In this homework, we regard $\mu$-integrable as $\int |f| \, d\mu = \int f^+ \, d\mu + f^- \, d\mu < \infty$.

**Problem 1.**

**Proof.** Since $f$ is $\mu$-integrable, then $|f|$ is $\mu$-integrable as well by definition.

Consider set $A_n = \{ x \in X : f(x) > n \}$ and define $f_n = |f|1_{A_n}$. Then $f_n \leq |f|$. Moreover, since $|f|$ is $\mu$-integrable, then $f_n \to 0$ almost everywhere as $n \to \infty$. By dominated convergence theorem, we have:

$$\lim_{n \to \infty} \int_X f_n \, d\mu = \int_X \lim_{n \to \infty} f_n \, d\mu = 0.$$

That is, for every $\varepsilon > 0$, there exists an $N > 0$ such that $\int_X f_n \, d\mu \leq \frac{\varepsilon}{2}$ for any $n \geq N$.

Problem 2. $L^1$ convergence does not imply pointwise convergence.

**Proof.** Let’s define $A_n$ as following measurable subsets of $[0, 1]$:

$$[0, 1], \quad [0, \frac{1}{2}], \quad \frac{1}{2}, 1], \quad [0, \frac{1}{3}], \quad \frac{1}{3}, \frac{2}{3}], \quad \frac{2}{3}, 1], \quad \cdots$$

Then we have

$$||1_{A_n} - 0||_1 = \mathcal{L}(A_n) \to 0.$$

However, for any $x \in [0, 1]$, $x$ belongs infinitely many sets in $\{A_n\}$, which implies that $1_{A_n} \not\to 0$ for every $x \in [0, 1]$.

**Problem 3.**

**Proof.**

1. Let $\varepsilon > 0$ and $M$ be such that $|g| \leq M$. Since $C^0_c(\mathbb{R})$ is dense in $L^1(\mathbb{R})$, that is, there exists a sequence $\{f_n\}$ in $C^0_c(\mathbb{R})$ such that $\int_{\mathbb{R}} |f_n - f| \, dx \to 0$ as $n \to \infty$. Assume for $n = N$, $\int_{\mathbb{R}} |f_N - f| \, dx \leq \frac{\varepsilon}{3M}$. Furthermore, since $f_N$ is continuous and compactly supported, then $f_N$ is uniformly continuous on $\text{supp}(f)$. That is, there exists $\delta > 0$ such that for every $|x - y| \leq \delta$, $|f_N(x) - f_N(y)| \leq \frac{\varepsilon}{3M\mathcal{L}(\text{supp}(f))}$. 


\[ |(f * g)(x) - (f * g)(y)| \]
\[ = |(f * g)(x) - (f_N * g)(x) + (f_N * g)(x) - (f_N * g)(y) + (f_N * g)(y) - (f * g)(y)| \]
\[ \leq \begin{align*}
I & |(f * g)(x) - (f_N * g)(x)| \\
II & |(f_N * g)(x) - (f_N * g)(y)| \\
III & |(f_N * g)(y) - (f * g)(y)| \\
\end{align*} \]

\[ I = \int \limits_{\mathbb{R}} |(f(x) - f_N(x))g(y)|dy \leq M \int \limits_{\mathbb{R}} |f(x) - f_N(x)||dy \leq M \frac{\varepsilon}{3} M = \frac{\varepsilon}{3}; \]

Based on the same reason, \( III \leq \frac{\varepsilon}{3}. \)

\[ II = \int \limits_{\mathbb{R}} |(f_N(x - z) - f_N(y - z))g(z)|dz \leq M \int \limits_{\mathbb{R}} |f_N(x - z) - f_N(y - z)|dz \]
\[ \leq M \frac{\varepsilon}{3M \mathcal{L}(\text{supp}(f))} \mathcal{L}(\text{supp}(f)) = \frac{\varepsilon}{3}. \]

Combine the above estimations, we get the result.

2. Recall Problem 1 in Homework 2, if \( f \) is uniformly continuous and integrable, then \( f(x) \to 0 \) as \( x \to \infty \). In part 1, we prove \( (f * g)(x) \) is uniformly continuous, once we prove \( (f * g)(x) \) is integrable, then we complete the result. By Tonelli’s theorem,

\[ \int \limits_{\mathbb{R}} |(f * g)(x)|dx = \int \limits_{\mathbb{R}} \int \limits_{\mathbb{R}} |f(x - y)g(y)|dydx \]
\[ = \int \limits_{\mathbb{R}} \int \limits_{\mathbb{R}} |f(x - y)g(y)|dxdy \]
\[ = \int \limits_{\mathbb{R}} |g(y)| \left( \int \limits_{\mathbb{R}} |f(x - y)|dx \right) dy \]
\[ = \int \limits_{\mathbb{R}} |g(y)| \left( \int \limits_{\mathbb{R}} |f(x)|dx \right) dy \]
\[ = \|g\|_1 \|f\|_1 < \infty. \]

\[ \square \]

Problem 4.

*Proof.* Since \( |f_n| \leq g \) and \( f_n \) converges to \( f \) pointwise, then \( |f| \leq g \). By definition, \( f_n \) and \( f \) belong to \( L^p(X) \). Furthermore,

\[ |f_n - f|^p \leq |g - (-g)|^p = (2g)^p. \]
Apply the dominated convergence theorem on the sequence \( h_n(x) = |f_n - f|^p \) and its upper bound \((2g)^p\), we have

\[
\lim_{n \to \infty} \|f_n - f\|_p = \lim_{n \to \infty} \left( \int_X |f_n - f|^p d\mu \right)^{1/p} = \left( \int_X \lim_{n \to \infty} |f_n - f|^p d\mu \right)^{1/p} = 0.
\]

**Problem 5.**

**Proof.** Given any \( L^p \) function \( f \), we may find a sequence of simple function \( f_n \) such that \( f_n \to f \) a.e. and \( |f_n| \leq |f| \). By definition, \( f_n \in L^p(X, \mu) \). Apply problem 4 for \( g = |f| \), we have \( f_n \to f \) in \( L^p(X, \mu) \). \( \square \)