Problem 1. Cauchy sequence and its subsequence

Proof. (a) Let \((x_{n_k})\) denote a convergent subsequence of \((x_n)\), let \(x_{n_k} \to x\) as \(k \to \infty\), i.e., for any \(\varepsilon > 0\), there exists a \(K\) such that for all \(k \geq K\), we have \(d(x_{n_k}, x) < \varepsilon\).

Since \((x_n)\) is Cauchy, for any \(\varepsilon > 0\), there exists a \(N\) such that for any \(n, m \geq N\), we have \(d(x_n, x_m) < \varepsilon\).

For any \(\varepsilon > 0\), pick \(N_0 = \max (N, n_K)\), for any \(n, m \geq N_0\), we have
\[ d(x_n, x) \leq d(x_n, x_m) + d(x_m, x) \leq \varepsilon + \varepsilon = 2\varepsilon. \]

Thus \(x_n \to x\) as \(n \to \infty\).

(b) For \(\varepsilon_1\), since \((x_n)\) is Cauchy, there exists a \(N_1\) such that for any \(m > n \geq N_1\), we have \(d(x_n, x_m) \leq \varepsilon_1\). We then define \(n_1 = N_1\). For any \(n_2 > n_1\), we have \(d(x_{n_1}, x_{n_2}) \leq \varepsilon_1\).

For each \(\varepsilon_k\), suppose \(n_k = N_k\), where for any \(m > n_k \geq N_k\), \(d(x_{n_k}, x_m) \leq \varepsilon_k\), we then define \(n_{k+1}\) recursively. For \(\varepsilon_{k+1}\), there exists a \(N_{k+1}\) such that for \(m > n \geq N_{k+1}\), we have \(d(x_n, x_m) \leq \varepsilon_{k+1}\).

If \(N_{k+1} > N_k\), then we define \(n_{k+1} = N_{k+1}\). Since \(n_{k+1} > n_k = N_k\), \(d(x_{n_k}, x_{n_{k+1}}) \leq \varepsilon_k\).

And for any \(m > n_{k+1} = N_{k+1}\), \(d(x_{n_k}, x_m) \leq \varepsilon_{k+1}\).

If \(N_{k+1} \leq N_k\), then we define \(n_{k+1} = N_k + 1\). Since \(n_{k+1} > n_k = N_k\), \(d(x_{n_k}, x_{n_{k+1}}) \leq \varepsilon_k\).

And for any \(m > n_{k+1} = N_k + 1 > N_{k+1}\), \(d(x_{n_k}, x_m) \leq \varepsilon_{k+1}\).

Problem 2.

Proof. (a) Assume there is a point \(z\) in both \(B_R(x)\) and \(B_r(y)\), i.e., \(d(x, z) < R\) and \(d(y, z) < r\).

Thus \(R + r > d(x, z) + d(y, z) \geq d(x, y) \geq R + r\), contradiction!

(b) Let \(z \in B_r(y)\), i.e., \(d(y, z) < r\). Then \(d(z, x) \leq d(z, y) + d(y, x) < r + R - r = R\), i.e., \(z \in B_R(x)\).

(c) Given \(\mathbb{R}^2\) associated with metric \(d(x, y) = \frac{e(x, y)}{1 + e(x, y)}\), where \(e(x, y)\) denotes the Euclidean metric. For \(x = (0, 0), y = (5, 0)\) and \(R = \frac{2}{3}, r = \frac{2}{3}\). We notice that
\[
B_{2/3}(x) = \left\{ z \in \mathbb{R}^2 : d(z, x) < \frac{2}{3} \right\} = \left\{ z \in \mathbb{R}^2 : e(z, x) < 2 \right\};
\]
\[
B_{2/3}(y) = \left\{ z \in \mathbb{R}^2 : d(z, y) < \frac{2}{3} \right\} = \left\{ z \in \mathbb{R}^2 : e(z, y) < 2 \right\},
\]

which yields that \(B_{2/3}(x)\) and \(B_{2/3}(y)\) are disjoint. However \(d(x, y) \leq 1 < \frac{2}{3} + \frac{2}{3}\).

2. Given \(\mathbb{R}^2\) associated with discrete metric \(d(x, y) = 1\) if \(x \neq y\) and \(d(x, y) = 0\) if \(x = y\). For \(x = (0, 0), y = (5, 0)\) and \(R = r = 2\). We note that \(B_2(x) = \mathbb{R}^2\) so that \(B_2(y) \subset B_2(x) = \mathbb{R}^2\), however \(d(x, y) = 1 > 0 = 2 - 2\).

Proof. (a) [Existence] For any \( x \in \bar{X} \setminus X \), since \( X \) is a dense subset, so there is a sequence \((x_n)\) in \( X \) such that \( x_n \to x \). We define \( \tilde{f}(x) := \lim_{n \to \infty} f(x_n) \). For any \( x \in X \), we define \( \tilde{f}(x) = f(x) \). It satisfies \( \tilde{f}|_X = f \) obviously.

First we want to show \( \tilde{f}(x) \) is well-defined on \( \bar{X} \).

For any \( x \in \bar{X} \setminus X \), since \( x_n \to x \), for any \( \varepsilon > 0 \), there exists a \( N \) such that for any \( m, n \geq N \), we have

\[
d_{\bar{X}}(x_n, x_m) \leq d_{\bar{X}}(x_n, x) + d_{\bar{X}}(x, x_m) \leq 2\varepsilon,
\]

Furthermore, due to the fact that \( f \) is Lipschitz continuous function, we have

\[
d_Y(f(x_n), f(x_m)) \leq Lip(f)d_{\bar{X}}(x_n, x_m) \leq 2Lip(f)\varepsilon,
\]

which yields that \( (f(x_n)) \) is a Cauchy sequence in \( (Y, d_Y) \). Since \( Y \) is complete, \( (f(x_n)) \) converges in \( Y \).

On the other hand, assume that there exists a different sequence \((y_m)\) in \( X \) that \( y_m \to x \). For any \( \varepsilon \), there exists \( N, M \) such that for \( n \geq N \) and \( m \geq M \), we have \( d(x_n, x) \leq \varepsilon \) and \( d(y_m, x) \leq \varepsilon \). Then

\[
d_Y(f(x_n), f(x_m)) \leq Lip(f)d_{\bar{X}}(x_n, y_m) \leq Lip(f) (d_{\bar{X}}(x_n, x) + d_{\bar{X}}(x, y_m)) \leq 2Lip(f)\varepsilon,
\]

which shows that the limit \( \tilde{f}(x) \) does not depend the choice of sequences.

Second we want to show \( \tilde{f}(x) \) is Lipschitz and \( Lip(\tilde{f}) = Lip(f) \). (which implies it is continuous, as well.)

We show \( d_Y(y, \cdot) : Y \to \mathbb{R} \) is continuous with respect to \( y \). Then the same holds for \( d_Y(\cdot, y) \) and for \( d_{\bar{X}} \). In fact,

\[
-d_Y(y, y_0) \leq d_Y(y, \cdot) - d_Y(y_0, \cdot) \leq d_Y(y, y_0).
\]

Furthermore, we have

\[
\lim_n d_Y(y_n, \cdot) = d_Y(\lim y_n, \cdot).
\]

For any \( x, y \in \bar{X} \), assume that \( x_n \to x \) and \( y_n \to y \) with \((x_n), (y_n)\) in \( X \).

\[
d_Y(\tilde{f}(x), \tilde{f}(y)) = d_Y(\lim f(x_n), \lim f(y_n)) = \lim_n d_Y(\tilde{f}(x_n), f(y_n)) \leq \lim_n Lip(f)d_{\bar{X}}(x_n, y_n) = Lip(f)d_{\bar{X}}(\lim x_n, \lim y_n) = Lip(f)d_{\bar{X}}(x, y),
\]

which implies that \( \tilde{f} \) is Lipschitz and \( Lip(\tilde{f}) = Lip(f) \).

[Uniqueness] Suppose there is another continuous map \( g : \bar{X} \to Y \) such that \( g|_X = f \). Then there exists a \( x \in \bar{X} \setminus X \) such that \( g(x) \neq \tilde{f}(x) \). Since \( g \) is continuous, then for every \( \varepsilon > 0 \), there exists a \( \delta > 0 \), such that \( d_{\bar{X}}(x_n, x) < \delta \) implies \( d_Y(g(x), g(x_n)) < \varepsilon \). Note that \( d_Y(g(x), g(x_n)) = d_Y(g(x), f(x_n)) \) and Cauchy sequence \((f(x_n))\) in \( Y \) has the unique limit. Contradiction!
(b) [Sharpness] Let \( X = (-\infty, 0) \cup (0, \infty) \subset \bar{X} = \mathbb{R} \) with Euclidean metric \( e \). Then \( X \) is a dense subset of \( \bar{X} \). Let \( f(x) = \frac{1}{|x|} \) on \( X \). Then \( f(x) : (X, e) \rightarrow (\mathbb{R}, e) \) is a continuous function, however there is no way to extend \( f \) from \( X \) to \( \bar{X} \) continuously.

\( \square \)

**Problem 4.** “lower semi-continuous + coercive ⇒ minimum”

**Proof.** For any \( M > 0 \), there exists a \( R > 0 \) such that for \( \|x\| \geq R \) implies \( f(x) \geq M \). Thus

\[
\inf_{x \in \mathbb{R}^d} f(x) = \inf_{\|x\| > R} f(x), \inf_{\|x\| \leq R} f(x) \geq \inf(M, \inf_{\|x\| \leq R} f(x)).
\]

Suppose \( f(x) \) on \( B_R(0) := \{x : \|x\| \leq R\} \) is not bounded from below, i.e., for any \( n > 0 \), there exist an \( x_n \in B_R(0) \) such that \( f(x_n) < -n \). Since \( B_R(x) \) is compact, there exists a subsequence \( \{x_{n_k}\} \) such that \( x_{n_k} \rightarrow x \) in \( B_R(0) \). By lower semi-continuity, we have

\[
f(x) \leq \liminf_{k \rightarrow \infty} f(x_{n_k}) \rightarrow -\infty,
\]

which makes \( f \) is not defined at \( x \), contradiction! Thus there exists a \( B \in \mathbb{R} \) such that \( \inf_{x \in B_R(0)} f(x) = B \). Then

\[
\inf_{x \in \mathbb{R}^d} f(x) \geq \min(M, B),
\]

which implies \( f \) is bounded from below on \( \mathbb{R}^d \).

Let \( (x_n) \) denote a minimizing sequence such that \( f(x_n) \rightarrow \inf_{x \in \mathbb{R}^d} f(x) \) (why exists?). Due to coercivity of \( f \), \( (x_n) \) is bounded, thus there exists a subsequence \( \{x_{n_k}\} \) such that \( x_{n_k} \rightarrow x \). By lower semi-continuity of \( f \), we have

\[
f(x) \leq \liminf_{k \rightarrow \infty} f(x_{n_k}) = \inf_{x \in \mathbb{R}^d} f(x).
\]

By the definition of \( \inf \), we know \( x \) is the minimizer. \( \square \)

**Problem 5.**

**Proof.** Suppose \( S \) is an \( \varepsilon \)--net of \( A \), then

\[
A \subset \bigcup_{s \in S} B_\varepsilon(s) = \left( \bigcup_{s \in S \cap A} B_\varepsilon(s) \right) \cup \left( \bigcup_{s' \in S \setminus A} B_\varepsilon(s') \right).
\]

Note that for any \( s' \in S \setminus A \), if \( B_\varepsilon(s') \cap A = \emptyset \), then \( S \setminus \{s'\} \) is still an \( \varepsilon \)--net of \( A \).

Let \( S' = \{s \in S \cap A : B_\varepsilon(s) \cap A \neq \emptyset \} \). For any \( s' \in S' \), we pick any \( k \in B_\varepsilon(s') \cap A \), thus \( d(k, s') < \varepsilon = 2\varepsilon - \varepsilon \). By Problem 2b, \( B_\varepsilon(s') \subset B_{2\varepsilon}(k) \). In this way, we induce a map from set \( S' \) to set \( K' \) such that \( \bigcup_{s' \in S'} B_\varepsilon(s') \subset \bigcup_{k \in K} B_{2\varepsilon}(k) \) with cardinality \( |K'| \leq |S'| \).

Now we define \( K := \{s : s \in S \cap A\} \cup K' \). By construction, \( K \subset A \) and \( K \) is a \( 2\varepsilon \)--net of \( A \), with cardinality \( |K| \leq |S \cap A| + |K'| \leq |S| \).

\( \square \)

**Problem 6.**
Proof.

1. Assume $A$ has a finite $\varepsilon$-net $X_\varepsilon$ for every $\varepsilon > 0$, then for any subset $B$ of $A$, we have

   $$B \subset A \subset \bigcup_{x \in X_\varepsilon} B_\varepsilon(x),$$

   i.e., $X_\varepsilon$ is also a finite $\varepsilon$-net of $B$.

2. Assume $A$ is nonempty and totally bounded in $(X, d)$, i.e., for every $\varepsilon > 0$, there exist a finite $\varepsilon$-net $Y_\varepsilon \subset X$ of $A$.

   Define $D := \max_{(y_1, y_2) \in Y_\varepsilon \times Y_\varepsilon} d(y_1, y_2)$. Since $Y_\varepsilon$ is finite, $D$ exists and is finite.

   Pick $x \in A$ and for any $a \in A$, assume $x \in B_\varepsilon(y_1)$ and $a \in B_\varepsilon(y_2)$ for some $y_1, y_2 \in Y_\varepsilon$.

   Then

   $$d(x, a) \leq d(x, y_1) + d(y_1, y_2) + d(y_2, a) \leq 2\varepsilon + D.$$

   We prove $A$ is bounded.

   Take $\mathbb{N}$ with discrete metric $d(x, y) = 1$ when $x \neq y$ and $d(x, y) = 0$ when $x = y$. Then $\mathbb{N}$ is bounded because for any integer $n \in \mathbb{N}$, $d(0, n) \leq 1$.

   However $\mathbb{N}$ is not totally bounded, because for $\varepsilon \leq 1$, there does not exist a finite $\varepsilon$-net $X_\varepsilon$.

3. We assume $\mathbb{R}^n$ equipped with the associated metric $d$ of Euclidean norm or other equivalent norm.

   Let $A \subset \mathbb{R}^n$ is bounded, i.e., there exists a $r > 0$ and $x \in A$ such that for any $a \in A$, $d(x, a) \leq r$.

   For any small $\varepsilon > 0$, define $X_\varepsilon = \{x + \sum_{j=1}^{n} k_j \cdot e_j : -(\lfloor \frac{r}{2\varepsilon} \rfloor + 1) \leq k_j \leq (\lfloor \frac{r}{2\varepsilon} \rfloor + 1)\}$, which is finite. Furthermore, we have

   $$A \subset \bigcup_{x \in X_\varepsilon} B_\varepsilon(x),$$

   which implies $X_\varepsilon$ is a finite $\varepsilon$-net.

Problem 7.

Proof. Since $d$ is continuous with respect to each argument, let $(a_n, b_n)$ be a minimizing sequence such that $d(a_n, b_n) \to \text{dist}(A, B)$, by lower-semi continuity of $d$. Since $A$ and $B$ are compact, there exists a subsequence $(a_{n_k}, b_{n_k})$ such that $(a_{n_k}, b_{n_k}) \to (a, b)$. By the lower semi-continuity of $d$, we have

   $$d(a, b) \leq \liminf_{k} d(a_{n_k}, b_{n_k}) = \text{dist}(A, B).$$