Completeness is not a topological property.

Homeomorphic spaces could have different completeness.

An topological space that has different metrics inducing its topology is complete under one metric, is not complete under the other one.

e.g. $\mathbb{R}$ could be complete with $d(x, y) = |x - y|$, but induce a discrete topology. The same is true for the discrete topology.

e.g. $(\mathbb{R}, d(x, y) = |x - y|)$ is complete with $d(x, y) = 1$ if $x \neq y$.

$(-1, 1)$ is complete with hyperbolic metric $d(x, y) = \frac{1}{|x - y|}$.

Introduce some topology.

The open ball in terms of a metric is open in terms of the other one, but complete, because $(-1, 1)$ behave like $(0, \infty)$. 

∀ $x$, $d(0, x)$
A metrizable topological space is called complete-metrizable if there is at least one complete metric inducing its topology.

\[ (X, \tau) \leftrightarrow (Y, \varnothing) \text{ homemorphism} \]

- \( X \) is not complete-metrizable \( \Rightarrow \) \( Y \) is not complete-metrizable.
- \( Y \) is not complete-metrizable.
- \( X \) is complete under some metric but not the other \( \Rightarrow \) \( Y \) is complete-metrizable but not complete under some metric.

Discrete topology on a countable set

\[ (\mathbb{N}, \text{lex}) \leftrightarrow (\mathbb{N}, \{0, \infty\}) \leftrightarrow (\mathbb{Q}, \text{lex}) \leftrightarrow (\mathbb{Q}, \{0, \infty\}) \]

compact not complete not complete complete.

- \( X \) is complete under every metric. So is \( Y \).
- Discrete topology on a finite set.

Recall if \( f \) is uniformly continuous, \( \{a_n\} \) is Cauchy, \( \{f(a_n)\} \) is Cauchy.

Uniformly continuous homeomorphism preserves completeness.

Generalization

1. two topological vector space \( V \) and \( W \)
   \[ f: V \rightarrow W \text{ is uniformly continuous if for any neighborhood } B \text{ of zero in } W, \]
   then exists a neighborhood \( A \) of zero in \( V \) such that
   \[ \forall v, v' \in A \Rightarrow \| f(v) - f(v') \| < B \]

2. generalizes to "uniform space."