1. For any \( f \in C([0,1]) \), define
\[
||f||_1 := \left( \int_0^1 |f(x)|^2 \right)^{1/2}
\]
and
\[
||f||_2 := \left( \int_0^1 (1 + x)|f(x)|^2 \right)^{1/2}.
\]
Show that \( ||\cdot||_1 \) and \( ||\cdot||_2 \) are equivalent norms in \( C([0,1]) \).

2. Let \( X \) be the space of all sequences of real numbers with only finitely many nonzero terms. Consider the following two norms on \( X \):
\[
||x_n||_1 := \sum_{n=1}^{\infty} |x_n| \quad \text{and} \quad ||x_n||_2 := \sqrt{\sum_{n=1}^{\infty} |x_n|^2}.
\]
Are the norms \( ||\cdot||_1 \) and \( ||\cdot||_2 \) equivalent? Justify your answer.

3. Let \( X = C_b([0, \infty)) \) be the space of all bounded and continuous functions on \([0, \infty)\). For any \( a > 0 \), define
\[
||f||_a := \left( \int_0^\infty e^{-ax}|f(x)|^2 \right)^{1/2}, \quad \forall f \in X.
\]
(a) Show that \( ||\cdot||_a \) is a norm on \( X \).
(b) For any \( a > b > 0 \), show that \( ||\cdot||_a \) and \( ||\cdot||_b \) are not equivalent norms on \( X \).

4. Let \( e_1, e_2, \ldots, e_n \) be any given vectors in a real linear space \( X \), and let \( ||\cdot|| \) be a norm on \( X \). Show that for any \( x \in X \), there exists \( (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{R}^n \) such that
\[
||x - \sum_{i=1}^{n} \lambda_i e_i|| = \min_{(a_1, a_2, \ldots, a_n) \in \mathbb{R}^n} ||x - \sum_{i=1}^{n} a_i e_i||.
\]

5. Let \( X = (C([0,1]), ||\cdot||_\infty) \). Define \( T : X \to X \) by
\[
(Tf)(x) = x \int_0^x f(t) dt, \quad \forall f \in X.
\]
Show that \( T \in \mathcal{B}(X) \) and compute \( ||T|| \). Also prove that the inverse \( T^{-1} : ran(T) \to X \) exists but is not bounded.