} \frac{1}{2}\ln\left(\frac{A+1}{A}\cdot\frac{1}{\sqrt{1+1/A^2}}\right) + \frac{\pi}{4}$$

$$= \lim_{A\to\infty} \frac{1}{2}\ln\left(\left(1+\frac{1}{A}\right)\cdot\frac{1}{\sqrt{1+1/A^2}}\right) + \frac{\pi}{4}$$

$$= \frac{1}{2}\cdot\overset{0}{\cancel{\ln 1}} + \frac{\pi}{4} = \frac{\pi}{4}$$

37. From the graph we see that



$\int_0^1 \frac{\ln x}{x^2}\,dx < \int_0^{1/e} \frac{\ln x}{x^2}\,dx$;

and $\int_0^{\frac{1}{e}} \frac{\ln x}{x^2}\,dx < \int_0^{1/e} \frac{-1}{x^2}\,dx$

$= \lim_{A \to 0^+} \int_A^{1/e} -x^{-2}\,dx$

$= \lim_{A \to 0^+} \frac{1}{x}\Big|_A^{1/e} = \lim_{A \to 0^+} \left(\frac{1}{\frac{1}{e}} - \frac{1}{A}\right)$

$= e - \frac{1}{0^+} = e - \infty = -\infty$ (diverges)

so by comparison $\int_0^{\frac{1}{e}} \frac{\ln x}{x^2}\,dx$ diverges so by comparison $\int_0^1 \frac{\ln x}{x^2}\,dx$ diverges.

OR $\displaystyle\int_0^1 \frac{\ln x}{x^2}\,dx = \lim_{A\to 0^+}\int_A^1 \frac{\ln x}{x^2}\,dx$

(Let $u=\ln x$, $dv=\dfrac{1}{x^2}\,dx$

$\to du=\dfrac{1}{x}\,dx$, $v=\dfrac{-1}{x}$)

$$= \lim_{A\to 0^+}\left[\frac{-\ln x}{x}\Big|_A^1 - -\int_A^1 \frac{1}{x^2}\,dx\right]$$

$$= \lim_{A\to 0^+}\left[\frac{-\cancel{\ln 1}^{\,0}}{1} - \frac{-\ln A}{A} + \frac{-1}{x}\Big|_A^1\right]$$

$$= \lim_{A\to 0^+}\left[\frac{\ln A}{A} + \frac{-1}{1} - \frac{-1}{A}\right]$$

$$= \lim_{A\to 0^+}\left[\frac{\ln A}{A} + \frac{1}{A} - 1\right]$$

$$= \lim_{A\to 0^+}\left[\frac{1+\ln A}{A} - 1\right]$$

$$= \frac{-\infty}{0^+} - 1 = -\infty$$

(diverges)

41.) $0 \le \dfrac{1}{\sqrt{t}+\sin t} \le \dfrac{1}{\sqrt{t}}$ so

$$\int_0^{\pi} \frac{1}{\sqrt{t}+\sin t}\,dt \le \int_0^{\pi} \frac{1}{\sqrt{t}}\,dt$$

$$= \lim_{A\to 0^+} \int_A^{\pi} t^{-1/2}\,dt = \lim_{A\to 0^+} 2\sqrt{t}\,\Big|_A^{\pi}$$

$$= \lim_{A\to 0^+} \left(2\sqrt{\pi} - 2\sqrt{A}\right) = 2\sqrt{\pi} - 0 = 2\sqrt{\pi}$$

42.) $0 \le t - \sin t \le t \longrightarrow$

$$0 \le \frac{1}{t} \le \frac{1}{t-\sin t} \longrightarrow$$

$$\int_0^1 \frac{1}{t-\sin t}\,dt \ge \int_0^1 \frac{1}{t}\,dt = \lim_{A\to 0^+} \int_A^1 \frac{1}{t}\,dt$$

$$= \lim_{A\to 0^+} \ln|t|\,\Big|_A^1 = \lim_{A\to 0^+} \left(\overset{0}{\cancel{\ln 1}} - \ln A\right)$$

$$= -(-\infty) = \infty$$; so

$\displaystyle\int_0^1 \frac{1}{t-\sin t}\,dt$ diverges.

51.) $\int_0^\infty \frac{1}{\sqrt{x^6+1}}\,dx = \int_0^1 \frac{1}{\sqrt{x^6+1}}\,dx + \int_1^\infty \frac{1}{\sqrt{x^6+1}}\,dx$

$= C + D$;

$C = \int_0^1 \frac{1}{\sqrt{x^6+1}}\,dx \le \int_0^1 \frac{1}{\sqrt{0+1}}\,dx = \int_0^1 1\,dx$

$= x\Big|_0^1 = 1$ so $\int_0^1 \frac{1}{\sqrt{x^6+1}}\,dx$ converges;

$D = \int_1^\infty \frac{1}{\sqrt{x^6+1}}\,dx \le \int_1^\infty \frac{1}{\sqrt{x^6+0}}\,dx$

$= \int_1^\infty \frac{1}{x^3}\,dx = \lim_{A\to\infty} \frac{-1}{2x^2}\Big|_1^A$

$= \lim_{A\to\infty} \left(\frac{-1}{2A^2} - \frac{-1}{2}\right) = 0 + \frac{1}{2} = \frac{1}{2}$

so $\int_1^\infty \frac{1}{\sqrt{x^6+1}}\,dx$ converges ; then

$\int_0^\infty \frac{1}{\sqrt{x^6+1}}\,dx$ converges.

53.) $\int_1^\infty \frac{\sqrt{x+1}}{x^2}\,dx \le \int_1^\infty \frac{\sqrt{x+x}}{x^2}\,dx$

$= \lim_{A\to\infty} \int_1^A \frac{\sqrt{2}\cdot\sqrt{x}}{x^2}\,dx = \lim_{A\to\infty} \sqrt{2}\int_1^A \frac{1}{x^{3/2}}\,dx$

$= \lim_{A\to\infty} \sqrt{2}\,\frac{x^{-1/2}}{-1/2}\Big|_1^A = \lim_{A\to\infty} -2\sqrt{2}\left(\frac{1}{\sqrt{A}} - \frac{1}{\sqrt{1}}\right)$

$= -2\sqrt{2}\,(0-1) = 2\sqrt{2}$, so

$\int_1^\infty \frac{\sqrt{x+1}}{x^2}\,dx$ converges.

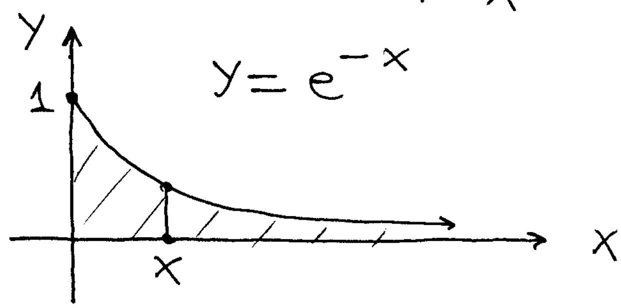
59.) $0 \le \frac{1}{x} \le \frac{e^x}{x}$ for $x \ge 1$ so

$$\int_1^\infty \frac{e^x}{x}\,dx \ge \int_1^\infty \frac{1}{x}\,dx = \lim_{A\to\infty} \int_1^A \frac{1}{x}\,dx$$

$$= \lim_{A\to\infty} \ln x \Big|_1^A = \lim_{A\to\infty} (\ln A - \overset{0}{\cancel{\ln 1}})$$

$= \infty$; then $\int_1^\infty \frac{e^x}{x}\,dx$ diverges.

67.)



$$\text{Area} = \int_0^\infty e^{-x}\,dx = \lim_{A\to\infty} \int_0^A e^{-x}\,dx$$

$$= \lim_{A\to\infty} -e^{-x} \Big|_0^A = \lim_{A\to\infty} \left(\frac{-1}{e^A} - -e^0\right)$$

$= 0 + 1 = 1$.

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70.) (DISC METHOD)

$$\text{Vol} = \pi \int_0^\infty (\text{radius})^2\,dx$$

$$= \pi \int_0^\infty (e^{-x})^2\,dx$$

$$= \lim_{A\to\infty} \pi \int_0^A e^{-2x}\,dx = \lim_{A\to\infty} \pi \cdot \frac{e^{-2x}}{-2} \Big|_0^A$$

$$= \lim_{A\to\infty} -\frac{\pi}{2}\left(\frac{1}{e^{2A}} - e^0\right) = -\frac{\pi}{2}(0-1)$$

$$= \frac{\pi}{2}$$

69.) (SHELL METHOD)

$$\text{Vol} = 2\pi \int_0^\infty (\text{radius})(\text{height})\, dx$$

$$= 2\pi \int_0^\infty (x)(e^{-x})\, dx$$

$$= \lim_{A\to\infty} 2\pi \int_0^A xe^{-x}\, dx$$

(Let $u = x$, $dv = e^{-x}dx$
$du = dx$, $v = -e^{-x}$)

$$= \lim_{A\to\infty} 2\pi \left\{ -xe^{-x}\Big|_0^A - -\int_0^A e^{-x}\, dx \right\}$$

$$= \lim_{A\to\infty} 2\pi \left\{ \frac{-A}{e^A} - e^{-x}\Big|_0^A \right\}$$

$$= \lim_{A\to\infty} 2\pi \left\{ \frac{-A}{e^A} - \left(\frac{1}{e^A} - 1\right) \right\}$$

"$\frac{\infty}{\infty}$" (L'Hopital)

$$= \lim_{A\to\infty} 2\pi \left\{ \frac{-1}{e^A} - \frac{1}{e^A} + 1 \right\}$$

$$= 2\pi \{ 0 - 0 + 1 \}$$

$$= 2\pi$$

75.) b.) Show that $\int_0^\infty \frac{\sin t}{t}\,dt < \infty$:

$$\int_0^\infty \frac{\sin t}{t}\,dt = \int_0^1 \frac{\sin t}{t}\,dt + \int_1^\infty \frac{\sin t}{t}\,dt$$

$$= \quad B \quad + \quad C \; ;$$

by MVT applied to $f(x) = \sin x$ on $[0, t]$ we get $\frac{f(t) - f(0)}{t - 0} = f'(c)$ for $0 < c < t$

$$\to \quad \frac{\sin t - \sin 0}{t} = \cos c < 1$$

$$\to \quad \frac{\sin t}{t} \leq 1 \; ; \text{ thus,}$$

$$B = \int_0^1 \frac{\sin t}{t}\,dt \leq \int_0^1 1\,dt = t\Big|_0^1 = 1 < \infty .$$

For $C = \int_1^\infty \frac{\sin t}{t}\,dt$

$$\left(\text{Let } u = 1/t, \; dv = \sin t\,dt \right.$$

$$\left. \to du = \frac{-1}{t^2}\,dt, \; v = -\cos t \right)$$

$$= -\frac{\cos t}{t}\Big|_1^\infty - \int_1^\infty \frac{\cos t}{t^2}\,dt \; ;$$

$$-\frac{\cos t}{t}\Big|_1^\infty = \lim_{A\to\infty} -\frac{\cos t}{t}\Big|_1^A$$

$$= \lim_{A\to\infty} \underbrace{-\frac{\cos A}{A}}_{\to 0} - \frac{-\cos 1}{1}$$

by Squeeze Principle

$$= 0 + \cos 1 < \infty \quad ;$$

$$\int_1^\infty \left| \frac{\cos t}{t^2} \right| dt = \int_1^\infty \frac{|\cos t|}{t^2}\, dt$$

$$\leq \int_1^\infty \frac{1}{t^2}\, dt = \lim_{A\to\infty} \int_1^A \frac{1}{t^2}\, dt$$

$$= \lim_{A\to\infty} \left. \frac{-1}{t} \right|_1^A = \lim_{A\to\infty} \left(\frac{-1}{A} - \frac{-1}{1} \right)$$

$$= 0 + 1 < \infty \quad ; \quad \text{then}$$

$$C = \int_1^\infty \frac{\sin t}{t}\, dt = \left. -\frac{\cos t}{t} \right|_1^\infty - \int_1^\infty \frac{\cos t}{t^2}\, dt$$

$$< \infty \quad , \quad \text{and}$$

$$\int_0^\infty \frac{\sin t}{t}\, dt = B + C < \infty \quad .$$