**Problem 1.** Exercise 5.6

**Proof.** 1. For any nonzero $x \in X$, let $Y = \{ kx : k \in \mathbb{R} \}$. Then $Y$ is a linear subspace of $X$.

We define a functional $\psi : Y \mapsto \mathbb{R}$ by $\psi(kx) = k\|x\|$.

It is linear since $\psi(kx + lx) = (k + l)\|x\| = \psi(kx) + \psi(lx)$.

It is bounded and $\|\psi\| = 1$.

By Hahn-Banach theorem, there exist a bounded linear function $\phi : X \mapsto \mathbb{R}$ such that $\|\phi\| = \|\psi\| = 1$. Moreover, since $x \in Y$, $\phi(x) = \psi(1 \cdot x) = \|x\|$.

For $x = 0 \in X$, we pick any $\psi_y$ with nonzero $y$ defined as above. The corresponding $\phi_y(0) = \|0\| = 0$ with $\|\phi_y\| = 1$.

2. Assume $x \neq y$, let $z = x - y$ and $z$ is nonzero. We define $Z = \{ kz : k \in \mathbb{R} \}$ and a bounded linear functional $\psi : Z \mapsto \mathbb{R}$ by $\psi(kz) = k\|z\|$. It can be extended to a bounded linear functional $\phi : X \mapsto \mathbb{R}$ such that $\phi(z) = \phi(x - y) = \|x - y\| > 0$, however $\phi(x - y) = \phi(x) - \phi(y) = 0$. Contradiction!

**Problem 2.** Lower semi-continuity of weak convergence.

**Proof.** From Problem 1, we know that there exists a bounded linear functional $\phi \in X^*$ such that $\phi(x) = \|x\|$ and $\|\phi\| = 1$.

$x_n \rightharpoonup x$ implies that $\phi(x_n) \rightharpoonup \phi(x)$. Thus

$$\|x\| = \phi(x) = |\phi(x)| = \lim_{n \to \infty} |\phi(x_n)| \leq \liminf_{n \to \infty} \|\phi\| \|x_n\| = \liminf_{n \to \infty} \|x_n\|.$$ 


**Problem 3.** Isometric embedding of a separable Banach space in $\ell^\infty$.

**Proof.** Let $X$ denote the separable Banach space and $Y$ denote its countable dense subset. For any $x \in X$, let $(x_n)$ denote the sequence in $Y$ that converges to $x$. By Exercise 5.6, there exists a sequence $(\phi_n)$ in $X^*$ such that $\|\phi_n\| = 1$ and $\phi_n(x_n) = \|x_n\|$. Now we define the isometric embedding $J : X \mapsto \ell^\infty$ by

$$J(x) = (\phi_1(x), \phi_2(x), \cdots, \phi_n(x), \cdots)$$

$J$ is well-defined since $|\phi_k(x)| \leq \|\phi_k\| \|x\| = \|x\|$ for all $k$.

$J$ is linear and we will prove $\|J(x)\|_\infty = \|x\|$ to show it is an isometry. Especially we prove that $\|J(x)\|_\infty \geq \|x\|$. By the continuity of norm and continuity of linear functional $\phi_n$, we have

$$\|x\| = \lim_{n \to \infty} \|x_n\| = \lim_{n \to \infty} \phi_n(x_n) = \lim_{n \to \infty} \phi_n(x_n - x) + \phi_n(x)$$

$$\leq \lim_{n \to \infty} \|\phi_n\| \|x_n - x\| + \sup_n \phi_n(x) = \|J(x)\|_\infty.$$ 


Remark: Thanks Matthew Corbelli for pointing this out. I think the assumption of completeness is not necessary in this proof. It might suggest that there is an isometric isomorphism between any separable Banach space with a closed linear subspace of $\ell^\infty$. However, we just need an embedding in this problem.

**Problem 4.** Exercise 5.7

Proof.

$$Kf(x) = \int_0^1 \sin \pi(x - y) f(y) dy$$
$$= \int_0^1 (\sin \pi x \cos \pi y - \cos \pi x \sin \pi y) f(y) dy$$
$$= \sin \pi x \int_0^1 \cos \pi y f(y) dy - \cos \pi x \int_0^1 \sin \pi y f(y) dy$$
$$= a(f) \sin \pi x + b(f) \cos \pi x,$$

where $a(f) = \int_0^1 \cos \pi y f(y) dy$ and $b(f) = -\int_0^1 \sin \pi y f(y) dy$.

Thus the range $\text{ran}(K) \subset \text{span} \{\sin \pi x, \cos \pi x\}$.

On the other hand, since for $f \equiv 1$, $Kf(x) = -\frac{2}{\pi} \cos \pi x$; for $f = \cos \pi x$, $Kf(x) = \frac{1}{2} \sin \pi x$.

Thus for $f(x) = 2a \cos \pi x - \frac{b \pi}{2}$, $Kf(x) = a \sin \pi x + b \cos \pi x$, which yields that $\text{ran}(K) = \text{span} \{\sin \pi x, \cos \pi x\}$.

$$Kf(x) = 0 \iff \begin{cases} \int_0^1 \cos \pi y f(y) dy = 0; \\ \int_0^1 \sin \pi y f(y) dy = 0. \end{cases}$$

$$\text{ker}(K) = \left\{ f \in C[0, 1] : \int_0^1 \cos \pi y f(y) dy = 0 \quad \text{and} \quad \int_0^1 \sin \pi y f(y) dy = 0. \right\}$$

**Problem 5.** Exercise 5.11

Proof.

$$0 \leq \|T_n\| - \|T\| \leq \|T_n - T\| \to 0.$$

**Problem 6.** Weak limit is unique.

Proof. This is the result from Problem 1 question b.

**Problem 7.** Exercise 5.17

Proof.
• $I - K$ is one-to-one.
  
  For any $x \neq y$, $(I - K)(x - y) = (x - y) - K(x - y)$.
  
  And $\|K(x - y)\| \leq \|K\|\|x - y\| \leq \|x - y\|$. Thus
  
  $$\|(I - K)(x - y)\| = \|(x - y) - K(x - y)\| \geq \|x - y\| - \|K(x - y)\| > 0.$$  

• $I - K$ is onto.
  
  For any $y \in X$, we define $x_n = \left(I + \sum_{i=1}^{n-1} K^i\right)y$. Now we show that $(x_n)$ is a Cauchy sequence in the Banach space $X$.

  $$\|x_n - x_m\| = \left\|\sum_{i=1}^{m-1} K^i y\right\| \leq \sum_{i=1}^{m-1} \|K^i\|\|y\| \leq \sum_{n} \|K^n\|\|y\| \leq \frac{\|K\|^n}{1 - \|K\|}\|y\| \leq \varepsilon,$$

  for large enough $N$ and $m > n \geq N$.
  
  Let $x_n \to x$ with respect to $\|\cdot\|$. By the continuity of $I - K$, we have

  $$(I - K)x = \lim_{n \to \infty} (I - K)x_n = \lim_{n \to \infty} (I - K)(I + \sum_{i=1}^{n-1} K^i)y = \lim_{n \to \infty} (I - K^n)y = y,$$

  since $\lim_{n \to \infty} \|K^n y\| = 0$.
  
  In this way, for any $y \in X$, we can find a $x \in X$ such that $(I - K)x = y$, that is, $I - K$ is onto.

• $(I - K)^{-1} = I + K + K^2 + K^3 + \cdots$ and the right hand side converges uniformly.

  $$\left\|I - (I - K)(I + \sum_{i=1}^{n-1} K^i)\right\| = \left\|I - (I - K^n)\right\| = \|K^n\| \leq \|K\|^n \to 0,$$

  which implies that

  $$(I - K)(I + \sum_{i=1}^{n-1} K^i) \to I \quad \text{with respect to } \|\cdot\|.$$  

Based on a similar process, one can also show that:

$$(I + \sum_{i=1}^{n-1} K^i)(I - K) \to I \quad \text{with respect to } \|\cdot\|.$$  

Therefore, we have $(I - K)^{-1} = I + K + K^2 + K^3 + \cdots$.  

\qed