Problem 1.

Proof. To show $\Sigma$ is a $\sigma$-algebra.

- Clearly $\emptyset \in \Sigma$.

- If $E \in \Sigma$, then by definition of $\Sigma$, either $E$ is countable or $X - E$ is countable.
  
  When $E$ is countable, by definition of $\Sigma$, $X - E \in \Sigma$.

  When $X - E$ is countable, by definition of $\Sigma$, $X - E \in \Sigma$.

- Assume $\{E_i\}_{i=1}^{\infty} \subset \Sigma$, if $\bigcup_{i=1}^{\infty} E_i$ is countable, then $\bigcup_{i=1}^{\infty} E_i \in \Sigma$. Otherwise, there must be a $k$ such that $E_k$ is uncountable, which yields that $X - E_k$ is countable. Note that de Morgan’s laws hold for any family of sets:

$$X - \bigcup_{i=1}^{\infty} E_i = \bigcap_{i=1}^{\infty} (X - E_i),$$

which is at most countable due to $X - E_k$ countable. Thus $\bigcup_{i=1}^{\infty} E_i \in \Sigma$.

Problem 2.

Proof.

a) Let us define $B_1 = A_1$ and $B_{i+1} = A_{i+1} - A_i$ for $i \geq 2$. Note that $B_i \in \Sigma$ and $\{B_i\}_{i=1}^{\infty}$ are pairwise disjoint. Moreover, $\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} B_i$ and $A_n = \bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} B_i$. Therefore,

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \mu\left(\bigcup_{i=1}^{\infty} B_i\right) = \sum_{i=1}^{\infty} \mu(B_i) = \lim_{n \to \infty} \sum_{i=1}^{n} \mu(B_i) = \lim_{n \to \infty} \mu\left(\bigcup_{i=1}^{n} B_i\right) = \lim_{n \to \infty} \mu(A_n).$$

b) Let us define $B_i = A_1 - A_i$. Note that $B_i \in \Sigma$, $B_i \subset B_{i+1}$ for each $i$. By (a), we have:

$$\mu\left(\bigcup_{i=1}^{\infty} B_i\right) = \lim_{n \to \infty} \mu(B_n) = \lim_{n \to \infty} \mu(A_1 - A_n) = \mu(A_1) - \lim_{n \to \infty} \mu(A_n),$$

which is well-defined if $\mu(A_1) < \infty$. Thus

$$\mu\left(\bigcap_{i=1}^{\infty} A_i\right) = \mu(A_1 - \bigcup_{i=1}^{\infty} B_i) = \mu(A_1) - \mu\left(\bigcup_{i=1}^{\infty} B_i\right) = \lim_{n \to \infty} \mu(A_n).$$

Counter-example: Take the measure space as Lebesgue measure on $\mathbb{R}$. Take $A_i = (i, \infty]$. For each $i$, $\mu(A_i) = \infty$ thus $\lim_{i \to \infty} \mu(A_i) = \infty$ however $\mu\left(\bigcap_{i=1}^{\infty} A_i\right) = 0$. 

$\square$
Problem 3.

Proof.

a) To show \( \mu \) is a measure on \((X, \Sigma)\):

- Since \( \mu_n(\emptyset) = 0 \) for all \( n \), then \( \mu(\emptyset) = \lim_{n \to \infty} \mu_n(\emptyset) = 0 \).

- Given a sequence of pairwise disjoint measurable sets \( \{E_i\}_{i=1}^\infty \subset \Sigma \),

\[
\mu\left(\bigcup_{i=1}^\infty E_i\right) = \lim_{n \to \infty} \mu_n\left(\bigcup_{i=1}^\infty E_i\right) = \lim_{n \to \infty} \sum_{i=1}^\infty \mu_n(E_i).
\]

By monotone convergence theorem, we can exchange the limit with the summation:

\[
\mu\left(\bigcup_{i=1}^\infty E_i\right) = \sum_{i=1}^\infty \lim_{n \to \infty} \mu_n(E_i) = \sum_{i=1}^\infty \mu(E_i).
\]

b) To show \( \mu \) is a measure on \((X, \Sigma)\):

- Since \( \mu_n(\emptyset) = 0 \) for all \( n \), then \( \mu(\emptyset) = \sum_{n=1}^\infty \mu_n(\emptyset) = 0 \).

- Given a sequence of pairwise disjoint measurable sets \( \{E_i\}_{i=1}^\infty \subset \Sigma \),

\[
\mu\left(\bigcup_{i=1}^\infty E_i\right) = \sum_{n=1}^\infty \mu_n\left(\bigcup_{i=1}^\infty E_i\right) = \sum_{n=1}^\infty \sum_{i=1}^\infty \mu_n(E_i) = \sum_{i=1}^\infty \sum_{n=1}^\infty \mu_n(E_i) = \sum_{i=1}^\infty \mu(E_i).
\]

\[\square\]

Problem 4.

Proof. When \( f \) is \( \Sigma \)-measurable, we have \( f^{-1}((a, \infty)) \in \Sigma \) for any \( a \in \mathbb{R} \). Since Borel set in \( \mathbb{R} \) are generated by open intervals. It suffices to show \( f^{-1}((a, b)) \in \Sigma \).

Let us recall some facts from set theory, which can be proved in one line by the definition of sets. Given a function \( f : X \rightarrow Y \), we have:

\[
f^{-1}(Y - B) = X - f^{-1}(B) \quad \text{for any} \ B \subset Y; \quad (1)
\]

\[
f^{-1}\left(\bigcap_{i \in I} B_i\right) = \bigcap_{i \in I} f^{-1}(B_i) \quad \text{for any} \ \{B_i\}_{i \in I} \quad \text{for any index set} \ I; \quad (2)
\]

\[
f^{-1}\left(\bigcup_{i \in I} B_i\right) = \bigcup_{i \in I} f^{-1}(B_i) \quad \text{for any} \ \{B_i\}_{i \in I} \quad \text{for any index set} \ I; \quad (3)
\]

Since \( f^{-1}((b, \infty)) \in \Sigma \), then by (1) we have

\[
f^{-1}((\infty, b]) = f^{-1}((b, \infty)^c) = X - f^{-1}((b, \infty)) \in \Sigma.
\]
Now we have both $f^{-1}((a, \infty)) \in \Sigma$ and $f^{-1}((\infty, b]) \in \Sigma$, by (2):
\[
f^{-1}((a, b]) = f^{-1}((a, \infty) \cap (-\infty, b]) = f^{-1}((a, \infty)) \cap f^{-1}((-\infty, b]) \in \Sigma.
\]
Note that $(a, b) = \bigcup_{i=1}^{\infty} (a, b - \frac{1}{i}]$, by (3):
\[
f^{-1}((a, b)) = f^{-1}(\bigcup_{i=1}^{\infty} (a, b - \frac{1}{i}]) = \bigcup_{i=1}^{\infty} f^{-1}((a, b - \frac{1}{i}]) \in \Sigma,
\]
which completes the proof. \(\square\)

**Problem 5.** To define the equivalence relationship on the space of measurable function $f : (X, \Sigma, \mu) \to (\mathbb{R}, \mathcal{B})$, we DO NOT require $\mu$ is complete.

**Proof.** The reflexive and symmetric properties are clearly satisfied. Now we prove the transitive property.

Let $f, g, h$ be measurable function on $(X, \Sigma, \mu)$ such that $f \sim g$ and $g \sim h$, i.e., there exists a measurable set $X_{fg}$ and $X_{gh}$ of measure zero, such that $f(x) = g(x)$ on $X_{fg}$, $f(x) \neq g(x)$ on $X_{fg}$, $g(x) = h(x)$ on $X_{gh}$, $g(x) \neq h(x)$ on $X_{gh}$.

As a result, $f(x) = h(x)$ on $(X \setminus X_{fg}) \cap (X \setminus X_{gh}) = X \setminus (X_{fg} \cup X_{gh})$. In the meanwhile, $f(x) \neq h(x)$ on some subset of $X_{fg} \cup X_{gh}$. Note that $X_{fg} \cup X_{gh} \in \Sigma$ and $\mu(X_{fg} \cup X_{gh}) \leq \mu(X_{fg}) + \mu(X_{gh}) = 0$.

Since $f, h$ are measurable function, then $f - h$ is measurable function as well, which yields that $(f - h)^{-1}(0) \in \Sigma$. By (1), $X_{fh} := (f - h)^{-1}(0^*) = X - (f - h)^{-1}(0) \in \Sigma$. Moreover, $X_{fh} \subset X_{fg} \cup X_{gh}$ which implies that $\mu(X_{fh}) < \mu(X_{fg} \cup X_{gh}) = 0$. \(\square\)

**Remark:** $f = g \mu$-almost everywhere is not necessary to be defined for measurable functions. For functions $f, g : X \to \mathbb{R}$, we say $f = g \mu$-almost everywhere if there exists a set $X_1$ with $\mu(X_1) = 0$ such that:
\[
f(x) = g(x) \text{ on } X \setminus X_1 \text{ and } f(x) \neq g(x) \text{ on } X_1.
\]
In this case $\{x : f(x) \neq g(x)\}$ is not necessary a measurable set (an element in $\Sigma$).

However, for measurable function $f, g$, the set $X_{fg} = \{x : f(x) \neq g(x)\}$ is always measurable (an element in $\Sigma$).

Furthermore, there is a theorem which might lead this mess as well:

$\mu$ is complete if and only if the following implication is valid: If $f$ is measurable and $f = g$ $\mu$-almost everywhere, then $g$ is measurable.

Furthermore, other form of definition to “almost everywhere” might increase the mess: We say $f = g \mu$-almost everywhere if there exists a set $X_1$ with $\mu(X_1) = 0$ such that: $\{x : f(x) \neq g(x)\} \subset X_1$.

**Problem 6.** Finite measure space has at most countable many atoms.

**Proof.** Consider the set: $X_n = \{x^* \in X : \frac{1}{n} \leq \mu(x^*) \leq \frac{1}{n-1}\}$. Suppose on contrary there are uncountable number of atoms, then the above set must be uncountable for at least one $n$. Let $X_N$ denote the uncountable set of atoms. Then
\[
\mu(X_N) \geq \sum_{x \in X_N} \frac{1}{N} \to \infty \quad \text{if } X_N \text{ is uncountable},
\]
which contradicts with $\mu(X_N)$ is finite from $\mu$ is finite measure. \(\square\)