Problem 1. Induced metric of a space, which is the product space of metric spaces.

Proof. We want to show

- **Non-negativity:** for any $p \geq 1$ and $i = 1, \cdots, n$, since $d_i(x_i, y_i) \geq 0$ for any $x_i, y_i \in X_i$, then we have

$$D_p(x, y) = \left( \sum_{i=1}^{n} d_i(x_i, y_i)^p \right)^{1/p} \geq 0;$$

and by non-negativity,

$$D_p(x, y) = 0 \iff \forall i, \quad d_i(x_i, y_i) = 0 \iff \forall i, \quad x_i = y_i \iff x = y.$$

- **Symmetry:** for $i = 1, \cdots, n$, we know $d_i(x_i, y_i) = d_i(y_i, x_i)$, then we have

$$D_p(x, y) = \left( \sum_{i=1}^{n} d_i(x_i, y_i)^p \right)^{1/p} = \left( \sum_{i=1}^{n} d_i(y_i, x_i)^p \right)^{1/p} = D_p(y, x).$$

- **Triangle inequality:** Recall Minkowski’s inequality for $p \geq 1:

$$(\sum_{i=1}^{n} |a_i + b_i|^p)^{1/p} \leq \left( \sum_{i=1}^{n} |a_i|^p \right)^{1/p} + \left( \sum_{i=1}^{n} |b_i|^p \right)^{1/p}, \quad (1)$$

for any numbers $\{a_i\}_{i=1}^{n}$ and $\{b_i\}_{i=1}^{n}$, thus take $a_i = d_i(x_i, y_i)$ and $b_i = d_i(y_i, z_i)$

$$D_p(x, y) + D_p(y, z) = \left( \sum_{i=1}^{n} d_i(x_i, y_i)^p \right)^{1/p} + \left( \sum_{i=1}^{n} d_i(y_i, z_i)^p \right)^{1/p} \geq \left( \sum_{i=1}^{n} (d_i(x_i, y_i) + d_i(y_i, z_i))^p \right)^{1/p} \geq \left( \sum_{i=1}^{n} d_i(x_i, z_i)^p \right)^{1/p} = D_p(x, z),$$

since $d_i(x_i, y_i) + d_i(y_i, z_i) \geq d_i(x_i, z_i)$ and $x \mapsto x^p$ is increasing.

Problem 2. Induced metric of pre-image space under injection. (pull-back metric)

Proof. Let $y_i = f(x_i) \in Y$. On the other hand, due to injection, for any $y_i \in Y$, there exists a unique $x_i$ such that $f(x_i) = y_i$. And we want to show

- **Non-negativity:**

$$d_X(x_1, x_2) = d_Y(f(x_1), f(x_2)) = d_Y(y_1, y_2) \geq 0,$$

and $d_X(x_1, x_2) = 0 \iff d_Y(y_1, y_2) = 0 \iff y_1 = y_2 \iff x_1 = x_2.$
• Symmetry:
\[ d_X(x_1, x_2) = d_Y(y_1, y_2) = d_Y(y_2, y_1) = d_X(x_2, x_1). \]

• Triangle inequality:
\[ d_X(x_1, x_2) + d_X(x_2, x_3) = d_Y(y_1, y_2) + d_Y(y_2, y_3) \geq d_Y(y_1, y_3) = d_X(x_1, x_3). \]

Problem 3.

Proof. We want to show

• Non-negativity: Since \( f(0) = 0 \) and \( f \) is non-decreasing, \( f(x) \geq 0 \) on \([0, \infty)\). And \( d(x, y) \geq 0 \), thus \( f(d(x, y)) \) is well-defined.
\[ d_f(x, y) = f(d(x, y)) \geq 0 \quad \text{and} \quad d_f(x, y) = 0 \iff f(d(x, y)) = 0 \iff d(x, y) = 0 \iff x = y. \]

• Symmetry:
\[ d_f(x, y) = f(d(x, y)) = f(d(y, x)) = d_f(y, x). \]

• Triangle inequality:
\[ d_f(x, y) + d_f(y, z) = f(d(x, y)) + f(d(y, z)) \geq f(d(x, y) + d(y, z)) \geq f(d(x, z)) = d_f(x, z). \]

Problem 4. Metric induced by norm.

Proof. First, assume \( d \) is associated metric of norm, i.e., \( d(x, y) = \|x - y\|, \forall x, y \in X \). Then
\[ d(x + z, y + z) = \|x + z - (y + z)\| = \|x - y\| = d(x, y) \]
\[ d(\lambda x, \lambda y) = \|\lambda x - \lambda y\| = |\lambda|\|x - y\| = |\lambda|d(x, y). \]

Second, assume \( d \) is a metric satisfying translation invariance and positive homogeneity, then we define a function \( f : X \mapsto \mathbb{R} \) by \( f(x) = d(x, 0) \), since \( 0 \in X \). We verify that:

• Non-negativity: \( f(x) = d(x, 0) \geq 0 \). And \( f(x) = 0 \iff d(x, 0) = 0 \iff x = 0. \)

• Homogeneity: \( f(\lambda x) = d(\lambda x, 0) = |\lambda|d(x, 0) \), by positive homogeneity of \( d \).

• Triangle inequality: \( f(x) + f(y) = d(x, 0) + d(y, 0) = d(x, 0) + d(y - y, 0 - y) = d(x, 0) + d(0, -y) \geq d(x, -y) = d(x + y, -y + y) = d(x + y, 0) = f(x + y). \)

Problem 5. Alternative characterization of closed sets in a metric space.
Proof. “⇒”: Assume $F \subset X$ is closed and $x_n \to x$ for $x_n \in F$, i.e., for any $\varepsilon > 0$, there exists an integer $N > 0$ such that for $n \geq N$, $d(x_n, x) < \varepsilon$. Define $B(x, \varepsilon) = \{y \in X : d(y, x) < \varepsilon\}$. Then for $n \geq N$, $x_n \in B(x, \varepsilon)$.

Prove by contradiction. Assume there exists a sequence $\{y_n\} \subset F$ and $y_n \to y$, but $y \notin F$. Since $F$ is closed, $X \setminus F$ is open and $y \in X \setminus F$. There exist a $\varepsilon > 0$ such that $B(y, \varepsilon) \subset X \setminus F$. On the other hand, there exists a $N' > 0$ such that for $n \geq N'$, $y_n \in B(y, \varepsilon) \subset X \setminus F$, contradicted with the fact that every $y_n \in F$.

“⇐”: Assume for every sequence $\{x_n\} \subset F$, if $x_n \to x$, then $x \in F$. We want to show $F$ is closed.

Prove by contradiction. Assume $F$ is not closed, then $X \setminus F$ is not open. Therefore $X \setminus F$ is not empty and there exists a $x \in X \setminus F$ such that for every $\varepsilon = \frac{1}{n} > 0$, $B(x, \frac{1}{n}) \notin X \setminus F$, i.e., there exists a $y_n \in B(x, \frac{1}{n})$ but $y_n \notin F$.

By construction, the sequence $\{y_n\}$ in $F$ converges to $x$ but $x \in X \setminus F$, contradiction! \hfill \Box

Problem 6. Composition of continuous functions

Proof. For any $x_0 \in X$, let $y_0 = f(x_0)$ and $y = f(x)$. For every $\varepsilon > 0$, since $g : Y \mapsto Z$ is continuous, there exists a $\theta = \theta(\varepsilon)$ such that if $d_Y(y, y_0) < \theta$, we have $d_Z(g(y), g(y_0)) < \varepsilon$.

For above $x_0$ and $\theta$, since $f : X \mapsto Y$ is continuous, there exists a $\delta = \delta(\theta) > 0$ such that if $d_X(x, x_0) < \delta$, we have $d_Y(f(x), f(x_0)) < \theta$.

Therefore, for any $x_0 \in X$ and every $\varepsilon > 0$, there exists a $\delta > 0$ defined above such that if $d_X(x, x_0) < \delta$, then $d_Y(f(x), f(x_0)) = d_Y(y, y_0) < \theta$, thus we have $d_Z(h(x), h(x_0)) = d_Z(g(f(x)), g(f(x_0))) = d_Z(g(y), g(y_0)) < \varepsilon$. \hfill \Box