Problem 1.

Proof. By direct computation,

\[
(\hat{f}_N)_n = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{T}} e^{-inx} \sum_{k=-N}^{N} e^{ikx} \, dx = \begin{cases} 
\frac{1}{\sqrt{2\pi}} & N \geq n; \\
0 & N < n.
\end{cases}
\]

Apply Theorem 7.5 in the textbook, for fixed \(n\), there exists sufficiently large \(N\), such that

\[
(\hat{f}_N \ast g)_n = \sqrt{2\pi} (\hat{f}_N)_n \hat{g}_n = \hat{g}_n.
\]

Apparently, the sequence \(\{ (\hat{f}_N \ast g)_n \}_N \) in \(l^2\) converges to \(\hat{g}_n\) with respect to \(\| \cdot \|_\ell^2\) norm, by Parseval’s identity, the sequence \(\{ f_N \ast g \}\) converges to \(g\) with respect to \(\| \cdot \|_{L^2}\) norm. \(\square\)

Problem 2.

Proof. We note that \(f_n = f_{n-1} \ast f\). We first compute \((\hat{f})_k\).

\[
(\hat{f})_k = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{T}} a(2\mathbb{1}_{[-\frac{\pi}{2}, \frac{\pi}{2}]}(x) - 1)e^{ikx} \, dx = \frac{a}{\sqrt{2\pi}} \left( \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{ikx} \, dx - \int_{-\frac{\pi}{2}}^{0} e^{ikx} \, dx - \int_{0}^{\frac{\pi}{2}} e^{ikx} \, dx \right)
\]

\[
= \begin{cases} 
\frac{a}{\sqrt{2\pi}} & k \neq 0; \\
4 \sin\left(\frac{k\pi}{2}\right) & k = 0.
\end{cases}
\]

As a result, by the mathematical induction, we find that

\[
(\hat{f}_n)_k = \begin{cases} 
\frac{4^n a^n \sin^n\left(\frac{k\pi}{2}\right)}{k^n \sqrt{2\pi}} & k \neq 0; \\
0 & k = 0.
\end{cases}
\]

For any \(n \geq 1\), to have \(f_n \in L^2(\mathbb{T})\), we need

\[
\sum_{k=-\infty}^{\infty} \frac{16^n a^{2n} \sin^{2n} \frac{k\pi}{2}}{2\pi k^{2n}} = \sum_{k=1}^{\infty} \frac{16^n a^{2n} \sin^{2n} \frac{k\pi}{2}}{\pi k^{2n}} < \infty.
\]

To pass the limit \(n \to \infty\), we need to require \(|a| \leq \frac{1}{4}\); otherwise the series diverge.

For \(|a| < \frac{1}{4}\), \((\hat{f}_n)_k \to 0\) as \(n \to \infty\) for each \(k\). Then \(f_n \to g \equiv 0\), which is excluded since \(g\) is nonzero.

For \(a = -\frac{1}{4}\), \((\hat{f}_n)_k = \frac{(-1)^n \sin^n\left(\frac{k\pi}{2}\right)}{\sqrt{2\pi} k^n}\) diverges as \(n \to \infty\) for each \(k\).

For \(a = \frac{1}{4}\), \((\hat{f}_n)_1 = \frac{1}{\sqrt{2\pi} 1^n} = \frac{1}{\sqrt{2\pi}}\) as \(n \to \infty\), \((\hat{f}_n)_{-1} = \frac{(-1)^n}{\sqrt{2\pi} (-1)^n} = \frac{1}{\sqrt{2\pi}}\) as \(n \to \infty\) and

\[
(\hat{f}_n)_k = \frac{\sin^n\left(\frac{k\pi}{2}\right)}{\sqrt{2\pi} k^n} \to 0 \text{ as } n \to \infty \text{ for all } |k| \geq 2.
\]

As the result, the limit function:

\[
g(x) = \frac{1}{\sqrt{2\pi}} \left( \frac{1}{\sqrt{2\pi}} e^{ix} + \frac{1}{\sqrt{2\pi}} e^{-ix} \right) = \frac{e^{ix} + e^{-ix}}{2\pi} = \frac{\cos(x)}{\pi}.
\]

\(\square\)
Problem 3.

Proof. Since \( g(x) = f(x - \tau) = f(x) \), apply Fourier transform on both sides, we have

\[
\hat{g}_n = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{T}} f(x - \tau) e^{-inx} dx = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{T}} f(y) e^{-in(y+\tau)} dy = e^{-in\tau} \hat{f}_n = \hat{n}.
\]

Since \( \frac{\tau}{n} \) is an irrational number, \( e^{-in\tau} = 1 \) only when \( n = 0 \). As a result, \( \hat{f}_n = 0 \) for all \( n \neq 0 \).

Therefore, \( f \) must be a constant function. \( \square \)

Problem 4.

Proof.

a) Note that \((\hat{S}_N)_k = \hat{f}_k\) for \( k \leq N \) and \((\hat{S}_N)_k = 0\) for \( k \geq N + 1 \). Now we compute the Fourier coefficients for \( D_N \).

First, we recall a trigonometric identity:

\[
\frac{\sin((N + \frac{1}{2})x)}{\sin(\frac{x}{2})} = 1 + 2 \sum_{n=1}^{N} \cos(nx) = \sum_{n=-N}^{N} e^{inx}.
\]

Then:

\[
(D_N)_k = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{T}} \frac{1}{2\pi} \frac{\sin((N + \frac{1}{2})x)}{\sin(\frac{x}{2})} e^{-ikx} dx = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{T}} \sum_{n=-N}^{N} e^{inx} e^{-ikx} dx.
\]

\[
= \begin{cases} 
\frac{1}{\sqrt{2\pi}} & k \leq N; \\
0 & k \geq N + 1.
\end{cases}
\]

Since \( \sqrt{2\pi}(D_N)_k \hat{f}_k = (\hat{S}_N)_k \), then \( S_N = D_N \ast f \).

b) \( N = 0 \) goes to the case in part a). For \( N \geq 1 \), we recall a trigonometric identity:

\[
\sum_{k=0}^{N} \frac{\sin((k + \frac{1}{2})x)}{\sin(\frac{x}{2})} = \left( \frac{\sin((N+1)x)/2}{\sin(x)/2} \right)^2.
\]

Then \( F_N(x) = \frac{1}{N+1} \sum_{k=0}^{N} D_N(x) \). Thus

\[
\sqrt{2\pi}(F_N)_k \hat{f}_k = \frac{\sqrt{2\pi}}{N+1} \sum_{k=0}^{N} (D_N)_k \hat{f}_k = \frac{1}{N+1} \sum_{k=0}^{N} (\hat{S}_N)_k = (\hat{T}_N)_k,
\]

which yields that \( T_N = F_N \ast f \).
c) \((D_N)\) are not approximation identities since they are not necessary nonnegative. From the book by Folland or Stein and Shakarchi, there exists continuous function \(f\) that \(S_N = D_N \ast f\) does not convergence to \(f\) pointwise, thus we do not expect further for uniform convergence.

Now we prove \((F_N)\) are indeed approximation identities. As a result, \(T_N = F_N \ast f\) converges to \(f\) uniformly, as a property of approximation identities.

(Nonnegative): It is obvious that \((F_N)\) are nonnegative.

(Unit mass):
\[
\int_{-\pi}^{\pi} \frac{1}{2\pi(N+1)} \left( \frac{\sin((N+\frac{1}{2})x)}{\sin(\frac{x}{2})} \right)^2 \, dx = \frac{1}{2\pi(N+1)} \sum_{k=0}^{N} \int_{-\pi}^{\pi} \frac{\sin((k+\frac{1}{2})x)}{\sin(\frac{x}{2})} \, dx
\]
\[
= \frac{1}{2\pi(N+1)} \sum_{k=0}^{N} \left( \sum_{j=-k}^{k} e^{ijx} \right) dx = \frac{1}{2\pi(N+1)} \sum_{k=0}^{N} \int_{-\pi}^{\pi} 1 \, dx = 1.
\]

(Mass stays around zero): Fix \(0 < \delta \leq \pi\),
\[
0 \leq \int_{0 \leq |x| \leq \pi} \frac{1}{2\pi(N+1)} \left( \frac{\sin((N+\frac{1}{2})x)}{\sin(\frac{x}{2})} \right)^2 \, dx = \frac{1}{\pi(N+1)} \int_{0 \leq x \leq \pi} \left( \frac{\sin((N+\frac{1}{2})x)}{\sin(\frac{x}{2})} \right)^2 \, dx
\]
\[
\leq \frac{1}{\pi(N+1)} \int_{0 \leq x \leq \pi} \left( \frac{1}{\sin(\frac{x}{2})} \right)^2 \, dx = \pi - \delta
\]

\[
= \frac{\pi - \delta}{\pi(N+1) \sin^2(\frac{\delta}{2})}.
\]

take \(N \to \infty\) on both sides, by sandwich’s theorem, we have
\[
\lim_{N \to \infty} \int_{0 \leq |x| \leq \pi} F_N(x) \, dx = 0.
\]

Problem 5.

Proof. a) Orthonormality:
\[
\langle e_m, e_n \rangle_{L^2[0,\pi]} = \int_{0}^{\pi} e_m \, e_n \, dx = \int_{0}^{\pi} \frac{2}{\pi} \sin(mx) \sin(nx) \, dx
\]
\[
= \frac{1}{\pi} \int_{0}^{\pi} \cos((m-n)x) - \cos((m+n)x) \, dx
\]
\[
= \begin{cases} 
  m = n : & \frac{1}{\pi} \int_{0}^{\pi} \cos(0) - \cos(2mx) \, dx \\
  m \neq n : & \frac{1}{\pi} \left( \frac{\sin((m-n)x)}{m-n} - \frac{\sin((m+n)x)}{m+n} \right) \bigg|_{0}^{\pi}
\end{cases}
\]
\[
= \begin{cases} 
  m = n : & 1 \\
  m \neq n : & 0
\end{cases}
\]
Completeness: Let \( f \in L^2([0, \pi]) \), we define \( \bar{f} \in L^2([-\pi, \pi]) \) by odd extension:

\[
\bar{f}(x) = \begin{cases} 
  f(x) & x > 0; \\
  0 & x = 0; \\
  -f(-x) & x < 0.
\end{cases}
\]

Since \( \bar{f} \in L^2([-\pi, \pi]) \), and \( \frac{1}{\sqrt{2\pi}} e^{inx} \) is an orthonormal basis of \( L^2([-\pi, \pi]) \), then there exists a sequence of coefficients, such that \( \bar{f}(x) = \sum_{n=-\infty}^{\infty} a_n e^{inx} \) holds with respect to \( \| \cdot \|_{L^2([-\pi, \pi])} \). Note that \( \bar{f} \) is an odd function, then the above right hand side is indeed in the form of \( \sum_{n=1}^{\infty} b_n \sin(n) \). Since

\[
\left\| f - \sum_{n=1}^{N} b_n \sin(n) \right\|_{L^2([0,\pi])} \leq \left\| \bar{f} - \sum_{n=1}^{N} b_n \sin(n) \right\|_{L^2([-\pi,\pi])} \rightarrow 0, \quad \text{as} \quad N \rightarrow \infty,
\]

which yields that \( (e_n) \) is complete orthonormal basis.

b) Orthonormality: For \( m \neq 0 \),

\[
\langle f_0, f_0 \rangle = \int_0^\pi \frac{1}{\pi} dx = 1;
\]

\[
\langle f_0, f_m \rangle = \int_0^\pi \frac{2}{\pi} \cos(mx)dx = \frac{2}{m\pi} \sin(mx)|_0^\pi = 0.
\]

For \( m, n \neq 0 \),

\[
\langle f_m, f_n \rangle = \int_0^\pi \frac{2}{\pi} \cos(mx) \cos(nx)dx = \frac{1}{\pi} \int_0^\pi (\cos((m+n)x) + \cos((m-n)x))dx
\]

\[
= \begin{cases} 
  m = n : & \frac{1}{\pi} \int_0^\pi \cos(2mx) + \cos(0)dx; \\
  m \neq n : & \frac{1}{\pi} \left( \frac{\sin((m+n)x)}{m+n} + \frac{\sin((m-n)x)}{m-n} \right)|_0^\pi.
\end{cases}
\]

\[
= \begin{cases} 
  m = n : & 1; \\
  m \neq n : & 0.
\end{cases}
\]

Completeness: Let \( f \in L^2([0, \pi]) \), we define \( \tilde{f} \in L^2([-\pi, \pi]) \) by even extension:

\[
\tilde{f}(x) = \begin{cases} 
  f(x) & x \geq 0; \\
  f(-x) & x < 0.
\end{cases}
\]

Since \( \tilde{f} \in L^2([-\pi, \pi]) \), and \( \frac{1}{\sqrt{2\pi}} e^{inx} \) is an orthonormal basis of \( L^2([-\pi, \pi]) \), then there exists a sequence of coefficients, such that \( \tilde{f}(x) = \sum_{n=-\infty}^{\infty} a_n e^{inx} \)
holds with respect to $\|\cdot\|_{L^2([-\pi,\pi])}$. Note that $\tilde{f}$ is an even function, then the above right hand side is indeed in the form of $\sum_{n=0}^{\infty} b_n \cos(nx)$. Since

$$\left\| f - \sum_{n=0}^{N} b_n \cos(nx) \right\|_{L^2([0,\pi])} \leq \left\| \tilde{f} - \sum_{n=0}^{N} b_n \cos(nx) \right\|_{L^2([-\pi,\pi])} \to 0, \quad \text{as} \quad N \to \infty,$$

which yields that $(f_n)$ is complete orthonormal basis.