BOUNDEDNESS CRITERIA FOR REAL QUIVERS OF RANK 3

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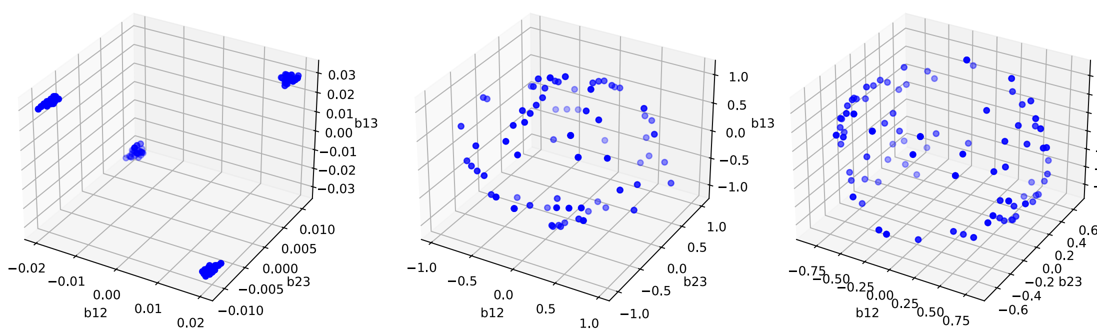
Abstract. We study the boundedness of a mutation class for quivers with real weights. The main result is a characterization of bounded mutation classes for real quivers of rank 3.

1. Introduction

The object of this note is to introduce and study the notion of boundedness for mutation classes of quivers with real weights. In short, a quiver mutation class is said to be bounded if the coefficients of any of its quivers are uniformly bounded. This is a subtler notion for quivers with real weights, as opposed to quivers with integer weights, as we show that there exists bounded mutation classes with infinitely many quivers in them. Our main contribution is a characterization of rank 3 quivers with bounded mutation class, leading to the classification of such quivers with a criterion that can be readily verified.

1.1. **Scientific context.**

Quivers and their mutations have acquired a prominent role in mathematics, especially since the introduction of cluster algebras by S. Fomin and A. Zelevinsky, cf. [10, 11, 12]. For a reference focused on the combinatorics of mutation, S. Fomin presented a number of known results and open problems on quiver mutations in his talk at OPAC 2022, cf. [6]. As witnessed by the number of basic questions that remain open, it might be fair to state that the combinatorics governing quiver mutations remain rather mysterious. Some recent efforts to understand the combinatorics of quiver mutations have been fruitful, e.g. studying long mutation cycles, cf. [3, 7], or constructing invariants of quiver mutation, cf. [2, 8, 14, 15].



$Q = (-0.02, -0.01, 0.03)$ 150 mutations

$Q = (-0.9, -0.22, 0.7106)$ 100 mutations

$Q = (-0.84, -0.26, 0.11)$ 125 mutations

Figure 1. Depiction of the $(p, q, r) \in \mathbb{R}^3$ coordinates for three quivers Q and a random sequence of mutations applied to them. In these cases, all three mutation classes $[Q]$ are bounded. The mutation sequences have been generated at random and are provided in the appendix.

Thinking of mutations of a quiver as a discrete group action, it is rather reasonable to wonder about the dynamical properties of quiver mutation. In turn, it is often productive to study discrete dynamical systems in relation to real continuous dynamical systems. From that perspective, we are naturally lead to study quivers with real weights and their orbits under mutations. We use the notation $Q = (p, q, r)$ for a quiver Q with real weights $p, q, r \in \mathbb{R}$. Here $|p|$ is the weight of the edge between vertices 1 to 2 and the sign of p indicates whether the arrow goes from vertex 1 to vertex 2, if p positive, or from vertex 2 to vertex 1, if p negative. Similarly, the weight q is for vertices 2 and 3 and r for 1 and 3. Visually, a quiver $Q = (p, q, r)$ with $p, q, r \in \mathbb{R}_{>0}$ and its exchange matrix B_Q are:

$$\begin{tikzcd} & 2 \arrow[dr, "q"] & \\ 1 \arrow[ur, "p"] \arrow[rr, "r"] & & 3 \end{tikzcd} \quad , \qquad B_Q = \begin{pmatrix} 0 & p & r \\ -p & 0 & q \\ -r & -q & 0 \end{pmatrix}.$$

To get a first sense, Figure 1 depicts the quivers obtained by applying three random mutation sequences $(\mu_{i_\ell} \circ \ldots \circ \mu_{i_1})(Q)$ to three randomly chosen quivers $Q = (p, q, r)$ with real weights.[1] Figure 2 depicts three different random sequences of mutations applied to the same quiver Q. Note that, independently, the study of quivers with real weights has also gained recent attention due to their connection to the metric geometry of surfaces, cf. e.g. [4, 5, 13] and references therein. In particular, mutation-finite quiver with real weights were beautifully classified in [5, Theorem A].

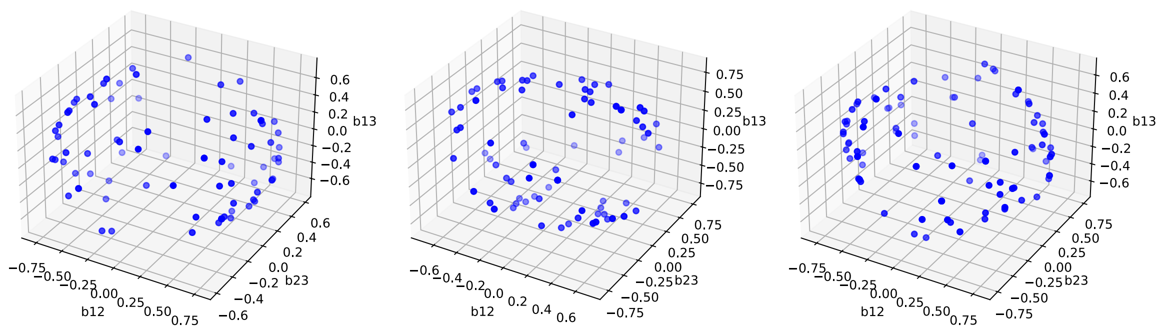


FIGURE 2. Depiction of the $(p, q, r) \in \mathbb{R}^3$ coordinates for quivers obtain by applying three different sequences to $Q = (-0.6, -0.43, 0.567)$. The mutation class $[Q]$ is bounded. The mutation sequences have been generated at random, cf. Section 3.

For quiver with real weights, the combinatorics of quiver mutation gains certain dynamical and geometric aspects that are not present for quivers with integer weights. The focus of this note is to begin the study of one such property: the notion of a mutation class being bounded. Namely, understanding whether the coefficients of a given quiver remain uniformly bounded under an arbitrarily long sequence of mutations. For instance, we shall prove that the quivers Q in Figure 1 and 2 have bounded mutation classes. For a quiver with integer weights, a quiver mutation class is bounded if and only if it is finite. For a quiver with real weights such simple characterization fails: there are mutation classes that are infinite, i.e. they contain infinitely many different quivers, and yet all quivers have their coefficients uniformly bounded.

A specific goal when studying a property of a quiver mutation class is to have a usable criterion to decide whether the mutation class of a given quiver possesses such property. In our case, the aim would be to be able to tell whether a given quiver with real weights has a bounded or unbounded mutation class. Since a mutation-finite quiver tautologically has a bounded mutation class, the focus is on deciding whether a mutation-infinite quiver has a bounded or unbounded mutation class. Our main result achieves this goal for quivers of rank 3.

1.2. **Main result.** Let Q be a quiver with real weights and $[Q]$ its mutation class. We refer to [9, Chapter 2] for background on quiver mutations and [13, Section 4] for the case of real weights. The classification of finite-mutation type quivers with real weights is established in [4, Thm. 5.9] for rank 3 and in [5, Theorem A] for arbitrary rank.

[1] The specific mutation sequences $(i_1, \ldots, i_\ell)$ for each of these three quivers are written in Section 3.2. We provide a code to generate such images in Section 3.1, with input the quiver Q and the desired length of a mutation sequence.

Given a quiver Q, we denote by $\|Q\| := \max\{|w| \in \mathbb{R} : w \text{ weight of } Q\}$ the maximum of the absolute values of the weights of Q. By definition, the norm of a quiver mutation class $[Q]$ is

$$\|[Q]\| := \sup_{Q\in[Q]} \|Q\|.$$

By definition, $[Q]$ is said to be bounded if and only if $\|[Q]\|$ is finite. If $[Q]$ is mutation-finite, then $\|[Q]\|$ is finite and thus $[Q]$ is bounded. The converse holds if Q has integer weights. In particular, there exist unbounded mutation-infinite classes $[Q]$. Interestingly, for quivers with real weights, we shall see that there are mutation-infinite classes $[Q]$ that are bounded.

Our exploration of boundedness focuses on the study of quivers of rank 3. As above, we write $Q = (p, q, r)$ to indicate that the quiver Q has weights $p, q, r \in \mathbb{R}$. The Markov constant $C(Q) \in \mathbb{R}$ of Q, as introduced in [1, Section 1], is a mutation invariant of rank 3 quivers whose value governs important aspects of the behavior of Q under mutation. To wit, [4, Section 4] illustrates how the threshold $C(Q) = 4$ marks a transition for the type of geometric realization of such mutation classes. The constant $C(Q)$ is defined as

$$C(Q) := \begin{cases} p^2 + q^2 + r^2 - |pqr| & \text{if } Q = (p, q, r) \text{ is cyclic,} \\ p^2 + q^2 + r^2 + |pqr| & \text{if } Q = (p, q, r) \text{ is acyclic.} \end{cases}$$

In addition to the notation $Q = (p, q, r)$, it is also convenient to refer to a quiver Q with real weights $p, q, r \in \mathbb{R}_{\geqslant 0}$, where here the quiver Q is given simply as a directed graph (with no weighted edges, so not a multi-graph), and $|p|, |q|, |r|$ denote the weights of those already oriented edges. In this sense, we can and do assume the weights $p, q, r \in \mathbb{R}_{\geqslant 0}$ are always non-negative by also specifying the underlying directed graph Q itself as part of the input.

The main result of this article is the following characterization:

Theorem 1.1. *Let Q be a quiver with real weights of rank 3 and $[Q]$ its mutation class. Then $[Q]$ is bounded if and only if one of the following holds:*

(1) $[Q]$ is of finite type.

(2) $[Q]$ is mutation acyclic and $C(Q) \leqslant 4$.

Theorem 1.1, as presented, might not appear to be an optimal characterization. Indeed, given a quiver Q it still requires being able to decide whether $[Q]$ is mutation acyclic. That said, we can conclude from Theorem 1.1 the following optimal characterization, complete and quickly verifiable:

Corollary 1.2. *Let Q be a quiver with real weights (p, q, r) with $p \geqslant q \geqslant r \geqslant 0$. Then*

$$[Q] \text{ is bounded} \iff p \leqslant 2 \text{ and } C(Q) \leqslant 4.$$

The strength of Corollary 1.2 is that it allows us to determine boundedness of a mutation class $[Q]$ for any given quiver Q by just using the given quiver itself, and not its mutations. Part of the non-trivial content of Theorem 1.1 is the assertion that a mutation-infinite quiver Q with $C(Q) > 4$ has an unbounded mutation class $[Q]$. Correspondingly, from the viewpoint of Corollary 1.2, that a quiver Q with $C(Q) > 4$ and real weights $p, q, r \leqslant 2$ has an unbounded mutation class $[Q]$. Namely, we show that it does not matter how small the given weights $p, q, r \in \mathbb{R}_{\geqslant 0}$ are: if $C(Q) > 4$ then $[Q]$ is unbounded. We prove these facts by exhibiting explicit mutation sequences that, under the hypothesis $C(Q) > 4$, result in quiver weights arbitrarily increasing, see e.g. the inequality in Equation (4).

Acknowledgments. We thank Melissa Sherman-Bennett for helpful comments on a first draft of the manuscript and Scott Neville for useful discussions. R.C. is supported by the NSF CAREER DMS-1942363, a Sloan Research Fellowship of the Alfred P. Sloan Foundation and a UC Davis College of L&S Dean's Fellowship.

2. Proof of main result

The proof of Theorem 1.1 and Corollary 1.2 are structured as follows. Section 2.1 establishes in Proposition 2.2 a first characterization of bounded mutation classes. Such characterization depends on the (non)existence of a cyclic quiver with particular weights: Section 2.2 proves two lemmas on the possible weights of rank 3 quiver depending on cyclicity and the Markov constant. Section 2.3

then uses the results from Section 2.1 and Section 2.2 to prove Theorem 1.1. Section 2.4 then proves Corollary 1.2 from Theorem 1.1. Throughout this section, Q is a rank 3 quiver with real weights.

2.1. **A preliminary characterization.** The goal of this subsection is to prove Proposition 2.2. The argument uses the following assertion:

Lemma 2.1. *Let $[Q]$ be an infinite mutation class. Then $[Q]$ contains a cyclic quiver.*

Proof. Since $[Q]$ is infinite, it must not contain any quiver with only one nonzero weight. Thus any quiver in $[Q]$ has at least two non-zero weights. Suppose that a quiver $Q \in [Q]$ is given with exactly two non-zero weights: if Q is cyclic we are done, and else we have an acyclic quiver Q with exactly two nonzero weights. Since the following three quivers are mutation equivalent, via the mutations μ_1 then μ_3 from left to right,

$$\begin{tikzcd} & 2 \arrow[dl] \arrow[dr] & \\ 1 & & 3 \end{tikzcd} \qquad \begin{tikzcd} & 2 \arrow[dr] & \\ 1 \arrow[ur] & & 3 \end{tikzcd} \qquad \begin{tikzcd} & 2 & \\ 1 \arrow[ur] & & 3 \arrow[ul] \end{tikzcd}$$

we can assume without loss of generality that the acyclic quiver Q is the middle quiver. In that case, mutation at vertex 2 yields the cyclic quiver:

$$\begin{tikzcd} & 2 \arrow[dr, "q"] & \\ 1 \arrow[ur, "p"] & & 3 \end{tikzcd} \quad \overset{\mu_2}{\longleftrightarrow} \quad \begin{tikzcd} & 2 \arrow[dl, "p"'] & \\ 1 \arrow[rr, "pq"] & & 3 \arrow[ul, "q"'] \end{tikzcd}$$

If Q had been an acyclic quiver with three nonzero weights then, up to relabeling vertices and reversing orientation, Q is mutation equivalent to

$$\begin{tikzcd} & 2 \arrow[dr, "q"] & \\ 1 \arrow[ur, "p"] \arrow[rr, "r"] & & 3 \end{tikzcd}$$

which is itself mutation equivalent, via μ_2, to the following cyclic quiver:

$$\begin{tikzcd} & 2 \arrow[dl, "p"'] & \\ 1 \arrow[rr, "pq+r"] & & 3 \arrow[ul, "q"'] \end{tikzcd}$$

Either case, we obtain a cyclic quiver in $[Q]$, as required. □

Here follows a preliminary classification of bounded mutation classes in rank 3:

Proposition 2.2. *Let Q be a rank 3 quiver with real weights. Then*

$$[Q] \text{ is unbounded} \iff \exists Q' \in [Q] \text{ with } Q' \text{ cyclic and one weight larger than } 2.$$

Proof. For ($\Longrightarrow$) we proceed as follows. Since $[Q]$ is unbounded, there exists a quiver $Q' \in [Q]$ with one weight greater than 2. If Q' is cyclic, then we are done. Suppose thus that Q' is acyclic. Since $[Q]$ is unbounded, Q' is mutation-infinite and the same proof of Lemma 2.1 implies that there must be a cyclic quiver Q'' with one weight greater than 2. Indeed, the two following two cases cover all such possibilities. If Q' is the following quiver on the left

$$\begin{tikzcd}
Q' & 2 \arrow[dr, "q"] & & & Q'' & 2 \arrow[dl, "p>2"'] & \\
1 \arrow[ur, "p>2"] & & 3 & \overset{\mu_2}{\longleftrightarrow} & 1 \arrow[rr, "pq"] & & 3 \arrow[ul, "q"']
\end{tikzcd}$$

then Q'' is chosen to be the quiver on its right. If Q' is instead the left quiver in

$$\begin{tikzcd}
Q' & 2 \arrow[dr, "q"] & & & Q'' & 2 \arrow[dl, "p>2"'] & \\
1 \arrow[ur, "p>2"] \arrow[rr, "r"] & & 3 & \overset{\mu_2}{\longleftrightarrow} & 1 \arrow[rr, "pq+r"] & & 3 \arrow[ul, "q"']
\end{tikzcd}$$

then Q'' is chosen to be the corresponding quiver to its right. This concludes ($\Longrightarrow$).

Let us prove ($\Longleftarrow$). For that, we assume that $Q' = (p_0, q_0, r_0) \in [Q]$ is the given cyclic quiver with weights satisfying $0 < r_0 \leqslant q_0 \leqslant p_0$ and $p_0 > 2$. Without loss of generality, we can assume that Q' is of the following form:

$$\begin{tikzcd}
Q' & 2 \arrow[dr, "q_0"] & \\
1 \arrow[ur, "p_0"] & & 3 \arrow[ll, "r_0"]
\end{tikzcd}$$

The goal is to show that $[Q]$ is unbounded. We prove that by iteratively mutating at vertices 1 and 2, which we momentarily show forces the weights of the quivers in the sequence to arbitrarily increase. Specifically, let us define the numbers $q_i, r_i \in \mathbb{R}_{\geqslant 0}$ recursively by

$$q_i := p_0 r_i - q_{i-1}, \quad r_i := p_0 q_{i-1} - r_{i-1}, \quad i > 0.$$

We claim that these numbers are the weights of the quivers appearing in the mutation sequence starting at Q' and alternately mutating at vertices 2 and 1. The beginning of the sequence is:

$$\begin{tikzcd}
Q' & 2 \arrow[dr, "q_0"] & & & & 2 \arrow[dl, "p_0"'] & & & & 2 \arrow[dr, "q_1=p_0r_1-q_0"] & \\
1 \arrow[ur, "p_0"] & & 3 \arrow[ll, "r_0"] & \overset{\mu_2}{\longleftrightarrow} & 1 \arrow[rr, "r_1=p_0q_0-r_0"] & & 3 \arrow[ul, "q_0"'] & \overset{\mu_1}{\longleftrightarrow} & 1 \arrow[ur, "p_0"] & & 3 \arrow[ll, "r_1"]
\end{tikzcd}$$

To verify this claim, we need to check that the arrows q_i, r_i are indeed going in the direction that retains cyclicity, i.e. $q_i, r_i > 0$ for all $i > 0$. In addition, we also assert that the inequalities $q_i > r_i > q_{i-1}$ hold for all $i > 0$. We prove such inequalities

$$q_i > r_i > q_{i-1} > 0 \tag{1}$$

by induction on $i \in \mathbb{N}$. The base case is $i = 1$: since $r_0 \leqslant q_0$ and $p_0 > 2$, we must have

$$r_1 = p_0 q_0 - r_0 > 2q_0 - q_0 = q_0 > 0, \text{ and } q_1 = p_0 r_1 - q_0 > 2r_1 - r_1 = r_1 > 0.$$

For the inductive step, we assume that $q_k > r_k > q_{k-1} > 0$ for some $k \in \mathbb{N}$. Then it follows that

$$r_{k+1} = p_0 q_k - r_k > 2q_k - q_k = q_k > 0, \text{ and } q_{k+1} = p_0 r_{k+1} - q_k > 2r_{k+1} - r_{k+1} = r_{k+1} > 0,$$

which proves Equation (1). To conclude that $[Q]$ is unbounded, it suffices to show that these weights q_i, r_i increase as we iterate the mutations at 2 and 1, i.e. as $i \to \infty$. This a consequence of the following:

Claim 1. $\lim\limits_{i\to\infty} q_i = \infty$.

Proof. Consider the quantity $\mathbf{p} := p_0^2 - p_0 - 1$. The hypothesis $p_0 > 2$ implies $\mathbf{p} > 1$ and thus $\lim\limits_{i\to\infty} \mathbf{p}^i = \infty$. We claim that the sequence of weights (q_i) satisfies

$$q_i \geqslant \mathbf{p}^i q_0, \quad \text{for all } i > 0. \tag{2}$$

Equation (2) can be established by induction on i, as follows. The base case is $i = 1$, and since $r_0 \leqslant q_0$, we have

$$q_1 = p_0 r_1 - q_0 = p_0^2 q_0 - p_0 r_0 - q_0 \geqslant p_0^2 q_0 - p_0 q_0 - q_0 = (p_0^2 - p_0 - 1) q_0 = \mathbf{p}^1 q_0.$$

For the inductive step, we assume that $q_k \geqslant \mathbf{p}^k q_0$ for some $k \in \mathbb{N}$. Given that we have shown previously that $r_k < q_k$, it follows that

$$q_{k+1} = p_0 r_{k+1} - q_k = p_0^2 q_k - p_0 r_k - q_k > p_0^2 q_k - p_0 q_k - q_k = (p_0^2 - p_0 - 1) q_k \geqslant \mathbf{p} \cdot \mathbf{p}^k q_0 = \mathbf{p}^{k+1} q_0,$$

which implies Equation (2). Since $\lim\limits_{i\to\infty} \mathbf{p}^i = \infty$, it follows that

$$\lim_{i\to\infty} q_i > q_0 \cdot \lim_{i\to\infty} \mathbf{p}^i = q_0 \cdot \infty = \infty,$$

as required. □

The fact that $[Q]$ is unbounded now follows from Claim 1, as we constructed a sequence of quivers in $[Q]$ whose coefficients increase arbitrarily as we iteratively mutate at the vertices 2 and 1. □

2.2. **Two quick lemmas.** A caveat of Proposition 2.2 is that it is in generally challenging to determine the existence or non-existence of a cyclic quiver with one weight larger than 2 in a given mutation class $[Q]$. This makes Proposition 2.2 difficult to use in practice. Hence, we are motivated to further explore the behavior of weights of rank 3 quivers, which will lead to Theorem 1.1, improving Proposition 2.2. The two necessary lemmas that we shall use in the proof Theorem 1.1 read as follows:

Lemma 2.3. *Let $Q = (p, q, r)$ be a cyclic quiver with real weights $0 < r \leqslant q \leqslant p$. If $C(p, q, r) > 4$, then $p > \sqrt{2}$.*

Proof. Let us argue by contradiction, assuming that $p \leqslant \sqrt{2}$. Since Q is cyclic, $C(Q)$ reads

$$C(p, q, r) = p^2 + q^2 + r^2 - pqr.$$

Considered as a real smooth function of p, q, r, we have $\partial_p C = 2p - qr > 0$ since $q, r \leqslant p$ and we are assuming $p \leqslant \sqrt{2}$. Thus, $C(p, q, r)$ is a strictly increasing function of p and it attains its maximum when p is maximized, that is, when $p = \sqrt{2}$. If we write $f(q, r) := C(\sqrt{2}, q, r) = 2 + q^2 + r^2 - \sqrt{2}qr$, this implies

$$C(p, q, r) \leqslant f(q, r).$$

Note that $\partial_q f = 2q - \sqrt{2}r > 0$ because $2 > \sqrt{2}$ and $q \geqslant r$. Thus $f(q, r)$ is a strictly increasing function of q and it attains its maximum when q is maximized. Since we have $q \leqslant p \leqslant \sqrt{2}$, we can write $g(r) := C(\sqrt{2}, \sqrt{2}, r) = 2 + 2 + r^2 - 2r = 4 + r^2 - 2r$ and the following inequality will hold

$$C(p, q, r) \leqslant f(q, r) \leqslant g(r).$$

Then $g(r)$ is maximize in the same way: $\partial_r g = 2r - 2 > 0$ if $r > 1$, and $\partial_r g < 0$ if $r < 1$. Hence, $g(r)$ is increasing for $r > 1$ and decreasing for $r < 1$. Since $0 < r \leqslant q \leqslant p \leqslant \sqrt{2}$, in the interval $[0, \sqrt{2}]$ the function $g(r)$ attains its global maximum at the boundary, i.e. either at $r = \sqrt{2}$ or $r = 0$. Note that $g(\sqrt{2}) = 4 + 2 - 2\sqrt{2} < 4 = g(0)$, thus we have that $g(r) < g(0) = 4$ in the interval $(0, \sqrt{2}]$. Hence,

$$C(p, q, r) \leqslant f(q, r) \leqslant g(r) < 4.$$

This contradicts the hypothesis $C(Q) > 4$, and thus we must have had that $p > \sqrt{2}$, as required. □

Lemma 2.3 gives a lower bound for p whenever $C(p, q, r) > 4$: this has an important role in the proof of the upcoming Lemma 2.4, and ultimately Theorem 1.1. Appropriately used, this next lemma will provide a way to construct a sequence of mutations with strictly increasing weights:

Lemma 2.4. *Let $Q = (p, q, r)$ be a cyclic quiver with weights $0 < r \leqslant q \leqslant p$. If $C(p, q, r) > 4$, then*

$$pq - r > \frac{C(Q)}{4} q.$$

Proof. Consider the smooth function

$$f : \mathbb{R}^3_{\geqslant 0} \longrightarrow \mathbb{R}, \quad f(p, q, r) := (pq - r) - \frac{C(Q)}{4} q,$$

which we need to show is positive. Let $g(p, q, r) := \partial_p f = q - q \cdot \frac{1}{4}(2p - qr)$. Since $p < 2$, we have $2p - qr < 4 - qr < 4$ and hence $g(p, q, r) > 0$ is positive and f is increasing as a function of p. By Lemma 2.3, the hypothesis $C(p, q, r) > 4$ implies the inequality $\sqrt{2} < p < 2$. Altogether, for any $p \in (\sqrt{2}, 2)$, $f(p, q, r) > f(\sqrt{2}, q, r)$ and so it suffices to show $f(\sqrt{2}, q, r) > 0$. By Lemma 2.3 again, if $p = \sqrt{2}$ then $C(p, q, r) \leqslant 4$, and thus we inspect this extremal case of $f(\sqrt{2}, q, r)$, for q, r satisfying $0 \leqslant r \leqslant q \leqslant \sqrt{2}$, and $C(\sqrt{2}, q, r) = 4$. Since $C(\sqrt{2}, q, r) = 2 + q^2 + r^2 - \sqrt{2}qr$, we have that

$$\partial_q C(\sqrt{2}, q, r) = 2q - \sqrt{2}r > 0.$$

Thus, $C(\sqrt{2}, q, r)$ is increasing as a function of q, hence it attains its maximum at $q = \sqrt{2}$ with value $C(\sqrt{2}, \sqrt{2}, r) = 4 + r^2 - 2r = 4 + r(r - 2)$. Since $r \in (0, \sqrt{2}]$, $r(r - 2) < 0$ and so $C(p, q, r) < C(\sqrt{2}, \sqrt{2}, 0) = 4$. In particular, the only remaining extremal point $(\sqrt{2}, q, r)$ satisfying $C(\sqrt{2}, q, r) = 4$ is $(\sqrt{2}, \sqrt{2}, 0)$, and the value of f is

$$f(\sqrt{2}, \sqrt{2}, 0) = 2 - \sqrt{2} \cdot \frac{4}{4} = 2 - \sqrt{2} > 0.$$

In conclusion, under the hypothesis $C(p, q, r) > 4$, we indeed have $f(p, q, r) > f(\sqrt{2}, \sqrt{2}, 0) > 0$. □

2.3. **Proof of Theorem 1.1.** Let us first prove the implication ($\Longleftarrow$). If $[Q]$ is of finite mutation type, then $[Q]$ is tautologically bounded. Therefore, we assume $[Q]$ is infinite, mutation acyclic, and $C(Q) \leqslant 4$. We will now show that $[Q]$ is bounded by $\sqrt{C(Q)}$.

Let $(\alpha, \beta, \gamma) \in [Q]$ be an acyclic quiver. Then $\alpha^2 + \beta^2 + \gamma^2 + \alpha\beta\gamma = C(\alpha, \beta, \gamma) = C(Q)$, and it follows that $\alpha^2, \beta^2, \gamma^2 \leqslant C(Q)$, hence $\alpha, \beta, \gamma \leqslant \sqrt{C(Q)}$. Thus the upper bound $\sqrt{C(Q)}$ holds for acyclic quivers. Let $Q = (p, q, r) \in [Q]$ be cyclic, and let $d(p, q, r)$ be the minimum number of mutations required to transform $Q = (p, q, r)$ into an acyclic quiver. Such d is finite because of the hypothesis that $[Q]$ is mutation acyclic. We show that $\sqrt{C(Q)}$ is an upper bound by induction on d.

The base case is $d = 1$, i.e. $Q = (p, q, r)$ is one mutation away from an acyclic quiver $Q' = (\alpha, \beta, \gamma)$. Since Q' is acyclic, we have already shown $\alpha, \beta, \gamma \leqslant \sqrt{C(Q)}$ and note that a single mutation to (p, q, r) preserves at least two weights. So, without loss of generality, we assume that $q, r \leqslant \sqrt{C(Q)}$ and we need to show $p \leqslant \sqrt{C(Q)}$. We argue by contradiction, so we assume that $p > \sqrt{C(Q)}$. We have $C(Q) = p^2 + q^2 + r^2 - pqr$: we now want to get a lower bound for $C(p, q, r)$, and ultimately for p. Let $f(p) := p^2 + q^2 + r^2 - pqr$ be considered as a smooth real function of p, then

$$f'(p) = 2p - qr > 0 \iff p > \frac{qr}{2}. \tag{3}$$

By assumption, $C(Q) \leqslant 4$ and thus $\sqrt{C(Q)} \leqslant 2$. Since we also have $q, r \leqslant \sqrt{C(Q)}$, we obtain

$$\frac{qr}{2} \leqslant \frac{C(Q)}{2} \leqslant \frac{C(Q)}{\sqrt{C(Q)}} = \sqrt{C(Q)} < p.$$

This implies 3, that is, $f(p)$ is strictly increasing and thus $f(p)$ is minimized when p is minimized. We are assuming that $p > \sqrt{C(Q)}$ and therefore we have the sequence of implications:

$$\begin{aligned} f(p) = p^2 + q^2 + r^2 - pqr &> C(Q) + q^2 + r^2 - \sqrt{C(Q)}qr \\ \implies 0 &> q^2 + r^2 - 2qr + 2qr - \sqrt{C(Q)}qr \\ \implies 0 &> (q-r)^2 + \left(2 - \sqrt{C(Q)}\right) qr. \end{aligned}$$

Since $(q-r)^2 \geqslant 0$, and $\sqrt{C(Q)} \leqslant 2$ implies $\left(2 - \sqrt{C(Q)}\right) qr \geqslant 0$, the last inequality implies

$$0 > (q-r)^2 + \left(2 - \sqrt{C(Q)}\right) qr \geqslant 0$$

which is a contradiction. Hence, $p \leqslant \sqrt{C(Q)}$ as required, concluding the base case.

For the induction step, we assume the statement to be true for any d with $1 \leqslant d \leqslant n \in \mathbb{N}$, and let $Q' = (p', q', r') \in [Q]$ be a cyclic quiver with $d(p', q', r') = n + 1$. Let $(\alpha, \beta, \gamma) \in [Q]$ be a cyclic quiver with $d(\alpha, \beta, \gamma) = n$ that is one mutation from (p', q', r'). By the induction hypothesis, we have $\alpha, \beta, \gamma \leqslant \sqrt{C(Q)}$. Since one mutation preserves at least two weights, we assume without loss of generality that $q', r' \leqslant \sqrt{C(Q)}$. Then, applying the same argument as in the base case, we readily conclude $p' \leqslant \sqrt{C(Q)}$.

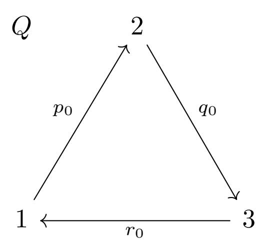
To summarize, we have shown that all acyclic and cyclic quivers in $[Q]$ are bounded by $\sqrt{C(Q)}$. This concludes the proof of the implication $(\Longleftarrow)$.

Let us prove the implication $(\Longrightarrow)$, which we show by contradiction. For that, we consider two cases, the most interesting being the second case:

Case 1. Suppose that $[Q]$ *is bounded, infinite, and mutation cyclic.* Since $[Q]$ is bounded and mutation cyclic, Proposition 2.2 implies that any quiver $Q' = (p, q, r) \in [Q]$ with $0 < r \leqslant q \leqslant p$ must have $p, q, r \leqslant 2$. If $p = q = r = 2$, then Q' is the Markov quiver, which is of finite mutation type. Thus, $[Q]$ being infinite, we must have $r < 2$. By [4, Lemma 3.3], it follows that $[Q]$ is mutation acyclic, which is a contradiction. Therefore, we conclude that $[Q]$ bounded and infinite implies mutation acyclic, establishing the mutation acyclic part of Theorem 1.1.(2) in this case.

Case 2. Suppose that $[Q]$ *is bounded, infinite, and* $C(Q) > 4$. The intuitive idea is that given any cyclic quiver $Q \in [Q]$, we will manage to use the inequality $C(Q) > 4$ to produce an arbitrarily long sequence of mutations such that the weights increase arbitrarily. Such mutation sequence depends on the starting quiver $Q \in [Q]$ to which we apply the procedure, which makes this argument finer than that in the proof of Proposition 2.2. This is done as follows.

Let $Q = (p_0, q_0, r_0) \in [Q]$ be a cyclic quiver with $0 < r_0 \leqslant q_0 \leqslant p_0 \leqslant 2$, and we can assume Q is



By mutating Q at vertex 2 we obtain the following quiver on the right:

$$
\begin{tikzcd}
Q & 2 \arrow[dr, "q_0"] & & & & \mu_2(Q) & 2 \arrow[dl, "p_0"'] & \\
1 \arrow[ur, "p_0"] & & 3 \arrow[ll, "r_0"] & {} \arrow[r, leftrightarrow, "\mu_2"] & {} & 1 \arrow[rr, "p_0q_0-r_0"] & & 3 \arrow[ul, "q_0"']
\end{tikzcd}
$$

For the new weights $p_0q_0 - r_0$, we must have either

$$p_0 \geqslant p_0q_0 - r_0 \geqslant q_0 \text{ or } p_0q_0 - r_0 \geqslant p_0 \geqslant q_0.$$

To describe the mutation sequence, we define $\mu^1(Q) = (p_1, q_1, r_1) := \mu_2(p_0, q_0, r_0)$ so that $0 < r_1 \leqslant q_1 \leqslant p_1 \leqslant 2$. In this notation we have $p_1 \geqslant p_0q_0 - r_0 > \frac{C(Q)}{4}q_0$. Now, recursively define

$$\mu^i(Q) = (p_i, q_i, r_i) := \mu^{i-1}(\mu^1(Q))$$

with the same properties that $0 < r_i \leqslant q_i \leqslant p_i \leqslant 2$, and $p_i \geqslant p_{i-1}q_{i-1} - r_{i-1}$. In words, at any step of this mutation sequence, we always choose to mutate at the vertex opposite to the arrow with the smallest weight. The quivers $\mu^i(Q) = (p_i, q_i, r_i)$ exactly record the quivers appearing in such mutation sequence. It suffices to show that at least one of such coefficients increases arbitrarily. Specifically, we will now show that $\lim\limits_{i\to\infty} \max(\mu^i(Q)) = \infty$. To prove this, it suffices to show $\lim\limits_{i\to\infty} p_iq_i - r_i = \infty$. This limit itself is a consequence of the following fact:

Claim. For all $i \geqslant 0$, the following inequality holds:

$$p_iq_i - r_i > \left(\frac{C(Q)}{4}\right)^{\left\lfloor \frac{i+2}{2} \right\rfloor} q_0. \tag{4}$$

Proof. Let us prove this inequality by induction on i. For the base case, consider $i = 0$ and $i = 1$. For $i = 0$, Lemma 2.4 implies that

$$p_0q_0 - r_0 > \frac{C(Q)}{4} \cdot q_0 = \left(\frac{C(Q)}{4}\right)^{\left\lfloor \frac{0+2}{2} \right\rfloor} q_0.$$

For $i = 1$, there are two cases to consider:

(1) If $\mu^1(Q) = (p_0, p_0q_0 - r_0, q_0) = (p_1, q_1, r_1)$, then Lemma 2.4 implies

$$p_1q_1 - r_1 > \frac{C(Q)}{4}q_1 = \frac{C(Q)}{4}(p_0q_0 - r_0) > \left(\frac{C(Q)}{4}\right)^2 q_0 > \left(\frac{C(Q)}{4}\right)^{\left\lfloor \frac{1+2}{2} \right\rfloor} q_0.$$

(2) If $\mu^1(Q) = (p_0q_0 - r_0, p_0, q_0) = (p_1, q_1, r_1)$, then we directly have

$$p_1q_1 - r_1 > \frac{C(Q)}{4}p_0 > \frac{C(Q)}{4}q_0 = \left(\frac{C(Q)}{4}\right)^{\left\lfloor \frac{1+2}{2} \right\rfloor} q_0.$$

This concludes the base cases $i = 0, 1$. For the induction step, we assume that

$$p_iq_i - r_i > \left(\frac{C(Q)}{4}\right)^{\left\lfloor \frac{i+2}{2} \right\rfloor} q_0$$

holds for all $0 \leqslant i \leqslant n$ and want to show it holds for $i = n+1$. The induction hypothesis for $i = n$ is

$$p_nq_n - r_n > \left(\frac{C(Q)}{4}\right)^{\left\lfloor \frac{n+2}{2} \right\rfloor} q_0$$

with the inequalities $p_n \geqslant p_{n-1}q_{n-1} - r_{n-1} > \left(\frac{C(Q)}{4}\right)^{\left\lfloor \frac{n+1}{2} \right\rfloor} q_0$. There are two cases for the next quiver $\mu^{n+1}(Q)$ in the mutation sequence:

(1) If $\mu^{n+1}(Q) = (p_n, p_nq_n - r_n, q_n) = (p_{n+1}, q_{n+1}, r_{n+1})$, then Lemma 2.4 applies to give

$$p_{n+1}q_{n+1} - r_{n+1} > \frac{C(Q)}{4}q_{n+1} = \frac{C(Q)}{4}(p_nq_n - r_n) > \left(\frac{C(Q)}{4}\right)^{\lfloor \frac{n+2}{2} \rfloor + 1} q_0 =$$
$$= \left(\frac{C(Q)}{4}\right)^{\lfloor \frac{n+4}{2} \rfloor} q_0 \geqslant \left(\frac{C(Q)}{4}\right)^{\lfloor \frac{(n+1)+2}{2} \rfloor} q_0$$

(2) If $\mu^{n+1}(Q) = (p_nq_n - r_n, p_n, q_n) = (p_{n+1}, q_{n+1}, r_{n+1})$, then we use Lemma 2.4 as follows:

$$\begin{aligned} p_{n+1}q_{n+1} - r_{n+1} &> \frac{C(Q)}{4}q_{n+1} \\ &= \frac{C(Q)}{4}p_n \\ &\geqslant \frac{C(Q)}{4}(p_{n-1}q_{n-1} - r_{n-1}) \\ &> \left(\frac{C(Q)}{4}\right)^{\lfloor \frac{n+1}{2} \rfloor + 1} q_0 \\ &\geqslant \left(\frac{C(Q)}{4}\right)^{\lfloor \frac{(n+1)+2}{2} \rfloor} q_0, \end{aligned}$$

which is indeed yields Equation (4) for $i = n + 1$. This concludes the claim. □

The claim implies that the desired limit satisfies

$$\lim_{i\to\infty} \max(\mu^i(Q)) \geqslant \lim_{i\to\infty} p_iq_i - r_i > \lim_{i\to\infty} \left(\frac{C(Q)}{4}\right)^{\lfloor \frac{i+2}{2} \rfloor} q_0 = \infty,$$

where we used the hypothesis $C(Q) > 4$ to conclude the last equality. Hence, $[Q]$ is unbounded, which is a contradiction. Therefore, it must have been that $C(Q) < 4$ if Q were bounded and infinite. □

2.4. **Proof of Corollary 1.2.** Let us prove ($\Longrightarrow$) by establishing the contrapositive. First, let us assume $p > 2$ and we want to conclude $[Q]$ is unbounded. If Q is cyclic, then Theorem 2.2 implies $[Q]$ is unbounded. We thus assume the given quiver Q is acyclic. Following the proof of Lemma 2.1, there exists a mutation sequence μ so that $\mu(Q)$ is cyclic and $\mathrm{Max}(\mu(Q)) \geqslant \mathrm{Max}(Q)$. Therefore $\mu(Q)$ is a cyclic quiver with $p > 2$ and Theorem 2.2 implies $[Q] = [\mu(Q)]$ is unbounded.

Second, let us assume $C(Q) > 4$ and we want to conclude $[Q]$ is unbounded. By contradiction, suppose $[Q]$ is bounded. By Theorem 1.1, it suffices to show that if $[Q]$ is of finite type, then $C(Q) \leqslant 4$. By [4, Theorem 6.11], Q must be mutation equivalent to one of the following quivers:

$$(2, 2, 2), (2, 2\cos(\pi/n), 2\cos(\pi/n)) \text{ for } n \in \mathbb{Z}_+, (1, 1, 0), (1, \sqrt{2}, 0),$$
$$(1, 2\cos(\pi/5), 0), (2\cos(\pi/5), 2\cos(2\pi/5), 0), (1, 2\cos(2\pi/5), 0).$$

It is then readily verified that all of these have Markov Constant $C(Q) \leqslant 4$. Hence $[Q]$ could not have been finite type if $C(Q) > 4$, and thus not bounded.

Let us prove ($\Longleftarrow$) directly from Theorem 1.1. We are assuming $p \leqslant 2$ and $C(Q) \leqslant 4$ and want to conclude $[Q]$ is bounded. First, we consider the case where the inequality $p < 2$ is strict. By [4, Lemma 3.3], such mutation class $[Q]$ must be mutation acyclic: since we already have $C(Q) \leqslant 4$, Theorem 1.1 implies that $[Q]$ is bounded. Second, consider the case of equality $p = 2$. If Q is acyclic, then $[Q]$ is mutation acyclic and Theorem 1.1 again proves $[Q]$ is bounded. Therefore, we assume Q is cyclic. In this case, if $p = q = r = 2$, then Q is the Markov quiver, and hence $[Q]$ is bounded. So it remains to check the cases where $r < 2$. By [4, Corollary 4.8], $[Q]$ must then be mutation acyclic and Theorem 1.1 implies that $[Q]$ is bounded. □

2.5. **Final remarks and questions.** A few comments on the results above:

(1) On the hypotheses of Corollary 1.2, it is not true in general that $C(Q) \leqslant 4$ implies $[Q]$ is bounded. For example, the cyclic quiver $Q = (3, 3, 3)$ has Markov constant $C(3, 3, 3) = 0 \leqslant 4$ and it is mutation cyclic and unbounded. Hence, the additional requirement $p \leqslant 2$ is necessary.

(2) The simpler part of the proof of Theorem 1.1 established that $\sqrt{C(Q)}$ is an actual bound, but for the statement it suffices to argue there exists a bound, without giving a particularly sharp one. Here is a more intuitive argument that a mutation acyclic class $[Q]$ with $C(Q) < 4$ must necessarily be bounded, which does nevertheless not provide such a sharp upper bound. We consider the weights of a quiver $Q = (p, q, r)$ as a point in $\mathbb{R}^3$ and think of mutations as a continuous action of $G = \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$ on $\mathbb{R}^3$. Here $G = \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$ is the quotient of the free group $\mathbb{F}_3 = \langle \mu_1, \mu_2, \mu_3 \rangle$ by the relations $\mu_i^2 = 1$.[2] The constraint $C(Q) < 4$ implies that the real weights of Q must be in the subset

$$X := \{(x, y, z) | x^2 + y^2 + z^2 + |xyz| \leqslant 4\} \subset \mathbb{R}^3, \quad \text{if } Q \text{ is acyclic,}$$
$$Y := \{(x, y, z) | x^2 + y^2 + z^2 - |xyz| \leqslant 4\} \subset \mathbb{R}^3, \quad \text{if } Q \text{ is cyclic.}$$

Note that X is bounded as a subset of $\mathbb{R}^3$. Thus, all acyclic quivers in $[Q]$ will remain in X, which is bounded. For the cyclic quivers, we note that Y contains 8 singular points $S = \{(\pm 2, \pm 2, \pm 2)\}$ such that $Y \backslash S$ has a unique bounded component. Thus, the fact that any acyclic quiver lies in such bounded connected component and continuity of the mutation G-action imply that any cyclic quiver must actually remain in that bounded component. This implies that $[Q]$ is bounded under these hypothesis.

(3) As obvious as it is, we record this goal: *Find a complete and verifiable characterization of bounded mutation classes for quivers of higher rank.* That is, prove a result such as Theorem 1.1 and Corollary 1.2 (or better) for quivers of rank 4 and above.

(4) It would be interesting to understand the dynamical properties of quiver mutations for real weights in rank 3 and beyond. That is, understanding the weights of quivers as points in $\mathbb{R}^{\binom{n}{2}}$, to characterize the distribution of points of a given orbit, either quantitatively or qualitatively. For instance, if the orbit is bounded, understanding what are the possible limit sets, e.g. whether they are dense in some positive-dimensional subset of $\mathbb{R}^{\binom{n}{2}}$. If the orbit is unbounded, it would be interesting to establish quantitative growth estimates, and study whether there are directions or cones of divergence.

Note that already in rank 3, it would be interesting to understand properties of mutation orbits beyond boundedness, e.g. density in the given two level sets (cyclic and acyclic) for a fixed value of the Markov constant, ergodicity, or any other type of measure-related properties, see e.g. Figures 1 and 2 above. The code in Section 3.1 can maybe be helpful to develop intuition.

3. Appendix: SageMath Code and details for figures

The following two subsections contain some of the experimental data displayed in the introduction and SageMath code that produces many such examples. These examples and the code are *not* logically needed for the mathematical results of the article. That said, we found running this code helpful to gain intuition on the dynamics of real quiver mutation, in particular towards guessing the statement of what ended up being Theorem 1.1 and the key inequality in Equation 4.

3.1. **SageMath code to generate and plot random mutation sequences of a quiver.** The main line of commands to produce a random sequence of mutations and their plot uses the following series of functions. The first function inputs a skew-symmetric matrix $B \in M_3(\mathbb{Q})$ and an index $k \in \{1, 2, 3\}$ and outputs the mutated exchange matrix $\mu_k(B)$, as follows:

```
def mutate_matrix_rational_mpl(B, k):

    if B.dimensions() != (3, 3):
        raise ValueError("Input matrix B must be a 3x3 matrix.")
    if not B ==-B.transpose():
        raise ValueError("Input matrix B must be skew-symmetric.")
    if k not in [1, 2, 3]:
        raise ValueError("Mutation index k must be 1, 2, or 3.")
    B_prime = matrix(QQ, 3, 3)
    for i in range(3):
        for j in range(3):
            if i == k- 1 or j == k- 1:
                B_prime[i, j] =-B[i, j]
            else:
```

[2] We are admittedly not allowing relabeling of vertices here, but that is irrelevant for the argument.

```
            bik = B[i, k- 1]
            kj = B[k- 1, j]
            if bik > 0 and kj > 0:
                B_prime[i, j] = B[i, j] + bik * kj
            elif bik < 0 and kj < 0:
                B_prime[i, j] = B[i, j]- bik * kj
            else:
                B_prime[i, j] = B[i, j]
return B_prime
```

This next function inputs a skew-symmetric matrix $B \in M_3(\mathbb{Q})$ and a sequence of indices $(i_1, \ldots, i_\ell)$ with $i_j \in \{1, 2, 3\}$. It applies this sequence of mutations to B and plots the resulting entries $(1, 2)$, $(2, 3)$ and $(1, 3)$ of the mutated exchange matrices $(\mu_{i_j} \circ \ldots \circ \mu_{i_1})(B)$. The output is the list of such coordinate entries for the mutate matrices and the plot:

```
def apply_mutation_sequence_plot_mpl(B_initial, mutation_sequence):

    mutated_data = []
    B_current = B_initial
    mutated_data.append((float(B_current[0, 1]), float(B_current[1, 2]),float(B_current[0, 2])))

    for k in mutation_sequence:
        B_current = mutate_matrix_rational_mpl(B_current, k)
        mutated_data.append((float(B_current[0, 1]), float(B_current[1, 2]),$\sqcup\xhookrightarrow$float(B_current[0, 2])))

    fig = plt.figure()
    ax = fig.add_subplot(111, projection='3d')
    x = [data[0] for data in mutated_data]
    y = [data[1] for data in mutated_data]
    z = [data[2] for data in mutated_data]
    ax.scatter(x, y, z, c='blue', marker='o')
    ax.set_xlabel('b12')
    ax.set_ylabel('b23')
    ax.set_zlabel('b13')
    ax.set_title('Mutation Sequence in (b12, b23, b13) Space')
return mutated_data, plt
```

The following function inputs a length $\ell \in \mathbb{N}$ and outputs a randomly generated sequence of indices $(i_1, \ldots, i_\ell)$ with $i_j \in \{1, 2, 3\}$, of length ℓ with no two consecutive indices being equal, i.e. $i_j \neq i_{j+1}$:

```
import random
def generate_alternating_sequence(length):

     if length <= 0:
       return []
    sequence = []
    first_number = random.choice([1, 2, 3])
    sequence.append(first_number)

    for _ in range(length- 1):
        possible_next = [num for num in [1, 2, 3] if num != sequence[-1]]
        next_number = random.choice(possible_next)
        sequence.append(next_number)
return sequence
```

This next function inputs a vector $v = (x, y, z) \in \mathbb{Q}^3$ and outputs the skew-symmetric matrix $M(v) \in M_3(\mathbb{Q})$ with entries $(1, 2)$ being x, $(2, 3)$ being y and $(1, 3)$ being z:

```
def skew_symmetric_matrix(v):

     x, y, z = v
     M = matrix(QQ, 3, 3)
```

```
    M[0, 1] = x
    M[1, 0] =-x
    M[1, 2] = y
    M[2, 1] =-y
    M[0, 2] = z
    M[2, 0] =-z
return M
```

The **main line of commands** that the user can choose to execute is as follows:

```
vector = (-0.6,-0.43,0.567)
B_rational_mpl = skew_symmetric_matrix(vector)
print("Initial Exchange Matrix is:")
show(B_rational_mpl)
length1 = 100
mutation_seq_mpl = generate_alternating_sequence(length1)
print(f"Random alternating sequence of length {length1}: {mutation_seq_mpl}")
mutation_history_mpl,plot_mpl=apply_mutation_sequence_plot_mpl(B_rational_mpl,
    mutation_seq_mpl)
plot_mpl.savefig("FigurePlot3D.pdf")
```

In the line of commands above, the user chooses the quiver $Q = (p, q, r)$ by selecting the tuple "vector". The user also selects the length $\ell \in \mathbb{N}$ of the mutation sequence by choosing the value of "length1". The plot is then saved in the document named "FigurePlot3D.pdf".

3.2. **Mutation sequences for Figures 1 and 2.** Figures 1 and 2 have been generated by the code in Section 3.1. This subsection displays the precise mutation sequences $(i_1, \ldots, i_\ell)$ used in each of the examples. Here ℓ is the length of the mutation sequence and the indices are always chosen such that $i_j \neq i_{j+1}$ for all indices, to avoid involutive steps in the sequence. To start, for the quiver $Q = (-0.02, -0.01, 0.03)$ in Figure 1 (left), the mutation sequence $(i_1, \ldots, i_\ell)$ has $\ell = 150$ and reads

$$(2, 3, 2, 1, 3, 2, 1, 3, 1, 2, 1, 2, 3, 2, 1, 3, 1, 2, 3, 1, 2, 1, 2, 1, 2, 3, 1, 2, 1, 3, 1, 3, 2, 3, 1, 3, 2, 1, 2, 1, 2, 1, 2, 1, 3, 1, 2, 1, 2, 3, 2,$$

$$1, 3, 2, 3, 1, 2, 3, 1, 2, 1, 2, 3, 1, 3, 2, 1, 2, 1, 3, 1, 3, 2, 3, 2, 3, 2, 3, 2, 3, 2, 1, 3, 1, 3, 1, 2, 1, 2, 3, 1, 2, 1, 3, 1, 2, 1, 2, 3, 2, 1, 3, 2, 1,$$

$$3, 2, 1, 2, 1, 3, 1, 2, 3, 2, 1, 2, 3, 2, 1, 2, 1, 3, 2, 3, 2, 3, 2, 3, 1, 3, 2, 1, 3, 2, 1, 2, 3, 2, 3, 2, 1, 2, 3, 1, 3, 2, 3, 1, 2, 1).$$

For the quiver $Q = (-0.9, -0.22, 0.7106)$ in Figure 1 (center), $\ell = 100$ and the sequence is

$$(2, 3, 1, 3, 1, 3, 2, 1, 3, 1, 3, 1, 2, 3, 1, 2, 1, 2, 1, 2, 1, 3, 1, 2, 1, 2, 3, 2, 3, 1, 3, 2, 1, 3, 2, 1, 3, 2, 3, 1, 3, 2, 1, 2, 3, 2, 1, 3, 2,$$

$$3, 1, 2, 3, 1, 2, 3, 2, 1, 2, 1, 3, 1, 2, 1, 2, 3, 2, 1, 3, 2, 1, 2, 3, 2, 3, 1, 2, 3, 1, 3, 2, 3, 1, 2, 1, 3, 1, 3, 1, 3, 1, 3, 1, 2, 3, 2, 1, 2, 3, 1).$$

For the quiver $Q = (-0.84, -0.26, 0.11)$ in Figure 1 (right), $\ell = 125$ and the sequence is

$$(3, 2, 3, 1, 2, 3, 2, 1, 2, 1, 2, 3, 2, 1, 3, 2, 3, 2, 3, 1, 2, 1, 3, 2, 1, 2, 3, 2, 3, 1, 3, 1, 3, 1, 3, 2, 1, 3, 2, 1, 3, 1,$$

$$2, 3, 1, 2, 1, 3, 1, 2, 3, 2, 1, 2, 1, 3, 1, 2, 1, 3, 2, 3, 2, 1, 2, 3, 1, 3, 1, 3, 2, 1, 2, 3, 1, 3, 1, 3, 2, 1, 3, 2, 1, 2,$$

$$3, 1, 3, 2, 3, 2, 1, 2, 1, 3, 1, 2, 1, 3, 1, 3, 1, 3, 2, 1, 2, 3, 1, 3, 2, 3, 2, 3, 2, 1, 2, 3, 1, 2, 3, 2, 1, 3, 2, 3, 1).$$

Figure 2 plots quivers in the mutation class of the quiver $Q = (-0.6, -0.43, 0.567)$. Specifically, it plots the images of Q after applying three different sequences of mutations. The sequence are described as follows. For Figure 2 (left), the mutation sequence has length $\ell = 100$ and it is

$$(3, 1, 2, 1, 2, 1, 3, 2, 3, 1, 2, 1, 2, 1, 3, 1, 3, 1, 2, 1, 3, 2, 1, 3, 2, 1, 3, 2, 1, 3, 1, 3, 1, 3, 1, 2, 1, 2, 3, 2, 1, 3, 2, 1, 3, 2, 1, 3, 2, 1,$$

$$3, 1, 2, 1, 3, 2, 1, 3, 1, 3, 2, 3, 2, 1, 2, 1, 3, 1, 2, 3, 2, 1, 3, 1, 3, 2, 3, 2, 3, 1, 2, 3, 1, 2, 3, 2, 3, 1, 2, 3, 2, 1, 3, 2, 3, 2, 1, 2, 3, 1).$$

For Figure 2 (center), $\ell = 100$ and the mutation sequence is

$$(1, 2, 1, 3, 2, 3, 2, 1, 3, 2, 1, 2, 1, 3, 1, 3, 2, 1, 3, 1, 2, 3, 1, 2, 3, 2, 3, 1, 2, 3, 1, 2, 1, 3, 1, 2, 1, 2, 1, 3, 2, 1, 2, 3, 1, 2, 3, 1, 3, 1,$$

$$2, 3, 2, 3, 1, 3, 2, 3, 1, 3, 1, 2, 3, 1, 2, 3, 2, 3, 1, 3, 2, 1, 2, 1, 3, 1, 2, 1, 3, 1, 3, 2, 1, 3, 1, 3, 2, 3, 2, 1, 2, 1, 3, 2, 3, 2, 1, 3, 1, 2).$$

For Figure 2 (right), $\ell = 125$ and the mutation sequence is

$$(2, 3, 2, 3, 2, 1, 2, 1, 2, 3, 2, 3, 1, 3, 1, 2, 1, 3, 1, 3, 1, 3, 1, 3, 2, 1, 2, 3, 1, 3, 1, 3, 2, 1, 3, 2, 1, 2, 3, 1, 3,$$

$$2, 3, 2, 1, 3, 2, 1, 3, 2, 3, 1, 3, 1, 2, 1, 3, 2, 3, 1, 3, 1, 3, 2, 1, 3, 2, 3, 1, 3, 2, 1, 2, 1, 2, 3, 2, 3, 1, 3, 2, 1, 2,$$

$$3, 1, 3, 2, 3, 2, 3, 1, 3, 1, 2, 3, 1, 3, 1, 3, 2, 1, 2, 3, 1, 2, 1, 2, 1, 3, 2, 3, 1, 3, 2, 3, 1, 3, 1, 2, 1, 3, 1, 2, 1, 3).$$

The reader can generate many other such figures by running the code in the Subsection 3.1.

References

[1] Andre Beineke, Thomas Brüstle, and Lutz Hille. Cluster-cyclic quivers with three vertices and the Markov equation. *Algebr. Represent. Theory*, 14(1):97–112, 2011. With an appendix by Otto Kerner.

[2] Roger Casals. A binary invariant of matrix mutation. *Journal of Combinatorial Algebra (to appear)*, 2024.

[3] Tucker J. Ervin and Scott Neville. Mutation cycles from reddening sequences, 2025.

[4] Anna Felikson and Pavel Tumarkin. Geometry of mutation classes of rank 3 quivers. *Arnold Math. J.*, 5:37–55, 2019.

[5] Anna Felikson and Pavel Tumarkin. Mutation-finite quivers with real weights. *Forum Math. Sigma*, 11:Paper No. e9, 22, 2023.

[6] Sergey Fomin. Quiver mutations, 2022.

[7] Sergey Fomin and Scott Neville. Long mutation cycles, 2023.

[8] Sergey Fomin and Scott Neville. Cyclically ordered quivers, 2024.

[9] Sergey Fomin, Lauren Williams, and Andrei Zelevinsky. Introduction to Cluster Algebras. *arXiv:2008.09189*.

[10] Sergey Fomin and Andrei Zelevinsky. Double Bruhat cells and total positivity. *J. Amer. Math. Soc.*, 12:335–380, 1999.

[11] Sergey Fomin and Andrei Zelevinsky. Cluster algebras I. *J. Amer. Math. Soc.*, 15(2):497–529, 2002.

[12] Sergey Fomin and Andrei Zelevinsky. Cluster algebras II. *Invent. Math.*, 154(1):63–121, 2003.

[13] Philipp Lampe. On the approximate periodicity of sequences attached to noncrystallographic root systems, 2016.

[14] Scott Neville. Mutation-acyclic quivers are totally proper, 2024.

[15] Ahmet I. Seven and İbrahim Ünal. Congruence invariants of matrix mutation. *J. Pure Appl. Algebra*, 229(3):Paper No. 107920, 14, 2025.