Operator Norm: $X, Y$ normed linear space

$$T: X \to Y \text{ bounded linear map}$$

$$\|T\| = \inf \left\{ M \mid \|T(x)\| \leq M\|x\| \right\}$$

$$= \sup_{x \in X, \|x\|=1} \|T(x)\| = \sup_{\|x\|=1} \|T\| = \sup_{\|x\|=1} \|T\|$$

1. A linear map is bounded iff it's continuous.

Theorem: $T: X \to Y$ linear, if $X$ is finite-dim, then $T$ is continuous.

Proof: $\dim(X) = n$

Then $\dim(TX) = n \Rightarrow \dim(TX) = k$

$$X = \sum_{i=1}^{n} (x_i)$$

$$T(x) = \sum_{i=1}^{n} c_i f(x_i) \quad \|T(x)\| = \sum_{i=1}^{n} \|c_i f(x_i)\| \leq CM\|x\| \Rightarrow \text{bounded (continuous)}$$

Let $M = \sup \|f(x_i)\|$, $\sum_{i=1}^{n} \|c_i\| \leq C\|x\|$. (Any two norms on finite dim are equivalent).

Compute operator norm can be hard.

$\forall f \in L^1(a,b)$ $T: L^1(a,b) \to L^1(a,b)$ by $Tf(x) = \int_a^x f(t) \, dt$.

$$\|Tf\|_1 \leq \int_a^b \left| \int_a^x f(t) \, dt \right| \, dx \leq \int_a^b \left| \int_a^b f(t) \, dt \right| \, dx \leq \int_a^b |f(t)| \, dt \cdot (b-a) = (b-a)\|f\|_1,$$

$$\|T\| \leq b-a.$$

$$f_{x}(x) = \begin{cases} n, & x \in \left[ a, \frac{a+b}{2} \right] \\ 0, & x \in \left( a, \frac{a+b}{2} \right) \end{cases}$$

$$Tf_{x}(x) = \begin{cases} \int_a^x n \, dt & \text{if } x \in \left[ a, \frac{a+b}{2} \right] \\ \int_a^{x+b} n \, dt & \text{if } x \in \left( a, \frac{a+b}{2} \right) \end{cases}$$

$$\|Tf_{x}\|_1 \leq \int_a^{a+b} n(x-a) \, dx + \int_a^b 2 \, dx = \frac{1}{2n} + \left( b-a - \frac{b-a}{2} \right) = b-a - \frac{b-a}{2n}.$$
Let \( X \) and \( Y \) be named linear space.

\( \mathcal{B}(X,Y) \) the space of all continuous/banded linear mapping from \( X \) to \( Y \).

- \( \|x\|_{X,Y} = \inf \left\{ M : M \|x\|_X \leq \|Mx\|_Y \right\} \) is a norm of \( \mathcal{B}(X,Y) \)

- \( \mathcal{B}(X,Y) \) is vector space.

- Let \( X \) and \( Y \) be named linear space. \( Y \) complete.

Then \( \mathcal{B}(X,Y) \) is Banach.

- (T.) Cauchy

\[ \|T_n - T_m\|_X \leq \|T_n - T_m\|_X \]

\[ T_n \to Y \Rightarrow T \text{ is linear Banach, } \|T\| = \sup_{\|x\|_X = 1} \|Tx\|_Y \]

- (Cauchy sequence in normed space is bounded)

\[ T_n \to T \Rightarrow \|T\| \leq \liminf \|T_n\| \]

1. Define limit \( T \)
2. \( T \in \mathcal{B}(X,Y) \)
3. \( T_n \to T \) w.r.t \( \| \| \)

- Do not depend on \( X \) is complete or not.

\( T: A \subset X \to Y \) is continuous linear mapping

\( F \) \( : X \to Y \) unique extension.

\( \text{Id}: \mathcal{B}(A,Y) \to \mathcal{B}(X,Y) \) is an isometric isomorphism (preserves norm)

\[ T \to F \]

- Every incomplete normed linear space \( X \) can be isometrically embedded as a dense linear subspace of its completion \( \hat{X} \).

- Every completion contain \( X \) as a dense subset.

\( \text{Id}: \mathcal{B}(X,Y) \to \mathcal{B}(\hat{X},Y) \\
T \to F \) bijective/isometric
What's the problem of an unbounded linear map?

1. Let \( l^2 = \{ x \mid \sum_{k=1}^{\infty} |x_k|^2 < \infty \} \), \( A = \text{diag}(1, 2, \ldots) \), \( \|x\|_A = \left( \sum_{k=1}^{\infty} |x_k|^2 \right)^{1/2} \), \( (Ax)_k = k \cdot x_k \).

1.1. \( A : l^2 \to l^2 \) is not true. \( \exists x \in l^1 : Ax \notin l^1 \) (unbounded operator cannot be defined on the whole Banach space).

1.2. \( A : X \to Y \) is unbounded.

\[ a \in M, \quad \|a\|_A = 1, \quad \|Ax\|_A = n \]

2. \( M \) is dense in \( l^1 \) (at most finitely many non-zero sequence already dense in \( l^1 \)).

\[ \forall x \in l^1, \exists y \in l^1 \text{ s.t. } \left\{ \begin{array}{l} \sum_{k=1}^{\infty} |x_k - y_k|^2 \leq \varepsilon \quad \text{and} \quad \sum_{k=1}^{\infty} x_k^2 = \infty \\ \sum_{k=1}^{\infty} k^2 y_k^2 < \infty \end{array} \right. \]

\[ \text{(Equation 1)} \]

\[ D \quad \text{unbounded.} \]

\[ f \to f' \]

\[ f_n = x_n \quad f'_n = nx_n \]

\[ \|f\|_{\text{full}} = 2, \quad \|f'\|_{\text{full}} = n \]

3. \( TX \to Y \) is closed if \( (X, \| \cdot \|) \subseteq (D(T), \| \cdot \|) \) with \( X \to X \) in \( X \) and \( TX \to Y \) in \( Y \).

Then \( \exists \Xi \in \text{BANACH} \) with \( Y \subseteq TX \) have both closed.

Closed graph theorem implies, if \( T : X \to Y \) is closed and \( D(T) = X \), then \( T \) is bounded.

Thus \( f \) is closed, densely defined operator, \( D(T) = X \) is equivalent with unboundedness.

\[ \Xi(T) = X \quad \Xi(D(T)) = X \quad \Xi(D(T)) = X \]

1.2; \( T : X \to Y \) unbounded.

\( \exists \Xi(D(T)) \neq X \) \( \Xi(D(T)) \) is dense in \( X \).

No extension like banded linear mapping.