A microlocal introduction to Legendrian submanifolds

(Draft in progress)

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June 9, 2025

Contents

Preface

These notes contain material to support and complement the lecture series at the CBMS Summer School 2025 "Legendrian Links and the Microlocal Theory of Sheaves" (June 9-13 2025) at the Department of Mathematics of the Georgia Institute of Technology. This event was organized by J. Etnyre (Georgia Inst. Tech.), Lenhard Ng (Duke U), Bulent Tosun (U Alabama) Lisa Traynor (Bryn Mawr) and the author.

Through these lectures, we aim at presenting key foundational results in the microlocal theory of sheaves and selected applications to the study of Legendrian links to a broad audience of contact and symplectic topologists. We have leaned towards an accessible presentation, highlighting the crucial examples and discussing in detail the most enlightening cases of an argument. Through the notes, the reader is referred to the appropriate references for details on general cases and possible generalizations of results. The content of each lecture aligns with these notes as follows:

1. **Lecture 1**: "Lagrangian fillings of Legendrian links"
2. **Lecture 2**: "Sheaves and their singular support"
3. **Recitation 1**: "Examples of sheaves and their singular supports"
4. **Group work**: "Problem Set 1"
5. **Lecture 3**: "Exodromy description and singular support"
6. **Recitation 2**: "Details on Exodromy Equivalence and examples of singular support"
7. **Lecture 4**: "Invariance and Sheaf Quantization"
8. **Group work**: "Problem Set 2"
9. **Lecture 5**: "A Case Study: Legendrians and positive braids"
10. **Recitation 3**: "Braid varieties as moduli of sheaves"
11. **Lecture 6**: "Weaves, $\mathbb{L}$-compressible systems and sheaves"
12. **Lecture 7**: "Cluster theory and contact topology"
13. **Recitation 4**: "Cluster mutation and Lagrangian disk surgery"
14. **Lecture 8**: "Cluster structures on moduli of Lagrangian fillings"
15. **Lecture 9**: "Applications to symplectic topology and cluster algebras"
16. **Recitation 5**: "Infinitely many fillings and surjectivity"
17. **Lecture 10**: "Progress and conjectures on the classification of Lagrangian fillings"
18. **Group work**: "Problem Set 3"

Recitations 1, 3 & 5 led by H. Gao (Tsinghua U), and Recitations 2&4 by J. Hughes (Duke U). Group work supervised by A.M. Rodríguez (UC Davis) and A. Wong (UC Davis).

The notes comment on scientific context, providing due references and citations to mathematicians who have developed work in this area. In this preface, we highlight the outstanding contributions of M. Kashiwara and P. Schapira to the microlocal theory of sheaves, especially [KS90], and subsequently S. Guillermou, with [GKS12, Gui23]. Along with many others, including C. Viterbo, D. Tamarkin, D. Nadler, D. Treumann, E. Zaslow and their students and collaborators, they have helped shaped key aspects of the microlocal theory of sheaves as it currently applies to contact and symplectic topology.

Recommended prerequisites

On contact and symplectic topology, familiarity with the following concepts will be helpful:

1. Canonical symplectic structure in T^*M and contact structure in its ideal contact boundary $T^\infty M$. Contact and symplectic Darboux balls. See [AG01, Chapters 2,4], [Gei08, Chapter 2] or [CdS01, Parts I-IV].
2. Legendrian and Lagrangian submanifolds and fronts, e.g. [AG01, Chapter 5].

Specific focus will be devoted on Legendrian links in the contact Darboux 3-ball and exact embedded Lagrangian surfaces in the symplectic 4-ball. See [Etn05, Chapter 2] for a fantastic resource on Legendrian links, and also [Gei08, Chapter 3].

On Morse theory and stratifications, it will be helpful to absorb the material to have seen the starting steps in finite-dimensional Morse theory:

1. Morse functions and properties, e.g. [Mil63, Chapter 1] or [GP10, Chapter 1.7].
2. Stratified spaces, with Whitney stratifications in $M = \mathbb{R}^2, \mathbb{R}^3$, e.g. [GM88, Chapter 1].

Another reference on Morse theory is [AD14, Part I], e.g. Chapter 1 and Appendix A.

In addition, there are some preliminary readings that will be helpful for the lectures. On the microlocal theory of sheaves, the texts [Gui23, Parts 1,2,10,12] and [KS90, Chapter V] provide a first introduction to the *microlocal* aspects of sheaf theory. There are a number of additional introductory surveys on the microlocal theory of sheaves (many by P. Schapira) that can be found online and C. Viterbo's 2011 Eilenberg Lectures are worth reading. Other resources are the notes from the "Groupe de travail" at Orsay in 2015, available online.[1] The algebraic structure of a "cluster algebra" will naturally appear from the study of Lagrangian fillings of Legendrian links. It might be helpful to have a quick read at Chapters 2&3 of the book "Introduction to Cluster Algebras" by S. Fomin, L. Williams and A. Zelevinsky.[2]

[1] Currently active url is https://www.imo.universite-paris-saclay.fr/∼ patrick.massot/en/gdtSS/gdt2013.html.
[2] This book is available at "https://people.math.harvard.edu/∼williams/book.html".

Chapter 1: Introduction

The object of these lectures is to study Legendrian submanifolds and their Lagrangian fillings via the microlocal theory of sheaves. The notes are focused on exact Lagrangian surfaces and their boundary Legendrian links, often in standard Darboux balls. There are two pillars in the way we present such study:

1. **The construction of Legendrian links and Lagrangian fillings**. An important ingredient is the construction of $\mathbb{L}$-compressing systems, which allow for Lagrangian disk surgeries to be performed, thus obtaining potentially new Lagrangian fillings from given ones. From this viewpoint, the contact and symplectic topology of Lagrangian fillings hints towards having interesting wall-crossing phenomena.

2. **Invariants of Legendrian links and Lagrangian fillings via sheaves**.The study of sheaves and their singular supports allow for the construction of invariants. Specifically, they can sometimes be used to distinguish Legendrian submanifolds, up to contact isotopy, and Lagrangian fillings, up to Hamiltonian isotopy.

Contact and symplectic topology is the study of smooth functions and their first derivatives, broadly understood. In (2) above, sheaves can be seen as an incarnation of families of smooth functions, and their singular supports record information related to first derivatives. By combining (1) and (2) in the case of Legendrian links and their Lagrangian fillings, the concept of a cluster algebra arises naturally. In addition to the two aspects above, these notes provide a few key ingredients explaining the relation between the study of cluster algebras and that of Lagrangian surfaces.

1.1. Legendrian submanifolds: a first goal

The classification of Legendrian links $\Lambda \subseteq (\mathbb{R}^3, \xi_{\mathrm{st}})$ is a foundational and subtle problem. Whereas any smooth link can be represented by a Legendrian link, there are many non-isotopic such representatives. In certain cases, such as the smooth type of the knots with small crossings or certain families[1], there exists a complete classification of all Legendrian representatives. In general, such classification remains wide open. Precise definitions on the concept of Legendrians and contact manifolds, with examples and more details, are provided in Chapter 2.

The classification of Lagrangian fillings of a given Legendrian link is also a foundational problem. While it relates to the classification of Legendrian links, there are many results connecting this problem to several different branches of mathematics: variational calculus, via Lagrangian and Hamiltonian mechanics, the study of differential equations, via the Riemann-Hilbert correspondence, or the study of cluster algebras, via weaves and plabic graphs, to name some. For specificity, we center the discussion on the following geometric goal:

Goal 1.1.1. *Given a Legendrian link $\Lambda \subseteq (\mathbb{R}^3, \xi_{st})$, classify the embedded exact Lagrangian fillings $L \subseteq (\mathbb{R}^4, \lambda_{st})$ of Λ, up to Hamiltonian isotopy.*

[1] E.g. the unknot, the figure-eight, the trefoil and, more generally, twist knots, or also certain cables.

Goal 1.1.1 is one of the simplest versions of the more ambitious goal of classifying Hamiltonian isotopy classes of Lagrangian submanifolds in a symplectic manifold, a problem at the core of modern symplectic topology. It is the case that progress towards Goal 1.1.1 has itself enlightened our understanding of Weinstein manifolds and also contributed significantly towards the computation of well-used invariants, such as various variants of Fukaya categories. To ease notation, we use the words *Lagrangian filling* to mean *embedded exact Lagrangian filling.*

Let us fix a Legendrian link $\Lambda \subseteq (\mathbb{R}^3, \xi_{\mathrm{st}})$ and think about Goal 1.1.1. As with any classification, natural steps are:

1. **Construct** as many Lagrangian fillings of Λ as possible. This often employs techniques with flavors from geometric topology.
2. **Obstruct** any isomorphisms between the fillings we constructed, i.e. distinguish as many of the Lagrangian fillings we built as possible. This often requires a certain degree of algebra, may it be commutative, non-commutative, homological or categorical.[2]
3. **Show that we are done**, i.e. any Lagrangian filling must have been already obtained in (1) and any two Lagrangian fillings that could not be distinguished by (2) must be Hamiltonian isotopic.

In a nutshell, the recent developments in low-dimensional contact topology regarding Lagrangian fillings address (1) and (2). To date, there is essentially no technique I know that can address (3), even in relatively simple instances, e.g. the $(2, 5)$-torus knot; methods from geometric analysis, such as studying PDEs that deform smooth maps to harmonic or holomorphic ones, might be the ones closest to achieving (3).

The focus of these lectures is on (1) and (2) above. For (1), we use Lagrangian disk surgery as our main ingredient. For (2), we employ the microlocal theory of sheaves. These are not the only choices, and it has certainly been beneficial for Goal 1.1.1 to approach (1) and (2) with different techniques, as well as relating such different approaches to each other.

Remark 1.1.2 (Higher dimensions)**.** Many results presented here have direct analogues for higher-dimensional Legendrians $\Lambda \subseteq (T^\infty M, \xi_{\mathrm{st}})$ and their Lagrangian fillings in $(T^*M, \lambda_{\mathrm{st}})$. We often present arguments in general dimension, so the reader can use the tools if needed, while focusing on dimensions 3 and 4 when it comes to examples and applications. $\square$

1.2. Microlocal theory of sheaves: a first toolkit

The microlocal theory of sheaves can be used in fruitful ways to understand Legendrian submanifolds. For instance, it can:

1. Achieve the obstruction part of Goal 1.1.1. That is, sheaves can be employed to distinguish Lagrangian fillings of a given Legendrian link, cf. Section 1.1.
2. Help understand the set of Lagrangian fillings and its geometric properties. This is a crucial aspect in these notes, when trying to build a moduli space of Lagrangian fillings.
3. Distinguish Legendrian submanifolds, up to contact isotopy.

[2] More broadly, it often requires a flavor of homotopy theory, understood contemporarily, e.g. including the study of higher algebra and higher topoi, or its dg and A_∞ variants.

We now provide a condensed version of the key aspects that we will discuss when developing the microlocal theory of sheaves with Goal 1.1.1 in mind.

Let M be a real analytic manifold, $\Lambda \subseteq (T^\infty M, \xi_{\mathrm{st}})$ a Legendrian submanifold of the ideal contact boundary of the cotangent bundle $(T^*M, \lambda_{\mathrm{st}})$, and $\mathcal{K}$ a compactly-generated dg-category, which serves as the category of coefficients for sheaves. The core results from the microlocal theory of sheaves that we discuss are:

1. The construction of a dg-derived category $\mathrm{Sh}_\Lambda(M;\mathcal{K})$ such that a contact isotopy $\varphi_t \in \mathrm{Cont}(T^\infty M, \xi_{\mathrm{st}})$, $t \in [0,1]$, induces an equivalence of dg-categories
$$F(\varphi_t) : \mathrm{Sh}_{\varphi_0(\Lambda)}(M;\mathcal{K}) \longrightarrow \mathrm{Sh}_{\varphi_t(\Lambda)}(M;\mathcal{K}), \quad \forall t \in [0,1].$$
In particular, the dg-equivalence type of $\mathrm{Sh}_\Lambda(M;\mathcal{K})$ is a Legendrian isotopy invariant of Λ. This category $\mathrm{Sh}_\Lambda(M;\mathcal{K})$ is referred to as the dg-category of $\mathcal{K}$-valued sheaves on M with singular support on Λ.

2. Properties of $\mathrm{Sh}_\Lambda(M;\mathcal{K})$, its variants, and their associated derived moduli stacks of pseudoperfect objects, that are helpful in the study of Legendrian submanifolds. For instance, given a Legendrian link $\Lambda \subseteq (T^\infty\mathbb{R}^2, \xi_{\mathrm{st}})$, there exists a functor of dg-categories
$$\mathfrak{m}_\Lambda : \mathrm{Sh}_\Lambda(M;\mathcal{K}) \longrightarrow \mathrm{Loc}(\Lambda;\mathcal{K}) \tag{1.2.1}$$
to the dg-category $\mathrm{Loc}(\Lambda;\mathcal{K})$ of $\mathcal{K}$-valued local systems on Λ. Such functor $\mathfrak{m}_\Lambda$ is referred to as the microlocal functor, and it is crucial in the microlocal study of Λ. To wit, it is necessary to perform sheaf quantization, it is used in the construction of cluster structures associated to Λ, it carries a relative Calabi-Yau structure (generalizing Poincaré duality in the context of contact topology), and its left adjoint can be used to provide generators for $\mathrm{Sh}_\Lambda(M;\mathcal{K})$.

For Part 1 above, we use two fundamental ideas in the microlocal theory of sheaves:

1. The notion of **singular support** and its properties. The singular support $\mu\mathrm{supp}(\mathscr{F})$ of a sheaf $\mathscr{F} \in \mathrm{Sh}(M;\mathcal{C})$ will be a closed conical coisotropic subset $\mu\mathrm{supp}(\mathscr{F}) \subseteq (T^*M, \omega_{\mathrm{st}})$ of the cotangent bundle. The sheaf $\mathscr{F}$ is (real) constructible if and only if $\mu\mathrm{supp}(\mathscr{F})$ is Lagrangian, and this case will be our focus. For a Legendrian $\Lambda \subseteq (T^\infty M, \xi_{\mathrm{st}})$, the constructibility is manifested through the stratification of M given by the wavefront $\pi(\Lambda) \subseteq M$ of the given Legendrian Λ, with respect to which elements in $\mathrm{Sh}_\Lambda(M;\mathcal{K})$ will be locally constant.

2. The technique of **sheaf quantization** which, coarsely put, aims at constructing a sheaf with a specified singular support. In effect, it studies whether the functor (1.2.1) is essentially surjective. For us, there are two important cases of sheaf quantization: that of a graph of a contact isotopy, and that of a Lagrangian filling. The former is used to show that $\mathrm{Sh}_\Lambda(M;\mathcal{K})$ is a Legendrian invariant of Λ, under contact isotopies, and the latter will allow us to study Lagrangian fillings of Λ through the dg-category $\mathrm{Sh}_\Lambda(M;\mathcal{K})$.

This table highlights concepts relating contact topology and microlocal sheaf theory that will be discussed in Part 1:

Contact Topology in $(T^\infty M, \xi_{\mathbf{st}})$	**Category of sheaves on** M
Legendrian Link $\Lambda \subseteq (T^\infty M, \xi_{\mathrm{st}})$	$\mathrm{Sh}_\Lambda(M;\mathcal{K})$
Microlocal monodromy along Λ	$\mathfrak{m}_\Lambda : \mathrm{Sh}_\Lambda(M;\mathcal{K}) \longrightarrow \mu\mathrm{Sh}(\Lambda;\mathcal{K})$
Contact isotopy $\varphi_t \in \mathrm{Cont}(T^\infty M, \xi_{\mathrm{st}})$	$F(\varphi_t) : \mathrm{Sh}_{\varphi_0(\Lambda)}(M;\mathcal{K}) \xrightarrow{\sim} \mathrm{Sh}_{\varphi_t(\Lambda)}(M;\mathcal{K})$
Lagrangian filling $L \subseteq (T^*M, \lambda_{\mathrm{st}})$, $\partial L = \Lambda$, endowed with $\mathcal{K}$-local system	object $\mathscr{F}_L$ in $\mathrm{Sh}_\Lambda(M;\mathcal{K})$ constructed via $\mathrm{Sh}_{L^\uparrow}(M \times \mathbb{R};\mathcal{K})$
Intersections of Lagrangian fillings	morphisms in $\mathrm{Sh}_\Lambda(M;\mathcal{K})$

Table 1.1: Table summary of notions from contact topology in $(T^\infty M, \xi_{\mathrm{st}})$ and the corresponding concepts induced in the microlocal study of sheaves in M.

In the case of Legendrian links and their Lagrangian fillings, i.e. M a surface, the category $\mu\mathrm{Sh}(\Lambda;\mathcal{K})$ in the codomain of $\mathfrak{m}_\Lambda$ in Table 1.2 is equivalent to $\mathrm{Loc}(\Lambda;\mathcal{K})$. The microlocal monodromy along Λ thus assigns a $\mathcal{K}$-local system in Λ to a sheaf in $\mathrm{Sh}_\Lambda(M;\mathcal{K})$. The notation $L^\uparrow \subseteq (J^1M, \xi_{\mathrm{st}})$ in Table 1.2 stands for the Legendrian lift of $L \subseteq (T^*M, \lambda_{\mathrm{st}})$.

1.3. A conjecture: Lagrangian fillings and cluster seeds

As we develop our understanding of Legendrian links and their Lagrangian fillings, we keep Goal 1.1.1 as a running motivation. In doing so, it might be helpful to suggest a possible answer to such classification of Lagrangian fillings. The necessary concepts, evidence and motivation for the following statement are provided throughout the notes:

Conjecture 1.3.1 (Lagrangian fillings classified by cluster seeds)**.** *Let $(\Lambda, \mathfrak{t}) \subseteq (\mathbb{R}^3, \xi_{st})$ be a pointed Legendrian link, and $\mathfrak{M}(\Lambda, \mathfrak{t})$ the derived stack of pseudoperfect objects of the category $\mathrm{Sh}^c_{\Lambda,\mathfrak{t}}(\Lambda, \mathrm{Perf}(\mathbb{Z}))$. Then*

1. *$\Gamma(\mathfrak{M}(\Lambda, \mathfrak{t}), \mathcal{O}_{\mathfrak{M}}(\Lambda, \mathfrak{t}))$ is a cluster algebra,*

2. *Any Lagrangian filling $L \subseteq (\mathbb{R}^4, \lambda_{st})$ induces a cluster seed,*

3. *Any cluster seed is induced by an Lagrangian filling $L \subseteq (\mathbb{R}^4, \lambda_{st})$,*

4. *Two Lagrangian fillings inducing the same cluster seed are Hamiltonian isotopic.*

In consequence, the classification of the embedded exact Lagrangian fillings $L \subseteq (\mathbb{R}^4, \lambda_{st})$ of Λ, up to Hamiltonian isotopy, is given by the cluster seeds of a cluster algebra structure on the ring of functions of $\mathfrak{M}(\Lambda)$.

The conceptual message of Conjecture 1.3.1, were it be proven to be true, is that the classification of exact Lagrangian fillings, up to Hamiltonian isotopy, can be reduced to a problem in commutative algebra. The algebraic problem is itself interesting and subtle but, in a way, entirely stripped of symplectic topology.[3] Conjecture 1.3.1 contains many implicit statements, some of which are:

1. $\Gamma(\mathfrak{M}(\Lambda), \mathcal{O}_{\mathfrak{M}}(\Lambda))$ is a commutative dg-algebra. The current notion of cluster algebra in the literature is typically reserved for commutative algebras, concentrated in degree 0, and there is not yet an analogous notion for the dg-setting. Similarly, $\mathfrak{M}(\Lambda, \mathfrak{t})$

[3] "In a way" here because insights from symplectic topology have been proven useful in studying such algebraic problem. Thus there is a case for studying geometry, even if the problem can be reduced to algebra!

has (infinitely) many connected components, e.g. labeled by microlocal rank. The current notion of cluster algebras should be modified to fit the mutation formulas for higher microlocal rank. Therefore Conjecture 1.3.1.(1) can either be understood as first needing to develop these more general notions of cluster algebras – to fit dg-algebras and higher microlocal rank – or just taken to mean that the commutative algebra $H^0(\mathfrak{M}_1(\Lambda,\mathfrak{t}),\mathcal{O}_{\mathfrak{M}_1(\Lambda,\mathfrak{t})})$ is a cluster algebra, in the current sense of the notion, where $\mathfrak{M}_1(\Lambda,\mathfrak{t})$ denotes the component corresponding to microlocal rank 1.

2. In my view, Conjecture 1.3.1.(2) could specifically be understood as saying that any Lagrangian filling $L\subseteq(\mathbb{R}^4,\lambda_{\rm st})$ admits an $\mathbb{L}$-compressing system, unique in an appropriately understood sense, whose intersection quiver and microlocal merodromies provide a cluster seed for the cluster algebra in Conjecture 1.3.1.(1).

3. Conjecture 1.3.1.(4) is, in a sense, a generalization of the nearby Lagrangian conjecture for surfaces. The nearby Lagrangian conjecture states that, up to Hamiltonian isotopy, the only embedded exact Lagrangian surface in $(T^*\Sigma,\lambda_{\rm st})$ is the zero section $\Sigma\subseteq T^*\Sigma$, where Σ is a closed surface. It is known to be true for genus $g(\Sigma)=0,1$ and it remains open for any other case $g(\Sigma)\geq 2$.

   A natural way to generalize this is to consider a Lagrangian skeleton
   $$\mathbb{L}:=\Sigma\cup(D_1\cup\ldots\cup D_n)$$
   given by a smooth surface Σ and a collection of smooth disks D_i attached to Σ along a smooth simple curve. There is a sense in which this defines a Weinstein 4-manifold $T^*\mathbb{L}$, possibly with a stop if $\partial\Sigma\neq\emptyset$. In the particular case where the ∂D_i are linearly independent in $H_1(\Sigma,\mathbb{Z})$, Conjecture 1.3.1.(4) states that, up to Hamiltonian isotopy, the only embedded exact Lagrangian surfaces in $(T^*\mathbb{L},\lambda_{\rm st})$ are given either by the zero section $\Sigma\subseteq T^*\Sigma$ or by applying a sequence of Lagrangian disk surgeries to that zero section by using the disks $D_1,\ldots,D_n$.

Evidence for Conjecture 1.3.1 is provided by the class of Legendrian links $\Lambda_\beta\subseteq(\mathbb{R}^3,\xi_{\rm st})$. This is the case study presented in Chapter 6. Throughout these notes, we shall explain how to prove some of the statements in Conjecture 1.3.1 for these Legendrian links.

Remark 1.3.2. Following comment (3) above, Conjecture 1.3.1 can be naturally stated in more generality for the classification of embedded exact Lagrangian surfaces in a Weinstein 4-manifold of the form $T^*\mathbb{L}$, possibly with a stop at its boundary. The statement would then conjecture that embedded exact Lagrangian surfaces in such Weinstein 4-manifolds are Hamiltonian isotopic to Σ or equivalent to it via Lagrangian disk surgeries along a fixed collection of Lagrangian disks. □

1.4. Other developments via the microlocal theory of sheaves

The focus of these notes is the study of Legendrian and Lagrangian submanifolds, and we use the microlocal theory of sheaves as a technique to help us in this endeavor. That said, the techniques developed within the microlocal theory of sheaves apply more broadly to contact and symplectic topology. The foundations of the microlocal theory of sheaves are developed in the book [KS90]. Since then, the works [GKS12, Gui23, Nad09, NZ09, Tam18, Vit10, Vit19] brought forth new results and applications, reinvigorating research activity in this area. By now, the microlocal theory of sheaves has been successfully used to prove many new results in contact and symplectic topology, as well as give new proofs of known theorems in the field. A short selection of references is:

1. The monograph [Gui23] contains an encouraging number of such applications, including C^0-rigidity of symplectic geometry, cf. [Gui23, Part VII], the 3-cusp conjecture in [Gui23, Part VIII], and the fact that the projection of a nearby Lagrangian onto the zero section is a homotopy equivalence, in [Gui23, Part XIII]. See also [GV24] for further relations to C^0-symplectic geometry and spectral invariants.

2. Non-displaceability results for Lagrangian submanifolds via the microlocal theory of sheaves can be found in [GKS12, Tam18, Vic13].

3. Distinguishing Hamiltonian isotopy classes of embedded exact Lagrangians, e.g. as in [CG22, STWZ19]. In particular, [CG22] established that (many) Legendrian links admit infinitely many Lagrangian fillings.

4. Connections between Legendrian knots and smooth knot homologies, e.g. triply-graded Khovanov–Rozansky homology and HOMFLY polynomials, are developed in [STZ17], see also [CGGS20, CGGS21].

5. Symplectic and contact non-squeezing have also been established via microlocal sheaf techniques, cf. [Gui23, Part VI] and [Chi17, Zha24], as well as estimates on the number of Reeb chords of a Legendrian submanifold, cf. [Li21].

6. The construction of a relative Calabi-Yau structure for the functor (1.2.1) is provided in [KL24b], see also [KL24a, KL22].

7. A proof that microlocalization yields the Voros–Iwaki–Nakanishi coordinates in WKB analysis is established in [Kuw24]. The microlocal theory of sheaves also directly relates to the Betti side of the irregular Riemann-Hilbert correspondence and non-abelianization, cf. [CL23] and also [CN25].

8. More recently, the construction of cluster algebra structures, cf. [CW24, CGG+22], including the proof that Richardson varieties are cluster in [CGG+22]. In addition, a proof that the Muller-Speyer twist on positroids equals the Donaldson-Thomas transformation, cf. [CLSBW23].

Further developments in the microlocal theory of sheaves are [GPS24, Kuo23, NS20].

Chapter 2: Legendrian and Lagrangian submanifolds

This chapter introduces Legendrian submanifolds and their Lagrangian fillings, the geometric objects that we study throughout the notes. The focus is on Legendrian links $\Lambda \subseteq (T^\infty\mathbb{R}^2, \xi_{\mathrm{st}})$ and their (unobstructed) exact Lagrangian fillings in $(T^*\mathbb{R}^2, \lambda_{\mathrm{st}})$.

2.1. Contact and symplectic manifolds

Legendrian submanifolds $\Lambda \subseteq Y$ are certain types of smooth $(n-1)$-dimensional submanifolds of a smooth $(2n-1)$-dimensional manifold Y, where Y is equipped with a geometric structure. This geometric piece of information on Y is a contact structure:

Definition 2.1.1 (Contact structures)**.** Let Y be a smooth manifold and $\xi \subseteq TY$ a hyperplane distribution. By definition, ξ is said to be a contact structure on Y if ξ is maximally non-integrable.

By a result of G. Frobenius [Fro77], see also [Gei08, Section 1.1], the condition of being maximally non-integrable is equivalent to the local existence of a 1-form $\alpha \in Y$ such that $\xi = \ker\alpha$ and the top form $\alpha(d\alpha)^{n-1} \neq 0$ being non-zero. That is, ξ is a contact structure on Y if $\xi = \ker\alpha$ where $\alpha(d\alpha)^{n-1} \in \Omega^{2n-1}(Y)$ is a local volume form in Y. A hyperplane distribution $\xi \subseteq TY$ is always locally of the form $\xi = \ker\alpha$ for a 1-form $\alpha \in \Omega^1(Y)$, the meaningful condition is $\alpha(d\alpha)^{n-1} \neq 0$. For more information on contact structures, see e.g. [AG01, Chapter 4], [McD90, Section 3.5] or [Etn03, Gei08], and references therein. In particular, a diffeomorphism $\varphi : (Y, \xi) \longrightarrow (Y', \xi')$ is said to be a contactomorphism if $\varphi_*\xi = \xi'$.

Example 2.1.2. *(1) The manifold* $(Y, \xi) = (\mathbb{R}^{2n-1}, \xi_{st})$ *with*

$$\xi_{st} = \ker\{dz - y_1 dx_1 - \ldots - y_{n-1} dx_{n-1}\} \tag{2.1.1}$$

is a contact manifold. It is a result of G. Darboux that every contact manifold is locally, around any point, of this form, cf. [Arn89, Appendix 4.H], [AG01, Section 4.1.1], [MS98, Exercise 3.5.20] or [Gei08, Theorem 2.5.1]. In other words, any point in a contact manifold admits a neighborhood contactomorphic to $(\mathbb{R}^{2n-1}, \xi_{st})$*, as given by Equation* (2.1.1)*.*

(2) Let M *be a smooth manifold,* T^*M *its cotangent bundle and* λ_{st} *the canonical Liouville 1-form. This is the unique 1-form* $\lambda_{st} \in \Omega$ *such that* $\eta^*(\lambda_{st}) = \eta$ *for every* $\eta \in \Omega^1(M)$*, cf. [MS98, Prop. 3.1.18] or [Gei08, Section 1.4]. Let* g *be a Riemannian metric on* M *and* $T^gM \subseteq T^*M$ *the* g*-unit cotangent bundle. Then*

$$(T^gM, \xi_{st}) := (T^gM, \ker\{\lambda_{st}|_{T^gM}\})$$

is a contact manifold. The contactomorphism type of T^gM *is independent of* g*, and a more intrinsic description of such contact structure is via the notion of ideal contact boundary, cf. [Gir17, Prop. 2]. To emphasize the independence on* g*, we often denote such contact manifold by* $(T^\infty M, \xi_{st})$*.*

(3) Let Q *be a smooth manifold,* T^*Q *its cotangent bundle and* λ_{st} *the canonical Liouville 1-form. Then the smooth manifold* $T^*Q \times \mathbb{R}$ *admits the following contact structure*

$$(J^1Q, \xi_{st}) := (T^*Q \times \mathbb{R}_z, \ker\{dz - \lambda_{st}\}).$$

This is known as a 1-jet space: it carries the information of germs of smooth functions $f : Q \longrightarrow \mathbb{R}$ and their 1st derivatives, with T^*Q keeping track of the graph of df and the choice of contact structure forces $\mathbb{R}_z$ to record the value of f. As an example, there is a contactomorphism $(T^\infty\mathbb{R}^n, \xi_{st}) \cong (J^1S^{n-1}, \xi_{st})$. This type of contactomorphisms, between unit cotangent bundles and 1-jet spaces, are scarce: typically J^1M, resp. $T^\infty M$, is not even diffeomorphic to $T^\infty Q$, resp. J^1Q, for a smooth manifold Q. $\square$

Remark 2.1.3. For context on Example 2.1.2.(1), the classification of regular k-plane distributions in dimension n that admit a discrete normal form is quick to state: non-zero vector fields, where $k = 1$, contact-type structures, where $k = n-1$, and Engel structures, which are $(k, n) = (2, 4)$. These are all the possible distributions that around a point have no functional moduli, and in fact a unique normal form. In this sense, the study of contact structures, along with the dynamics of non-zero vector fields, is the simplest and most generic type of distribution one can study. See [VG88, Section 1] or [AN94, Section 2.7] and note that this problem was first studied by G. Darboux [Dar82, Section 5.(17)] and É. Cartan [Car01]. $\square$

Lagrangian submanifolds $L \subseteq X$ are certain types of smooth n-dimensional submanifolds of a smooth $2n$-dimensional manifold X, where X is equipped with a certain geometric structure. This geometric piece of information on X is a symplectic structure:

Definition 2.1.4 (Symplectic structures)**.** Let X be a smooth manifold and $\omega \in \Omega^2(X)$ a 2-form. By definition, ω is said to be a symplectic structure on X if ω is closed and maximally non-degenerate, i.e. $d\omega = 0$ and $\omega^n \in \Omega^{2n}(X)$ is non-vanishing.

By definition, a symplectic manifold (X, ω) is exact if $\omega = d\lambda$ for some $\lambda \in \Omega^1(X)$.

A diffeomorphism $\phi : (X, \omega) \longrightarrow (X', \omega')$ is said to be a symplectomorphism if $\phi^*\omega' = \omega$. All symplectic manifolds discussed in these notes will be exact.

Example 2.1.5. *(1) The manifold $(X, \omega) = (\mathbb{R}^{2n}, \omega_{st})$ with*

$$\omega_{st} = dx_1dy_1 + \ldots + dx_ndy_n$$

is a symplectic manifold. Notice the similarity with Example 2.1.2.(1). The same result of G. Darboux implies that every symplectic manifold is locally, around any point, of this form, cf. [AG01, Section 2.1.1] or [MS98, Theorem 3.2.2]. In other words, any point in a contact manifold admits a neighborhood contactomorphic to $(\mathbb{R}^{2n-1}, \xi_{st})$, as given by Equation (2.1.1)*.*

(2) Let M be a smooth manifold, T^*M its cotangent bundle and λ_{st} the canonical Liouville 1-form, as in Example 2.1.2.(2). Then

$$(T^*M, \xi_{st}) := (T^*M, d\lambda_{st})$$

is a symplectic manifold. By construction, it is an exact symplectic manifold as we can take $\lambda = \lambda_{st}$ as the primitive. The previous example (1) corresponds to the case $M = \mathbb{R}^n$. $\square$

A recurring theme in contact and symplectic topology is that many interesting interactions occur when a contact manifold (Y, ξ) is, in the appropriate sense, the boundary of a symplectic manifold (X, ω). For these notes, the main instance of this is the contact manifold $(T^\infty M, \xi_{\mathrm{st}})$ in Example 2.1.2.(2) thought of as the boundary of the exact symplectic manifold $(T^*M, \lambda_{\mathrm{st}})$ in Example 2.1.5.(2). The intuition is that the contact structure $\xi_{\mathrm{st}} = \ker \lambda_{\mathrm{st}}$ is given by the kernel of the restriction of the primitive of the symplectic structure to the boundary. In general, and more rigorously, if (X, λ) is an ideal Liouville domain, we denote by $(Y, \xi) = \partial(X, \lambda)$ its ideal contact boundary, cf. [Gir17, Section A] and [EKP06, Section 1.5].

Remark 2.1.6. The infinitesimal study of symplectic structures appears as the type C family in the classification of simple Lie algebras. Intuitively, type A infinitesimally studies geometries that preserve volume, types B and D geometries that preserve inner products, which is a *symmetric* tensor, and type C covers geometries that preserve skew-symmetric forms, which are *skew-symmetric* tensors. In this sense, symplectic geometry is one of the building blocks for any type of geometry, in the spirit of the Erlangen program. □

2.2. Legendrian and Lagrangian submanifolds

Let (Y, ξ) be a $(2n-1)$-dimensional contact manifold. There are different ways in which an embedded submanifold $\Sigma \subseteq Y$ can interact with ξ. An important one is being tangent to it, i.e. $T\Sigma \subseteq \xi$. The maximal non-integrability of ξ, as in Definition 2.1.1, implies that any such submanifold must have $\dim(\Sigma) \leq n-1$. This leads to the following:

Definition 2.2.1 (Legendrian submanifolds)**.** Let (Y, ξ) be a $(2n-1)$-dimensional contact manifold. An embedded submanifold $\Lambda \subseteq (Y, \xi)$ is said to be Legendrian if $T\Lambda \subseteq \xi$ and its dimension is $\dim \Lambda = n-1$. Two Legendrians $\Lambda_0, \Lambda_1 \subseteq (Y, \xi)$ are said to be Legendrian isotopic if there exists a 1-parametric family $\{\Lambda_t\}_{t\in[0,1]}$ of Legendrians in (Y, ξ).

This is the maximal possible dimension of a smooth submanifold satisfying $T\Lambda \subseteq \xi$. A current goal of contact topology is the classification of Legendrian submanifolds, up to Legendrian isotopy, of a given contact manifold (Y, ξ). A first example of a Legendrian submanifold is the 1-jet

$$j^1(f) := \{(x, \xi; z) \in J^1M : \xi = df_x, \quad z = f(x)\} \subseteq (J^1M, \xi_{\mathrm{st}}) \tag{2.2.1}$$

of a smooth function $f : M \longrightarrow \mathbb{R}$. Such Legendrians are Legendrian isotopic to the zero section via the family $j^1(tf)$ for $t \in [0, 1]$. We will often construct and manipulate Legendrian submanifolds via fronts, as introduced in Section 2.4 below, but here are some first comments and examples:

Example 2.2.2. (i) *Consider a Legendrian knot $\Lambda \subseteq (\mathbb{R}^3, \xi_{st})$ whose underlying smooth isotopy class is that of the unknot. Then its Legendrian isotopy type is uniquely determined by a pair of integer numbers $(rot(\Lambda), \mathrm{tb}(\Lambda)) \in \mathbb{Z}^2$, known respectively as the rotation and Thurston-Bennequin numbers of Λ. See e.g. [Etn05, Section 2.6.1] or [Gei08, Section 3.5]. More generally, the Legendrian isotopy class of any Legendrian knot $\Lambda \subseteq (\mathbb{R}^3, \xi_{st})$ whose underlying smooth type is an algebraic knot is uniquely determined by these two such numbers. See [Etn05, Section 5] and references therein.*

(ii) *There exist exactly two Legendrian isotopy classes in $(\mathbb{R}^3, \xi_{st})$ of Legendrian knots whose underlying smooth type is $m(5_2)$ and have maximal* tb*, cf. Example 2.4.2.(1) and Figure 2.2 below. This is proven in [Che02, Section 4], where these Legendrian knots are shown to be non-isotopic, and [ENV13, Theorem 1.1.(4)], where it is proven that these are the only two ones. In fact, given any number $k \in \mathbb{N}$, there exist smooth knots $K \subseteq \mathbb{R}^3$ which admit at least k distinct Legendrian isotopy classes of Legendrian knots whose underlying smooth type is K, cf. [EFM01, Theorem 4.1].*

(iii) *Let $S \subseteq M$ be a cooriented immersed submanifold. Its conormal lift $\nu(S) \subseteq (T^\infty M, \xi_{st})$ is a Legendrian submanifold. In particular, any oriented immersed curve in the plane $\mathbb{R}^2$ determines a Legendrian link in $(T^\infty\mathbb{R}^2, \xi_{st})$.*

(iv) A central equation in thermodynamics, which follows from the 1st and 2nd laws, is that

$$dU = TdS - PdV,$$

where the local variables $(U, S, T, V, P) \in \mathbb{R}^5$ *measure internal energy, entropy, temperature, volume and pressure. A key problem in thermodynamics is to determine the equilibrium states, i.e. states for which the equation above holds, corresponding to a critical point of the Gibbs free energy. Thus, the subset* $\Lambda \subseteq \mathbb{R}^5$ *of equilibrium states is a Legendrian surface in*

$$(\mathbb{R}^5, \ker\{dU - TdS + PdV\}).$$

As V. Arnol'd opens in [Arn90a]: "Every mathematician knows it is impossible to understand an elementary course in thermodynamics. The reason is that thermodynamics is based – as Gibbs has explicitly proclaimed – on a rather complicated mathematical theory, on contact geometry.". Ibid is a much recommended survey for graduate students and see also [EPR25] for a more modern account in relation to thermodynamics. □

The classification of Legendrian isotopy classes in $(\mathbb{R}^3, \xi_{\mathrm{st}})$ has been an active subject of study, including salient results such as [EH05, ELT12, ENV13]. The classification of (non-loose) Legendrians in $(\mathbb{R}^{2n-1}, \xi_{\mathrm{st}})$ is yet to be successful explored. Note that in higher dimensions there are typically (infinitely) many Legendrian spheres smoothly isotopic to the Legendrian unknot but not Legendrian isotopic to it, as being smoothly isotopic in higher codimension is not much of a smooth constraint.

In the symplectic context, part of the tangency condition in Definition 2.2.1 translates to differential forms as the vanishing of ω upon restriction. Specifically, a submanifold $L \subseteq (X, \omega)$ of a symplectic manifold (X, ω) is said to be *Lagrangian* if $\omega|_L \equiv 0$ vanishes identically and $\dim(L) = n$. In these notes, we study the following specific class of Lagrangian submanifolds in exact symplectic manifolds:

Definition 2.2.3 (Exact Lagrangian submanifolds)**.** Let (X, λ) be an exact symplectic manifold. An embedded submanifold $L \subseteq (X, \lambda)$ is said to be an exact Lagrangian if the deRham class $[\lambda|_L] \in H^1(L, \mathbb{R})$ vanishes, i.e. if there exists a function $f : L \longrightarrow \mathbb{R}$ such that $df = \lambda|_L$. Two exact Lagrangians $L_0, L_1 \subseteq (X, \lambda)$ are said to be exact Lagrangian isotopic if there exists a 1-parametric family $\{L_t\}_{t\in[0,1]}$ of exact Lagrangians in (X, λ).

As above, a current goal of symplectic topology is the classification of exact Lagrangian submanifolds, up to exact Lagrangian isotopy, of a given exact symplectic manifold (Y, λ). For the reader familiar with Hamiltonian isotopies, note that two exact Lagrangians are exact Lagrangian isotopic if and only if they are Hamiltonian isotopic, see e.g. [MS98, Section 9.3] and [Oh15, Theorem 3.6.7].

Example 2.2.4. (*i*) *Let* $f : M \longrightarrow \mathbb{R}$ *be a smooth function, then the graph* $gr(df) \subseteq (T^*M, \lambda_{st})$ *of its differential* $df \in \Omega^1(M)$ *is an embedded exact Lagrangian submanifold. Any such Lagrangian is exact Lagrangian isotopic to the zero section, e.g. by considering the family* $gr(d(tf))$ *for* $t \in [0, 1]$. *Compare these examples with Equation* (2.2.1).

(*i'*) *Let* S *be a closed smooth surface and* (T^*S, λ_{st}) *its cotangent bundle. For genus* $g(S) = 0, 1$, *any exact Lagrangian* $L \subseteq (T^*S, \lambda_{st})$ *is exact Lagrangian isotopic to the zero section* S. *More generally, it is conjectured that any exact Lagrangian in* (T^*M, λ_{st}) *is exact Lagrangian isotopic to the zero section, for any closed smooth manifold* M. *This is known as* *the nearby Lagrangian conjecture and has been a driving motif for modern symplectic topology.*

(ii) Let $(W_\Lambda, \lambda_\Lambda)$ be the Weinstein 4-fold obtained by attaching a Weinstein handle along a Legendrian knot $\Lambda \subseteq (S^3, \xi)$, seen as the ideal contact boundary of $(\mathbb{R}^4, \lambda_{st})$. If Λ is the (unique) max-tb representative of a (k,n)-torus knot, $(k,n) = 1$ and $k, n \geq 2$, then $(W_\Lambda, \lambda_\Lambda)$ contains more than one isotopy class of embedded exact Lagrangians. There are infinitely many such exact Lagrangian isotopy classes if $k \geq 4$ and $n \geq 5$.[1]

(iii) In line with Example 2.2.2.(iii), let $S \subseteq M$ be a cooriented immersed submanifold. Its full conormal lift $\nu_c(S) \subseteq (T^*M, \lambda_{st})$, with scalars in the direction included, is an exact Lagrangian submanifold, with corners along the zero section. The intersection of $\nu_c(S)$ with the unit cotangent bundle is the Legendrian $\nu(S) \subseteq (T^\infty M, \xi_{st})$.

(iv) There are exact symplectic manifolds with no embedded exact Lagrangians: the Weinstein manifold $(W_\Lambda, \lambda_\Lambda)$ obtained by attaching a Weinstein handle along a stabilized Legendrian sphere $\Lambda \subseteq (S^{2n-1}, \xi)$ at the ideal boundary of $(\mathbb{R}^{2n}, \lambda_{st})$ contains no embedded exact Lagrangian.

(v) Let $f : \mathbb{C}^n \longrightarrow \mathbb{C}$ be a multivariate polynomial with an isolated singularity at the origin $0 \in \mathbb{C}^n$. This complex algebraic singularity can be analyzed via the Milnor fibration, cf. [Mil68]. This is a Lefschetz fibration $F : B_\varepsilon(0) \longrightarrow \mathscr{D}^2$ from a neighborhood $B_\varepsilon(0) \subseteq \mathbb{C}^n$ of the singularity to a disk $\mathscr{D}^2 \subseteq \mathbb{C}$, and its regular fiber $(F^{-1}(\delta), \omega_{F^{-1}(\delta)})$ inherits a natural symplectic structure from $\mathbb{C}^n$. Once certain paths are chosen in $\mathscr{D}^2$, connecting the regular value δ to the critical values, there is a collection of (exact) Lagrangian spheres in the regular fiber $F^{-1}(\delta)$, known as the vanishing cycles. Historically, via [Arn95, Section 2], these types of examples lead to the study of the symplectic aspects of Dehn twists, which sparked a significant part of modern symplectic topology, cf. [Sei99, Sei01, Sei08]. □

Another class of examples of Legendrian and Lagrangian submanifolds are graphs of contactomorphisms and symplectomorphism, respectively. Though we shall not use them prominently in these notes, these class of submanifolds have proven crucial to the development of contact and symplectic topology.

Remark 2.2.5. In both Definitions 2.2.1 and 2.2.3 we focused on the integrable case of maximal dimension. There is a theory for submanifolds that are tangential to ξ, or where ω vanishes, of dimension less than maximal. These are known as subcritical isotropic submanifolds and M. Gromov established in [Gro86, Section 3.4.2] a classification of them in terms of formal data (intuitively, strictly smooth and algebraic topology) as a consequence of his work on h-principles. See also [Gei08, Theorem 6.3.5] or [EM02, Theorem 12.4.1]. Intuitively, subcritical isotropic submanifold carry no symplectic topological information. This transpires in the computation of invariants: all known invariant in contact and symplectic topology are trivial (appropriately understood) for such submanifolds. Thus our focus on Legendrian and Lagrangian submanifolds. □

2.3. Lagrangian fillings and cobordisms

In the same manner that contact manifolds often arise as boundaries of symplectic manifolds, or at least provide suitable boundary conditions to do geometry, Legendrian subman-

[1]There are even an infinite number of exact Lagrangian classes within the same surface smooth isotopy class. In fact, many such classes are Lagrangian isotopic, though not through *exact* Lagrangians.

ifolds also provide abundant and suitable boundary conditions for Lagrangian submanifolds. An important concept for us is the following:

Definition 2.3.1 (Lagrangian filling)**.** Let (X, λ) be an exact symplectic manifold and $\Lambda \subseteq \partial(X, \lambda)$ a Legendrian submanifold of its (convex) contact boundary. By definition, a Lagrangian filling L of Λ is an embedded exact Lagrangian submanifold $L \subseteq (X, \lambda)$ s.t.:

1. L coincides with the cone of Λ outside a compact set $K \subseteq (X, \lambda)$,
2. Any primitive of $\lambda|_L$ that is constant on $L \setminus (L \cap K)$.

In this case we write $\partial L = \Lambda$.

Definition 2.3.1 is often used in practice locally near the contact boundary $(Y, \ker \alpha) = \partial(X, \lambda)$, in which case a Legendrian $\Lambda \subseteq (Y \times \{1\}, \xi)$ is given and L is a Lagrangian filling in the collar neighborhood

$$L \subseteq (Y \times [0, 1], \ker(e^t \alpha)).$$

The most important cases in these notes are of this type, with a Legendrian Λ in $(Y, \xi) = (\mathbb{R}^{2n+1}, \xi_{\text{st}})$ and $(T^\infty \mathbb{R}^n, \xi_{\text{st}})$.

Remark 2.3.2. (1) The condition that a Lagrangian submanifold on (X, λ) intersects a contact hypersurface on a Legendrian submanifold is not generic. Rather, it is a codimension-1 condition. The infinitesimal problem, modeling Lagrangians via their generating quadratic forms, illustrates that: a quadratic form intersects a contact hyperplane as a Legendrian if and only if the discriminant of the quadratic form vanishes.

(2) If one prefers the more intrinsic framework of ideal Liouville domains and their ideal contact boundaries, as in [Gir17], Definition 2.3.1 can phrased in those terms by introducing the notion of an ideal Lagrangian submanifold and its ideal Legendrian boundary. □

Remark 2.3.2 notwithstanding, many Lagrangian submanifolds that appear in applications do have a natural Legendrian boundary, or that condition can be achieved after a Hamiltonian isotopy, see e.g. [Cha10, Section 5.1]. Moving forward, we always have in mind that Legendrian submanifolds Λ serve as a boundary condition for Lagrangian submanifolds.

Example 2.3.3. *(1) Building on Example 2.2.4.(iii), for any point $x \in M$, its cotangent fiber $L = T_x^* M \subseteq (T^* M, \lambda_{st})$ is a disk Lagrangian filling of the Legendrian sphere $\partial L = T_x^\infty M \subseteq (T^\infty M, \xi_{st})$. For $M = \mathbb{R}^2$, [EP96, Theorem 1.1.A], any Lagrangian filling of the Legendrian $T_x^\infty \mathbb{R}^2 \subseteq (T^\infty \mathbb{R}^2, \xi_{st})$ is actually exact Lagrangian isotopic to the cotangent fiber $L = T_x^* M$. The analogous statement in $M = \mathbb{R}^n$, $n \geq 3$, is not known.*

(2) Following Example 2.2.4.(v), any Lefschetz thimble of an isolated singularity $f : \mathbb{C}^n \longrightarrow \mathbb{C}$ is a Lagrangian disk filling of its associated vanishing cycle. Here the vanishing cycle is understood as a Legendrian sphere in the contact boundary of a Milnor ball: in an open book presentation of that contact manifold, the Legendrian can be realized as an exact Lagrangian on the Weinstein page, which is symplectomorphic to the Milnor fiber.

(3) For an isolated plane curve singularity $f : \mathbb{C}^2 \longrightarrow \mathbb{C}$, any real Morsification $\tilde{f}$ of f provides a Lagrangian filling in the Milnor ball with Legendrian boundary given by the unique max-tb representative of the link of the singularity, cf. [Cas22, Section 2]. Smoothly, this Lagrangian filling is isotopic to the Milnor fiber. □

There is a generalization of Definition 2.3.1, allowing two types of boundaries for a Lagrangian L, its concave boundary $\partial_- L$ and its convex boundary $\partial_+ L$. We will not use the framework of ideal Liouville domains here as technically [Gir17] does not discuss concave boundaries, though similar considerations apply. We simply adopt the following more ad hoc description, briefly assuming the notion of Liouville cobordism:

Definition 2.3.4 (Lagrangian cobordism)**.** Let $(X,\lambda) : (\partial_- X, \xi_-) \longrightarrow (\partial_+ X, \xi_+)$ be a Liouville cobordism and $\Lambda_\pm \subseteq \partial_\pm(X,\lambda)$ Legendrian submanifolds of its contact boundaries. By definition, a Lagrangian cobordism $L : \Lambda_- \longrightarrow \Lambda_+$ from Λ_- to Λ_+ is an embedded exact Lagrangian submanifold $L \subseteq (X,\lambda)$ such that:

1. L coincides with the cones of $\Lambda_\pm$ away from a compact set $K \subseteq (X,\lambda)$,
2. Any primitive of $\lambda|_L$ is constant on $L \setminus (L \cap K)$.

Definition 2.3.4 specializes to Definition 2.3.1 is $\Lambda_- \subseteq \partial_- X = \emptyset$ are taken to be the empty set. Even if one if focused on Lagrangian fillings, the usefulness of Lagrangian cobordisms is that they can be concatenated, so if one has a Lagrangian cobordism $L_1 : \Lambda_- \longrightarrow \Lambda_+$ and a Lagrangian filling $L_0 : \emptyset \longrightarrow \Lambda_-$ of Λ_-, then the concatenation $L_1 \circ L_0 : \emptyset \longrightarrow \Lambda_+$ is a Lagrangian filling of Λ_+. In this manner, one can often construct Lagrangian fillings of a given Λ by first using Lagrangian cobordisms down to a simpler Legendrian submanifold and then filling the latter.

Example 2.3.5. *(1) The trace*

$$\mathrm{tr}(\Lambda_t) := \{(p,t) \in Y \times [0,1] : p \in \Lambda_t\} \tag{2.3.1}$$

of any Legendrian isotopy $\Lambda_t \subseteq (Y,\xi)$, $t \in [0,1]$, is a Lagrangian cobordism from $\Lambda_- := \Lambda_0$ to $\Lambda_+ := \Lambda_1$ in the Liouville manifold $(Y \times [0,1], e^t\alpha)$, where $\xi = \ker\alpha$.

(2) Following Example 2.3.3.(3), an adjacency of singularities endowed with real Morsifications, see e.g. [AdGiZV85, Section 15.0.2], yields a Lagrangian cobordism between the max-tb Legendrian representatives of the links of the singularities, arguing as in [Cas22]. Specifically, for a simple adjacency $f \leftarrow g$, there is a Lagrangian cobordism $L : \Lambda_f \longrightarrow \Lambda_g$ with a unique saddle point.

(3) The study of parametric Morse theory, through generating families, provides many interesting instances of Lagrangian cobordisms, see e.g. [BST15]. Intuitively, these are certain quotients of a relative and parametric generalization of Example 2.2.4, with boundaries described by quotients of a parametric version of Equation (2.2.1). □

Remark 2.3.6 (On obstructed fillings and cobordisms)**.** It is a subtle task to decide whether a given general Lagrangian filling or cobordism interacts well (and non-trivially) with other geometric objects, such as pseudoholomorphic curves or sheaves. This leads to the notion of a Lagrangian being *obstructed* or *unobstructed*, whose precise definition is context dependent but in essence tries to capture whether Floer-theoretic of sheaf-theoretic invariants, for example, are well-defined and non-trivial for such a Lagrangian. Important points are:

1. Embedded exact Lagrangian fillings and cobordisms are unobstructed. Therefore, if we are working with embedded and exact, there are no further considerations.
2. Immersed exact fillings (or cobordisms) might be unobstructed or obstructed. In general, more singular exact Lagrangians might be unobstructed or obstructed. To date, there is not a immediate criterion being able to assess whether a general immersed filling is obstructed or not. There are many important examples where unobstructed immersed Lagrangian fillings have a key role.

3. There is an h-principle for immersed Lagrangians, see [Gro86, Section 3.4.4.(G)]. This abundance of immersed Lagrangians that one can obtain by an h-principle almost always yields obstructed Lagrangians. The aim of finding unobstructed immersed Lagrangian fillings is a subtle and interesting task which, to our knowledge, does not abide by any type of h-principle. □

2.4. Legendrian fronts

A smooth fibration $\pi : (Y, \xi) \longrightarrow B$ is said to be Legendrian if the fibers of π are Legendrian submanifolds of Y. Legendrian fibrations are also known as front projections. Note that for a Legendrian fibration, we must have $\dim(B) = n$.

Definition 2.4.1. Let $\Lambda \subseteq (Y, \xi)$ be a Legendrian submanifold and $\pi : (Y, \xi) \longrightarrow B$ a Legendrian fibration. By definition, the front of the Legendrian Λ with respect to the fibration π is the subset $\pi(\Lambda) \subseteq B$. □

Legendrian fibrations are useful because, in important cases, we can recover a Legendrian $\Lambda \subseteq (Y, \xi)$ by studying the image $\pi(\Lambda) \subseteq B$:

1. The advantage of studying $\pi(\Lambda) \subseteq B$ is that it is an $(n-1)$-dimensional subset of an n-dimensional manifold B. This is often more amenable for study than an ambient $(2n-1)$-dimensional manifold (Y, ξ).

2. The cost of decreasing dimension in (1) is that $\pi(\Lambda) \subseteq B$ is typically singular, with singularities that often go beyond immersed points.

Intuitively, studying Legendrians $\Lambda \subseteq (Y, \xi)$ via their front projections $\pi(\Lambda)$ trades dimension for singularities.

Example 2.4.2. *(1) Consider* $(\mathbb{R}^{2n-1}, \xi_{st})$ *with* $\xi_{st} = \{dz - y_1 dx_1 - \dots y_{n-1} dx_{n-1}\}$ *as in Example 2.1.2.(1). Then*

$$\pi : (\mathbb{R}^{2n-1}, \xi_{st}) \longrightarrow \mathbb{R}^n, \quad (x_1, y_1, \dots, x_{n-1}, y_{n-1}, z) \longmapsto (x_1, \dots, x_{n-1}, z) \tag{2.4.1}$$

is a Legendrian fibration. Any Legendrian submanifold $\Lambda \subseteq (\mathbb{R}^{2n-1}, \xi_{st})$ *is uniquely determined by* $\pi(\Lambda)$*, as we can recover the coordinates* y_i *along* Λ *via* $y_i = \partial_{x_i} z$*. In these notes, the most important cases will be* $n = 2, 3$*. For* $n = 2$*, the front* $\pi(\Lambda) \subseteq \mathbb{R}^2_{x,z}$ *of a Legendrian link* $\Lambda \subseteq (\mathbb{R}^3, \xi_{st})$ *generically has the singularities depicted in Figure 2.1:*

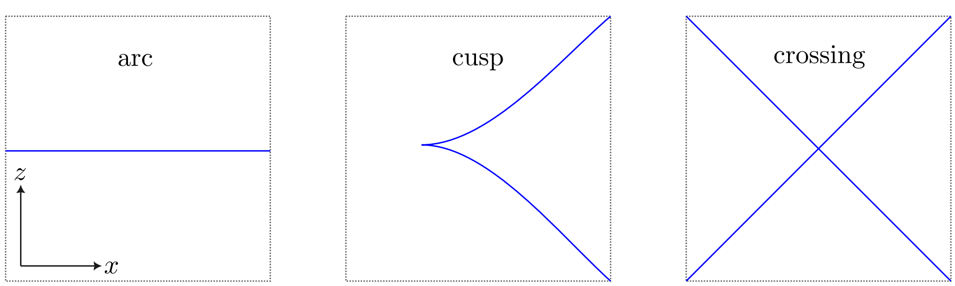


Figure 2.1: The generic singularities of a front of a Legendrian link $\Lambda \subseteq (\mathbb{R}^3, \xi_{\text{st}})$.

These singularities are locally modeled as

$$\textit{arc} := \{z = 0\}, \quad \textit{cusp} := \{x^3 = z^2\}, \quad \textit{crossing} := \{(x - z)(x + z) = 0\}.$$

For this front projection $\pi : \mathbb{R}^3_{x,y,z} \longrightarrow \mathbb{R}^2_{x,z}$ *the* y*-coordinate of* Λ *is uniquely determined as* $y = \partial_x z(x)$*, the* x*-slope of* z*. These local model can be glued together to form more elaborate Legendrian knots, see for instance two fronts depicted in Figure 2.2.*

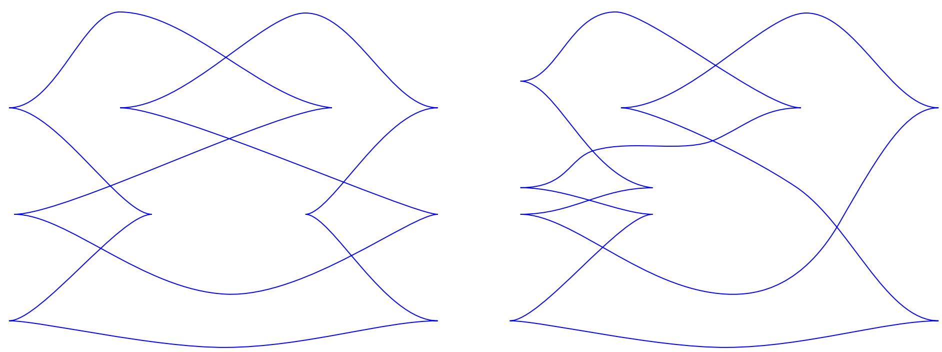


Figure 2.2: Two fronts for two Legendrian knots $\Lambda \subseteq (\mathbb{R}^3, \xi_{\mathrm{st}})$, fronts drawn in $\mathbb{R}^2_{x,z}$. The smooth type of both these knots in $\mathbb{R}^3$ is that of $m(5_2)$, the mirror of the three twist knot: the only 5-crossing knot apart from the $(2,5)$-torus knot. These Legendrian knots are Legendrian isotopic but not smoothly isotopic, as first proven in [Che02, Example 4.4].

(2) Consider $(T^\infty M, \xi_{st})$ *as in Example 2.1.2.(2). The restriction*

$$\pi : (T^\infty M, \xi_{st}) \longrightarrow M \tag{2.4.2}$$

of the projection $T^*M \longrightarrow M$ *to the zero section is a Legendrian projection. The front* $\pi(\Lambda) \subseteq M$*, with the additional information of a coorientation, recovers* $\Lambda \subseteq (T^\infty M, \xi_{st})$*. We indicate coorientation by drawing a segment in* $M \setminus \pi(\Lambda)$ *ortoghonal to* $\pi(\Lambda)$ *with one endpoint in* $\pi(\Lambda)$*. In particular, any cooriented immersed curve in* $\mathbb{R}^2_{q_1,q_2}$ *defines a Legendrian link in*

$$(T^\infty\mathbb{R}^2, \xi_{st}) \cong (\mathbb{R}^2_{q_1,q_2} \times S^1_\theta, \ker\{(\cos\theta)dq_1 + (\sin\theta)dq_2\}). \tag{2.4.3}$$

Specifically, a parametrization $\gamma : \mathbb{R}_t \longrightarrow \mathbb{R}^2_{q_1,q_2}$ *lifts to a Legendrian curve* $(\gamma(t), \theta(t))$*, where* $\theta(t)$ *is the unique angle such that* $(\sin(\theta(t)), -\cos(\theta(t))) = \partial_t\gamma(t)$*,* $t \in \mathbb{R}$*.*[2] *See Figure 2.3 for two more fronts for this projection.*

(3) The same contact manifold (Y, ξ) *might admit different Legendrian fibrations, and each of them might be useful in its own way. For instance, there is the following variation on the contactomorphism in Equation* (2.4.3)*:*

$$(\mathbb{R}^2_{q_1,q_2} \times S^1_\theta, \ker\{(\cos\theta)dq_1 + (\sin\theta)dq_2\}) \cong (J^1 S^1_\vartheta, \ker\{dz - \lambda_{st})), \tag{2.4.4}$$

[2] Here we use implicitly that an oriented curve in the oriented $\mathbb{R}^2_{q_1,q_2}$ is equivalent to a cooriented curve.

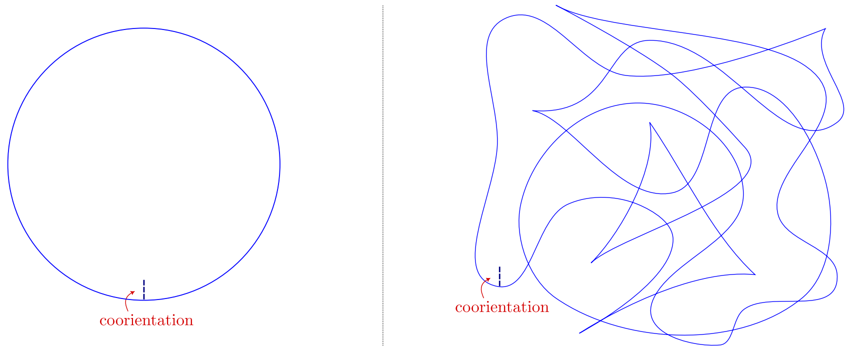


Figure 2.3: Two fronts in $\mathbb{R}^2$, which lifts to Legendrian knots in $(T^\infty\mathbb{R}^2, \xi_{\text{st}})$. Note the segment marking the coorientation needed to specify a unique Legendrian lift. (Left) This Legendrian knot is isotopic to the fiber of the front projection $T^\infty\mathbb{R}^2 \longrightarrow \mathbb{R}^2$. (Right) Any cooriented immersed curve in $\mathbb{R}^2$, possibly with simple cusp singularities, such as the one depicted here, determines a unique Legendrian knot in $(T^\infty\mathbb{R}^2, \xi_{\text{st}})$.

where $J^1S^1 := T^*S^1 \times \mathbb{R}$, $\lambda_{st} = \kappa d\vartheta \in \Omega^1(T^*S^1)$ the Liouville form, and the coordinates are $(\vartheta, \kappa; z) \in T^*S^1 \times \mathbb{R}_z$, cf. Example 2.1.2.(3). In the presentation of the right hand side, as a 1-jet space, the following projection is a Legendrian fibration

$$\Pi : (J^1S^1_\vartheta, \ker\{dz - \lambda_{st}\}) \longrightarrow S^1_\vartheta \times \mathbb{R}_z, \quad (\vartheta, \kappa; z) \longmapsto (\vartheta, z). \tag{2.4.5}$$

The Legendrian fibers of Π are diffeomorphic to $\mathbb{R}$, whereas the Legendrian fibration π from Equation (2.4.2) *on the contactomorphic contact manifold $(T^\infty\mathbb{R}^2, \xi_{st})$ has Legendrian fibers diffeomorphic to S^1.*

For instance, Figure 2.4 depicts the images under Π and π of two Legendrians $\Lambda_1, \Lambda_2 \subseteq (T^\infty\mathbb{R}^2, \xi_{st})$, where Λ_1 is a Legendrian fiber of Π, and Λ_2 is the result of applying the Euclidean lift of the geodesic flow to it for a few seconds. Note that the Legendrian fronts $\Pi(\Lambda_1), \Pi(\Lambda_2) \subseteq S^1 \times \mathbb{R}$ suffice to recover $\Lambda_1, \Lambda_2 \subseteq (T^\infty\mathbb{R}^2, \xi_{st})$, whereas $\pi(\Lambda_1), \pi(\Lambda_2)$ are not sufficient: the additional data of the coorientation, and more generally, the conormal lift directions, are necessary to recover Λ_1, Λ_2.

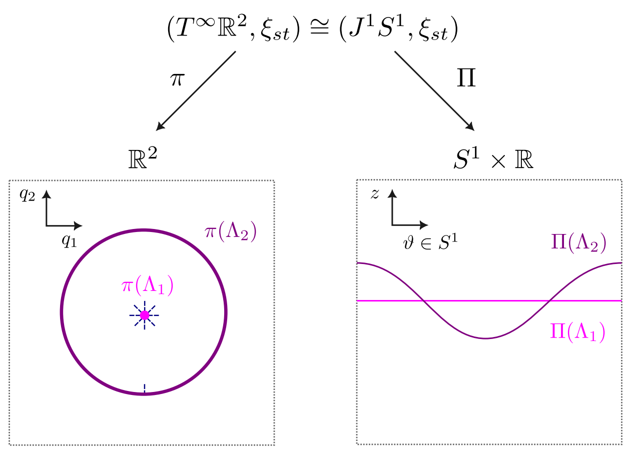


Figure 2.4: Two instances of Legendrian knots in $(T^\infty\mathbb{R}^2, \xi_{\text{st}})$ and their fronts under different Legendrian projections. (Left) According to the Legendrian fibration in Equation (2.4.2). (Right) Via the Legendrian fibration Equation (2.4.5), after using a contactomorphism to identify $(T^\infty\mathbb{R}^2, \xi_{\text{st}})$ with $(J^1S^1, \xi_{\text{st}})$.

(3) More generally, the contact manifolds (J^1M, ξ_{st}) *from Example 2.1.2.(3) admit the Legendrian fibrations*

$$\Pi : (J^1M, \xi_{st}) \longrightarrow M \times \mathbb{R}, \quad (q, p; z) \longmapsto (q, z),$$

where $(q, p) \in T^*M$ *is a point in the cotangent bundle. The contactomorphism obtained by concatenating Equation* (2.4.3) *and Equation* (2.4.4) *generalizes to a contactomorphism*

$$(T^\infty\mathbb{R}^n, \xi_{st}) \cong (J^1S^{n-1}, \xi_{st}),$$

as in Example 2.1.2.(3). The natural Legendrian projection on the left hand side, as in Equation (2.4.2)*, has* S^{n-1}*-fibers. The natural Legendrian projection on the right hand side, as in 2.4.5, has* $\mathbb{R}^{n-1}$*-fibers.* □

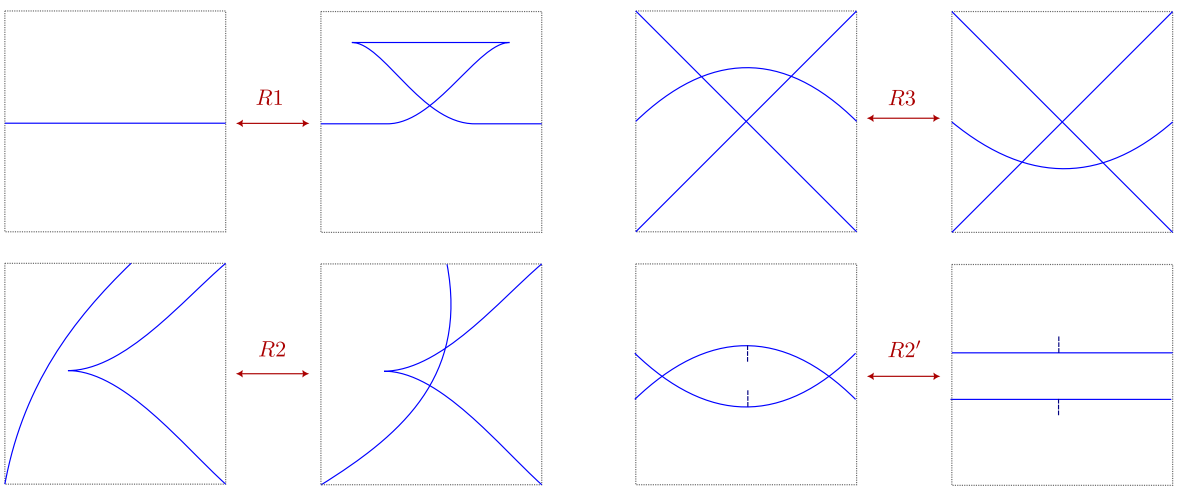


Figure 2.5: Legendrian Reidemeister moves in the front projection. Each of these moves can be realized by a Legendrian isotopy. Conversely, in $(\mathbb{R}^3, \xi_{\mathrm{st}})$ and $(T^\infty\mathbb{R}^2, \xi_{\mathrm{st}})$ every Legendrian isotopy is itself isotopic to a Legendrian isotopy whose front projections are exactly a sequence of these Reidemeister moves. There are symmetric versions of these moves which we always implicitly include: R1 upside-down, R2 with the cusp pointing east, and R2' with both coorientations in the bigon also pointing opposite but outwards.

2.4.1 Legendrian isotopies in front projections

Let (Y, ξ) be a contact manifold and $\pi : (Y, \xi) \longrightarrow B$ a front projection. It is often useful to visualize a Legendrian isotopy $\Lambda_t \subseteq (Y, \xi)$, $t \in [0, 1]$ via the fronts $\pi(\Lambda_t)$. Conversely, manipulating fronts in B is an effective way to construct Legendrian isotopies. In the local case of $(\mathbb{R}^3, \xi_{\mathrm{st}})$ and the front projection in Example 2.4.2.(1), the following result is helpful:

Theorem 1 (Legendrian link isotopies via front diagrams)**.** The following hold:

(1) Let $\Lambda_0, \Lambda_1 \subseteq (\mathbb{R}^3, \ker\{dz - ydx\})$ be two Legendrians whose fronts $\pi(\Lambda_0), \pi(\Lambda_1) \subseteq \mathbb{R}^2_{x,z}$ are generic. Then

$$\Lambda_0, \Lambda_1 \text{ Legendrian isotopic} \Longleftrightarrow \pi(\Lambda_0), \pi(\Lambda_1) \text{ related by the moves } R1, R2, R3,$$

where the Legendrian Reidemeister moves are depicted in Figure 2.5.

(2) Let $\Lambda_0, \Lambda_1 \subseteq (T^\infty \mathbb{R}^2_{q_1,q_2}, \xi_{st})$ be two Legendrians whose fronts $\pi(\Lambda_0), \pi(\Lambda_1) \subseteq \mathbb{R}^2_{q_1,q_2}$ are generic. Then

$$\Lambda_0, \Lambda_1 \text{ Legendrian isotopic} \Longleftrightarrow \pi(\Lambda_0), \pi(\Lambda_1) \text{ related by the moves } R1, R2, R2', R3,$$

where these Legendrian Reidemeister moves are depicted in Figure 2.5.

Theorem 1.(1) is proven in [Swi92, Theorem B], see also [Etn05, Section 2.3]. Theorem 1.(2) follows from its Part (1). For more results on moves for front diagrams, including higher-dimensions, see [Arn90b, Section 3.3]. Specifically, in Figure 48 therein, the Legendrian Reidemeister moves for Legendrian surface fronts are displayed as $A_4, D_4^{\pm}$.[3]

2.4.2 Lagrangian cobordisms in front projections

A practical way to construct Lagrangian cobordisms $L : \Lambda_- \to \Lambda_+$ between two Legendrian submanifolds Λ_-, Λ_+ is to use their front projections. The general theory of perestroikas of fronts is surveyed in [Arn90b, Section 3]. Since our focus is on Legendrian links and (surface) Lagrangian cobordisms between them, it suffices that we use the following result:

Theorem 2 (Lagrangian cobordism between Legendrian links via front diagrams)**.** Let $\Lambda_- \subseteq (\mathbb{R}^3, \xi_{st})$ be a Legendrian whose front $\pi(\Lambda_-)$ contains a piece as in the left diagrams of Figure 2.6. Then there exists an exact Lagrangian cobordism

$$L : \Lambda_- \longrightarrow \Lambda_+, \quad L \subseteq (\mathbb{R}^3 \times \mathbb{R}, d(e^t(dz - ydx))),$$

with concave end Λ_- and convex end Λ_+ such that Λ_+ admits a front given by $\pi(\Lambda_-)$ locally modified as depicted in right diagrams of Figure 2.6. In addition:

1. L is embedded for $S0, S1$ and D_4^-. The restriction of the function $t : \mathbb{R}^3 \times \mathbb{R} \longrightarrow \mathbb{R}$ to L has a unique critical point in these cases, of index 0 (i.e. a minimum) for $S0$ and of index 1 (i.e. a saddle) for $S1$ and D_4^-.

2. L is immersed for the clasp move and the restriction of the function t to L has no critical point (so L is diffeomorphic to $\Lambda_- \times \mathbb{R}$).

The move $S1$ can be refined with orientations: if the two strand on the right of Figure 2.6.(S1) are oriented in opposite directions, then the resulting Lagrangian saddle cobordism is oriented, otherwise it is a non-orientable Lagrangian cobordism.

[3] These are the strictly new moves for Legendrian surfaces in $(\mathbb{R}^5, \xi_{st})$: for a complete list, one should add those arising as traces of moves in lower dimensions as well, such as A_3 therein.

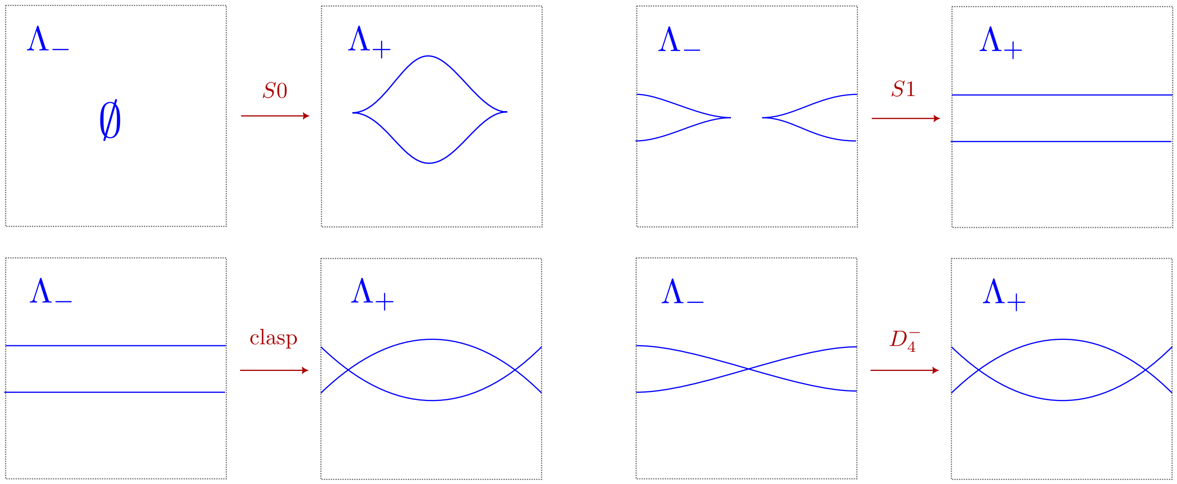


Figure 2.6: Ends of Lagrangian cobordisms, including two Legendrian surgeries, in the front projection. Each of these moves depicts the concave (bottom) end of the associated Lagrangian cobordism on the left and and convex (top) end on the right. The move $S0$ corresponds to a Lagrangian 0-handle, i.e. a minimum, and move $S1$ to a Lagrangian 1-handle, i.e. a saddle cobordism. These two are Legendrian surgeries and the associated Lagrangian cobordisms are embedded. The clasp move yields an *immersed* Lagrangian cobordism. Move D_4^- yields an embedded Lagrangian cobordism, with a unique saddle critical point, and it is obtained by performing Legendrian isotopies, then an $S1$-surgery and then further isotopies. The coorientations, which are not drawn explicitly, are all downwards.

Example 2.4.3. *(1) $S0$ states that the standard Legendrian unknot has a Lagrangian disk filling and, as stated in Example 2.3.3.(1), it is unique. Therefore, if $\Lambda_- \subseteq (\mathbb{R}^3, \xi_{st})$ is the standard Legendrian n-unlink, any Lagrangian cobordism $L : \Lambda_- \longrightarrow \Lambda$ can be filled at the bottom Λ_- to produce a Lagrangian filling of Λ.*

(2) Since the four moves in Figure 2.6 often suffice to simplify any Legendrian front to an unlink, it is frequently simple to produce a Lagrangian filling of a Legendrian link. The issue is that, if the clasp move has been used, such Lagrangian filling will likely be immersed and obstructed, cf. Remark 2.3.6. □

2.5. Examples of Legendrian submanifolds

Let us discuss a sample of sources for Legendrian submanifolds.

2.5.1 Conormal lifts

Expanding on Example 2.2.2.(iii), let $S \subseteq M$ be an immersed submanifold. Its conormal lift

$$\nu(S) := \{(q,p) \in T^\infty M : q \in S, p|_{TS} \equiv 0\} \subseteq (T^\infty M, \xi_{\mathrm{st}})$$

is a Legendrian submanifold. The front projection of $\nu(S)$ via the Legendrian projection $\pi : (T^\infty M, \xi_{\mathrm{st}}) \longrightarrow M$ from Equation (2.4.2) is the submanifold $S \subseteq M$ itself. If S is cooriented, we can further restrict p to be in the coorientation. The most used case is that of a cooriented hypersurface $S \subseteq M$, in which case we can choose the unique codirection aligning with the coorientation, known as the positive lift, or its opposite, known as the negative lift. These two conormal lifts of a cooriented hypersurface $S \subseteq M$ are denoted by $\nu_\pm(S) \subseteq (T^\infty M, \xi_{\mathrm{st}})$.

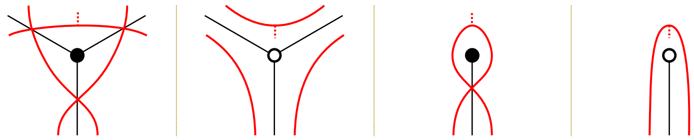


Figure 2.7: The local models for the front, a.k.a. alternating strand diagrams, for a plabic graph $\mathbb{G}$. The cooriented front defined from a plabic graph $\mathbb{G}$ in this manner yields a Legendrian link $\Lambda(\mathbb{G}) \subseteq (T^\infty\mathbb{R}^2, \xi_{st})$, or generally in $(T^\infty S, \xi_{st})$ if $\mathbb{G}$ is embedded in a surface S.

Plabic graphs

Alternating strand diagrams are specific type of cooriented immersed curve constructed from bicolored graphs, often planar. They are useful to relate Legendrian links to the algebraic combinatorics of total positivity and networks, as first introduced in [Pos06, Section 14]. For simplicity, we describe these as follows: let S be a smooth surface and $\mathbb{G} \subseteq S$ a graph with only univalent and trivalent vertices and a choice of color (black or white) on each vertex. To such $\mathbb{G} \subseteq S$, we can associated a cooriented immersed curve $\gamma(\mathbb{G}) \subseteq S$, known as the alternating strand diagram, using the local models in Figure 2.7. As in Example 2.4.2, such $\gamma(\mathbb{G})$ defines a Legendrian link

$$\Lambda(\mathbb{G}) \subseteq (T^\infty S, \xi_{st})$$

by declaring that $\gamma(\mathbb{G})$ is its cooriented front. Legendrian links of the form $\Lambda(\mathbb{G})$ in fact all admit embedded exact Lagrangian fillings, cf. [CL23, Section 2] and [STWZ19, Section 4.2]. See Figure 2.8 for some examples of such Legendrian link $\Lambda(\mathbb{G})$.

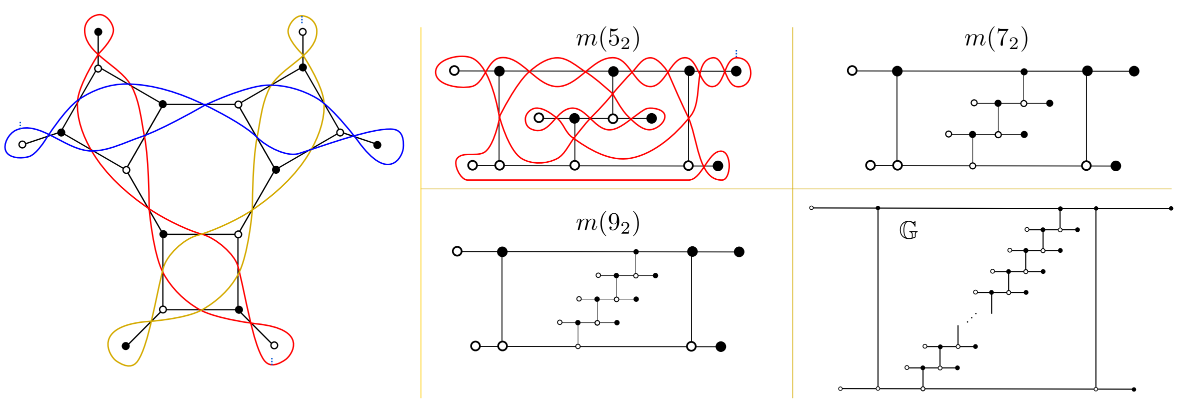


Figure 2.8: Instances of plabic graphs $\mathbb{G}$ and some of the fronts $\gamma(\mathbb{G})$ for their associated Legendrians $\Lambda(\mathbb{G}) \subseteq (T^\infty\mathbb{R}^2, \xi_{st})$. In these examples, all the Legendrian link $\Lambda(\mathbb{G})$ are actually contained in a Darboux ball $(\mathbb{R}^3, \xi_{st})$ inside of $(T^\infty\mathbb{R}^2, \xi_{st})$. (Left) The Legendrian $\Lambda(\mathbb{G})$ is a 3-component link, each component is a standard Legendrian unknot and the smooth link type is that of $(3,3)$-torus link. (Right) The Legendrians $\Lambda(\mathbb{G}) \subseteq (\mathbb{R}^3, \xi_{st})$ associated to these plabic graphs are all smoothly twist knots. For instance, the plabic graph labeled by $m(5_2)$ has $\Lambda(\mathbb{G})$ Legendrian isotopic to the Legendrian knot in Figure 2.2(left).

Remark 2.5.1. Another beautiful source of Legendrian submanifold is the theory of irregular singularities of algebraic differential equations. In the 1-dimensional case, where the differential equation occurs on a Riemann surface, the Stokes diagram of an irregular singularity is a cooriented immersed curve, and thus defines a Legendrian link, as above, see e.g. [CN25, Example 2.5] and references therein. □

2.5.2 Generating families

The study of Cerf diagrams, initiated in [Cer70], is a cornerstone of 1-parametric Morse theory, with Cerf's pseudo-isotopy theorem being a 1-parametric version of h-cobordism theorem, see [Cer70, Theorem 0] and respectively [Sma61, Theorem I], or [Sma62, Theorem 1.1]. One-parametric Morse theory is closely related to the study of Legendrian links: Cerf diagrams are understood as fronts. In general, we can consider families of generalized Morse functions $f_x : \mathbb{R}^N \longrightarrow \mathbb{R}$, for some $N \in \mathbb{N}$, parametrized by points $x \in M$ on smooth manifold M and obtain Legendrian submanifolds of $(J^1M, \xi_{\mathrm{st}})$. Specifically, given such a family $\mathfrak{f} := \{f_x\}_{x\in M}$ we can define the Legendrian submanifold

$$\Lambda_{\mathfrak{f}} := \{(x, df_x(q), f_x(q)) \in T^*M \times \mathbb{R}_z : q \in \mathbb{R}^N \text{ is a critical point of } f_x\} \subseteq (J^1M, \xi_{\mathrm{st}}).$$

The analytic condition on $q \in \mathbb{R}^N$ being a critical point of f_x is $df_x(q) = 0$. The front of $\Lambda_{\mathfrak{f}}$ under the Legendrian fibration from Example 2.4.2 is the set

$$\begin{aligned}\Pi(\Lambda_{\mathfrak{f}}) &= \{(x, z) \in M \times \mathbb{R}_z : z \text{ is a critical value of } f_x\}\\ &= \{(x, z) \in M \times \mathbb{R}_z : \exists q \in \mathbb{R}^N \text{ s.t. } df_x(q) = 0 \text{ and } f_x(q) = z\}.\end{aligned}$$

For instance, the fronts in Figure 2.9 are examples of fronts that arise in this manner, with $M = \mathbb{R}$. The set $\Pi(\Lambda_{\mathfrak{f}})$ is understood as a higher-dimensional analogue of a Cerf diagram. Technically, since $\mathbb{R}^N$ is non-compact, one often constraints the behavior of each f_x at infinity, e.g. linear or quadratic at infinity. Legendrian submanifolds of the form $\Lambda_{\mathfrak{f}}$, for a generating family $\mathfrak{f}$ linear at infinity, tend to have unobstructed Lagrangian fillings.[4] There are also more general constructions, allowing each Morse function $f_x : Q \longrightarrow \mathbb{R}$ to be an arbitrary family or considering non-trivial Q-bundles over M and Morse functions on them. The existence of a generating family for a given Legendrian (o Lagrangian) submanifold is still an active area of research. To wit, it was only recently established that nearby Lagrangians admit (twisted) generating functions if certain formal obstructions vanish, cf. [ACGK25, Section 1.2].

Remark 2.5.2. For generating families in contact topology, original sources are [Cha95, Section 1.3] and [Che96a, Section 2]. A starting place to learn is also [Tra01, Section 3] and [FR11, Section 4], plus see [BST15, Section 2] and references therein. □

2.5.3 The case study: Legendrian links from positive braids

Let $\beta \in \mathrm{Br}_n^+$ be a positive braid word in n-strands. If we draw the braid diagram in the plane, with crossings drawn as actual crossings (not overcrossings nor undercrossings), then we obtain a collection immersed curves. These can be closed up to a collection of immersed circles in different ways. For instance, we can consider a front as in Figure 2.9 (right), of which Figure 2.9 (left) is an instance where $\beta = \sigma_3\sigma_2^2\sigma_3\sigma_1^2\sigma_3\sigma_2\sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_3\sigma_1\sigma_3\sigma_1\sigma_2$. Legendrian in this class of examples are denoted by $\Lambda = \Lambda_\beta \subseteq (\mathbb{R}^3, \xi_{\mathrm{st}})$ and be studied thoroughly in Chapter 6. In particular we will establish that they have (often many) Lagrangian fillings if the braid is of the form βw_0 and the Demazure product of β is w_0, e.g. if the braid is of the form $w_0\beta w_0$.

[4] Appropriately understood, I would be inclined to argue that they *always* admit an unobstructed Lagrangian filling.

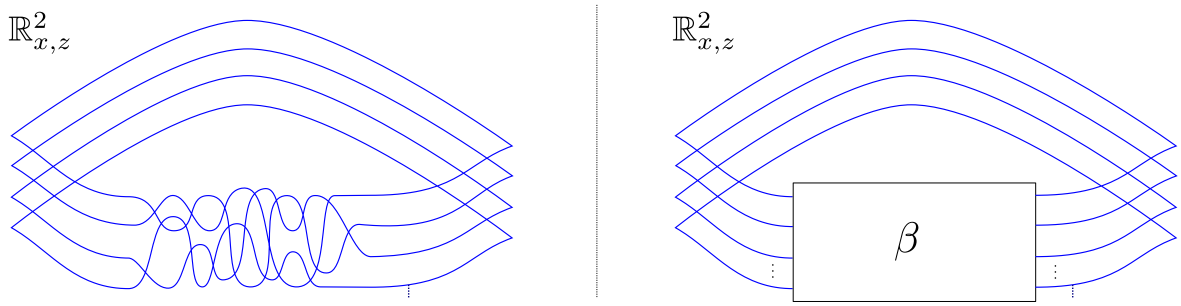


Figure 2.9: Two fronts in $\mathbb{R}^2_{x,z}$ that define Legendrian links in $(\mathbb{R}^3, \ker\{dz - ydz\}$. These types of Legendrian links will be studied in Chapter 6. They can be understood as Cerf diagrams of generating families linear at infinity, as the examples in Section 2.5.2.

2.5.4 Cone singularity and spinning

Consider the hyperspherical coordinates in $\mathbb{R}^n_{x_1,\dots,x_n}$ given by

$$\begin{aligned}
x_1(r,\varphi) &= r\cos(\varphi_1)\\
x_2(r,\varphi) &= r\sin(\varphi_1)\cos(\varphi_2)\\
x_3(r,\varphi) &= r\sin(\varphi_1)\sin(\varphi_2)\cos(\varphi_3)\\
&\vdots\\
x_{n-1}(r,\varphi) &= r\sin(\varphi_1)\sin(\varphi_2)\dots\sin(\varphi_{n-2})\cos(\varphi_{n-1})\\
x_n(r,\varphi) &= r\sin(\varphi_1)\sin(\varphi_2)\dots\sin(\varphi_{n-2})\sin(\varphi_{n-1})
\end{aligned}$$

where the radial coordinate is given by $r = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2} \in \mathbb{R}_{\geq 0}$ and the hyperspherical coordinates on S^{n-1} are:

$$\varphi_i \in [0,\pi] \quad i \in [1, n-2], \quad \varphi_{n-1} \in [0, 2\pi).$$

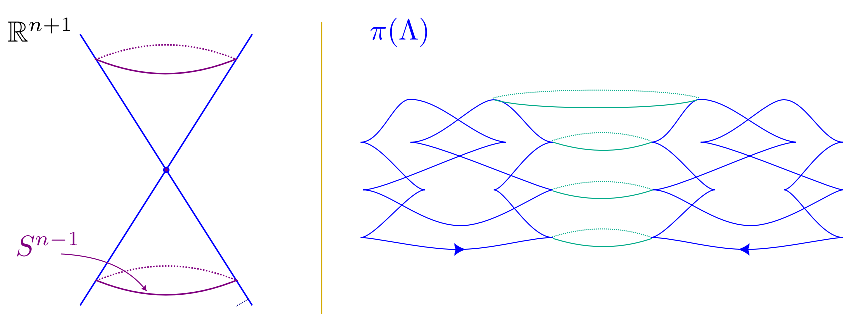


Figure 2.10: (Left) The front projection known as the cone singularity, which lifts to an embedded Legendrian $S^{n-1} \times \mathbb{R} \subseteq (\mathbb{R}^{2n+1}m, \xi_{\text{st}})$. (Right) An example of front spinning, in this case spinning the front in Figure 2.2 of a Legendrian knot.

The image $C_n \subseteq (\mathbb{R}^{2n+1}, \ker\{dz - y_1dx_1 - \dots y_ndx_n\})$ of the smooth embedding

$$c_n : S^{n-1}_\varphi \times \mathbb{R}_r \longrightarrow \mathbb{R}^{2n+1}$$

$$c_n(\varphi, r) := (x_1(r,\varphi), x_1(r,\varphi)/r, \dots, x_n(r,\varphi), x_n(r,\varphi)/r, r) \tag{2.5.1}$$

is a Legendrian submanifold, diffeomorphic to $S^{n-1} \times \mathbb{R}$. Its front projection is the cone

$$\pi(C_n) := \{z = \pm\sqrt{x_1^2 + \ldots + x_n^2}\} \subseteq \mathbb{R}^n_{x_1,\ldots,x_n} \times \mathbb{R}_z,$$

as depicted in Figure 2.10(left), with its conormal coorientation. Note that the entire sphere $S^{n-1} \times \{0\}$ maps onto a point in the front: the origin in $\pi(C_n)$. The germ of this front $\pi(C_n) \subseteq \mathbb{R}^{n+1}$ at the origin is known as the *cone singularity*, see [CM19, Section 2.4.1]. The study of the generic behavior of this front singularity dates back to A. Cayley's study on the shape of the surface containing all the centers of the principal curvatures of ellipsoids, cf. [Cay09, Chapter 145] and see also [Ali24, Section 3]. Specifically, the $n = 1, 2$ cases are reasonably understood, see e.g. [DR11, Section 3] for $n = 2$. The case $n = 3$ was started to be studied by A. Cayley and, in many ways, still remains to be understood. The higher-dimensional cases are yet to be explored.

The cone singularity can be completed to a front for many closed Legendrians. One of the simplest non-trivial fronts for a closed Legendrian submanifold that contains a cone point is as depicted in Figure 2.11. We denote by Λ_c its Legendrian lift and refer to it as the Cayley Legendrian, in honor of his work on caustics.

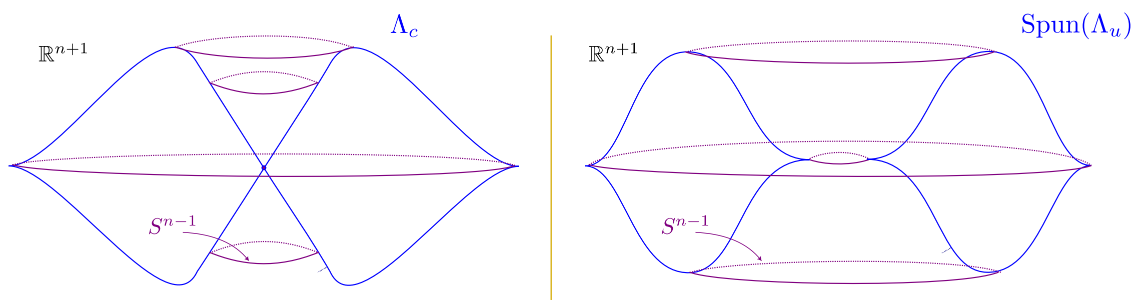


Figure 2.11: (Left) The front projection of a Legendrian $S^{n-1} \times S^1 \subseteq (\mathbb{R}^{2n+1}, \xi_{\mathrm{st}})$ with a cone singularity. We refer to its Legendrian lift as the Cayley Legendrian. (Right) The front projection of another Legendrian $\mathrm{Spun}(\Lambda_u) \subseteq (\mathbb{R}^{2n+1}, \xi_{\mathrm{st}})$, also diffeomorphic to $S^{n-1} \times S^1$, obtained by front spinning the front projection of a standard unknot in $(\mathbb{R}^3, \xi_{\mathrm{st}})$.

An independent construction of fronts that can be useful is front spinning, whereby one considers an axis disjoint from $\pi(\Lambda)$ and spins its front in higher-dimensions to create a new front for a new Legendrian, see e.g. Figure 2.10(right) and Figure 2.11(right). This is akin to the spinning of knots in smooth topology, as introduced in [Eps60, Spun knots] and [Zee65, Section 6]. See [EES05, Section 4.4] and [EES09, Example 5.3], also [Gol14, Section 2], for details of the construction in the Legendrian setting.

2.6. Lagrangian disk surgery

Let $L \subseteq (W, \lambda)$ be an embedded exact Lagrangian filling of a Legendrian submanifold $\Lambda = \partial L \subseteq \partial(W, \xi)$. To construct new Lagrangian fillings that are *not* Hamiltonian isotopic to L, we can try to perform a local operation to L. A local operation near a point $p \in L$ can be challenging to find and, for instance, [EP96, Theorem 1.1.A] implies that local knotting is *not* possible for Lagrangian surfaces.

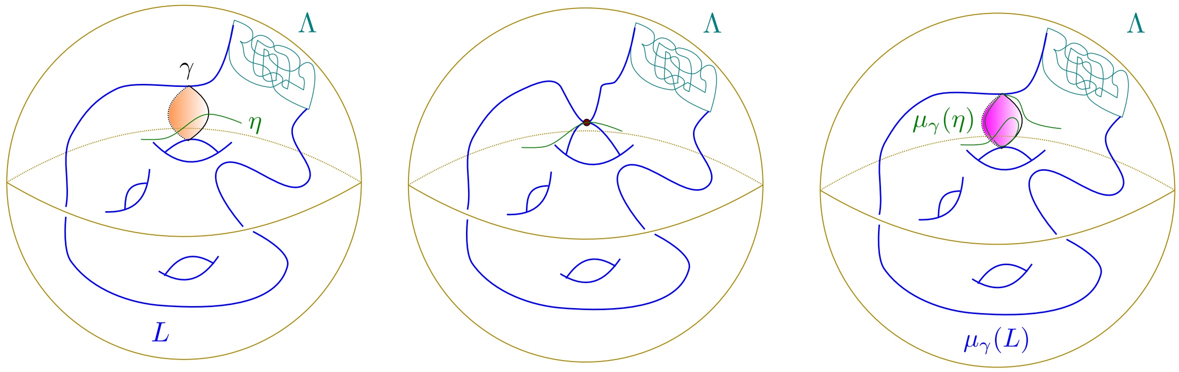


Figure 2.12: A Lagrangian filling $L \subseteq (\mathbb{D}^4, \lambda_{\text{st}})$ of a Legendrian link $(\Lambda \subseteq \partial\mathbb{D}^4, \xi_{\text{st}})$ undergoing Lagrangian disk surgery. (Left) The Lagrangian disk $\mathscr{D} \subseteq \mathbb{D}^4 \setminus L$ being surgered is depicted in orange, with boundary $\partial\overline{\mathscr{D}} = \gamma \subseteq L$. (Center) The critical moment of the surgery, where the filling L degenerates to an immersed exact Lagrangian. (Right) The Lagrangian filling $\mu_{\mathscr{D}}(L)$, also denoted $\mu_\gamma(L)$, result of the surgery. Here the new Lagrangian disk $\mu_{\mathscr{D}}(\mathscr{D}) \subseteq \mathbb{D}^4 \setminus \mu_\gamma(L)$ after the surgery is depicted in purple, with boundary $\gamma \subseteq \mu_\gamma(L)$ oriented in reverse.

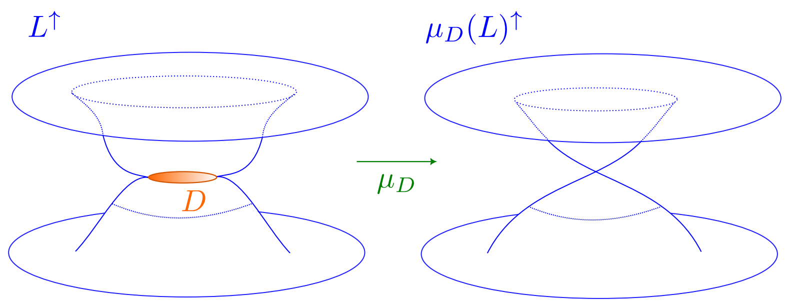
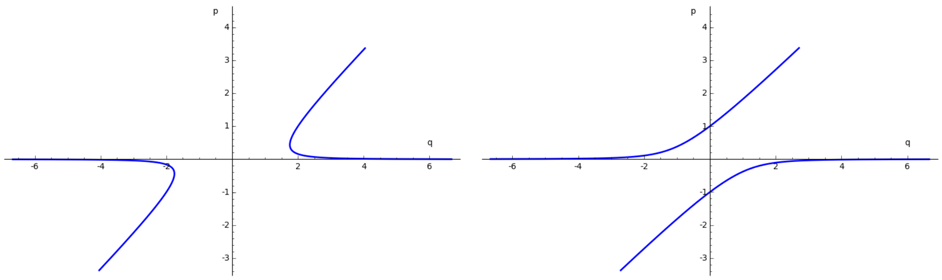


Figure 2.13: The effect of Lagrangian disk surgery seen in fronts for the Legendrian lifts $L^\uparrow$ and $\mu_{\mathscr{D}}(L)^\uparrow$ of L and $\mu_{\mathscr{D}}(L)$. These fronts can be understood in arbitrary dimensions, as spherical spuns of the cusp, for the left front, and the cone point, for the right front. The disk $\mathscr{D}$, in orange, is the same disk in orange as in Figure 2.12 (Left) and a neighborhood of the cone point corresponds to a neighborhood of the purple disk in Figure 2.12 (Right).

That said, L. Polterovich introduced a beautiful construction in [Pol91, Sections 3&4] that produces a (potentially) new Lagrangian filling $\mu_D(L)$ if one has an embedded Lagrangian disk $\mathscr{D} \subseteq W \setminus L$ whose boundary $\partial\mathscr{D} \subseteq L$ belongs to L. See also [LS91, Section 3.3] for the case of Lagrangian surfaces. This operation is local in a neighborhood of the disk $\mathscr{D}$, so the Lagrangian embeddings of L and $\mu_D(L)$ coincide outside of a neighborhood of $\partial\mathscr{D}$. Intuitively, this operation can be depicted as in Figure 2.12: the Lagrangian disk $\mathscr{D}$ is contracted to a point and expanded in a *different* way. Locally around $\mathscr{D}$, the Legendrian lifts of L and the surgered $\mu_D(L)$ can be represented via fronts as in Figure 2.13. This is justified by the computations that now follow, cf. also [CMP19, Section 6.2].

Local model for Lagrangian disk surgery

The local model for the Lagrangian surgery in [Pol91] is described as follows. Consider local coordinates $(q_1, \dots, q_n, p_1, \dots, p_n) \in \mathbb{R}^{2n}$ and model the two sheets of an immersed point in



(a) Lagrangian handle Γ^+. (b) Lagrangian handle Γ^-.

Figure 2.14: The Lagrangian handles $\Gamma^{\pm} \subseteq \mathbb{R}^{2n}(q,p)$.

a Lagrangian, as the one in Figure 2.12 (Center), and the Liouville form via

$$S_1 := \{p_1 = 0, \dots, p_n = 0\}, \quad S_2 := \{q_1 = p_1, \dots, q_n = p_{n-1}\}, \quad \lambda_{\mathrm{st}} = \sum_{i=1}^{n} p_i dq_i.$$

The two Lagrangian handles $\Gamma^{\pm}$ presented in [Pol91] provide two ways to resolve such an immersed point, resulting in an embedded exact Lagrangian. These Lagrangian handles are depicted in Figure 2.14, and in order to explicitly parametrize them we use coordinates $t = (t_1, \dots, t_n) \in \mathbb{R}^n$. First, the positive Lagrangian handle Γ^+ can be described via the parametrization $\Gamma^+ : \mathbb{R}^n \setminus \{0\} \longrightarrow \mathbb{R}^{2n}$ defined as

$$\Gamma^+(t_1, \dots, t_n) = \big((\mu + \mu^{-1})t_1, \dots, (\mu + \mu^{-1})t_n, \mu t_1, \dots, \mu t_n\big) \text{ where } \mu = \sum_{i=1}^{n} t_i^2.$$

Note that $\Gamma^{\pm}$ are diffeomorphic to $S^{n-1} \times \mathbb{R}$, we have the two asymptotics

$$\lim_{\mu \to \infty} \Gamma^+ \subseteq S_2, \quad \lim_{\mu \to 0} \Gamma^+ \subseteq S_1,$$

and in this model L and $\mathscr{D}$ are described as follows:

1. The given Lagrangian L is obtained by gluing the above positive Lagrangian handle Γ^+ to the Lagrangian sheet S_1 at the limit $\mu = 0$, and to the Lagrangian sheet S_2 at the limit $\mu = \infty$. Thus, intuitively, locally near the boundary $\partial\mathscr{D} \subseteq L$, the Lagrangian is modeled by Γ^+.

2. The Lagrangian disk $\mathscr{D}$ is modeled within the Lagrangian graph Γ_ℓ of the linear function

$$\ell : \mathbb{R}^n_{q_1,\dots,q_n} \longrightarrow \mathbb{R}^n_{p_1,\dots,p_n}, \quad p_i = \frac{q_i}{2}, \quad i \in [1,n].$$

Specifically, $\mathscr{D}$ is the image of the restriction of Γ_ℓ to $\{\|q\| < 1\}$. The boundary $\partial\mathscr{D} \subseteq L$ of $\mathscr{D}$ in this model can be taken to be the intersection $\Gamma_\ell \cap \Gamma^+$, which can be parametrized by $\Gamma^+(t_1, \dots, t_n)$ where $(t_1, \dots, t_n) \in S^{n-1}_{1/2}$ belong to the round sphere of radius 0.5.[5]

[5]Note that, generically, two Lagrangian submanifolds intersect in points, not in codimension 1 subsets of each other. This model is *not* a generic situation and $\Gamma_\ell \cap \Gamma^+ \cong S^{n-1}$ being diffeomorphic to a sphere does not persist under a generic Hamiltonian isotopy applied to either Lagrangian.

The surgered Lagrangian $\mu_{\mathscr{D}}(L)$ is obtained by using the negative Lagrangian handle $\Gamma^- : \mathbb{R}^{n-1} \setminus \{0\} \longrightarrow \mathbb{R}^{2n-2}$ parametrized by

$$\Gamma^-(t_1, \dots, t_n) = \left((\mu - \mu^{-1})t_1, \dots, (\mu - \mu^{-1})t_n, \mu t_1, \dots, \mu t_n\right).$$

This parametrization satisfies the asymptotics $\lim_{\mu \to \infty} \Gamma^- \subseteq S_2$ and $\lim_{\mu \to 0} \Gamma^- \subseteq S_1$, and can be glued to S_1 and S_2 in the asymptotic limits, thus constructing the Lagrangian $\mu_{\mathscr{D}}(L)$:

Definition 2.6.1 (Lagrangian disk surgery)**.** In the local model above, the Lagrangian disk surgery of L along $\mathscr{D}$ is the embedded exact Lagrangian $\mu_{\mathscr{D}}(L)$ obtained by exchanging the positive Lagrangian $\Gamma^+ \subseteq L$ by the negative Lagrangian handle Γ^-.

Lagrangian disk surgery has the following properties:

1. Γ^+ and Γ^- are smoothly isotopic and thus L and $\mu_{\mathscr{D}}(L)$ are smoothly isotopic. In fact, since Γ^+ and Γ^- are even Lagrangian isotopic, so are L and $\mu_{\mathscr{D}}(L)$.

2. Γ^+ and Γ^- are not Hamiltonian isotopic via a compactly supported Hamiltonian diffeomorphism of $\mathbb{R}^{2n}$. Equivalently, Γ^+ and Γ^- are not exact Lagrangian isotopic relative to their boundaries. This follows, for instance, from the fact that $\Gamma^\pm$ are Lagrangian fillings of the Hopf link inducing different augmentations, cf. [EHK16, Section 8.1] for the $n = 2$ case and [BST15, Section 6] for higher n. Alternatively, and historically preceding ibid., this follows from the fact that the Chekanov and Clifford tori, which differ by a unique Lagrangian disk surgery, are not Hamiltonian isotopic, as proven in [Che96b, Section 4].

3. In the model above, the new Lagrangian disk $\mu_{\mathscr{D}}(\mathscr{D})$ is given by

$$\mu_{\mathscr{D}}(\mathscr{D}) := \{(q,p) \in \mathbb{R}^{2n} : q_1 = 0, \dots, q_n = 0, \|p\| < 1\},$$

whose boundary $\partial\overline{\mathscr{D}}$ is indeed contained in $\mathrm{im}(\Gamma^-)$, as can be parametrized by $\Gamma^-(t_1, \dots, t_n)$ where $(t_1, \dots, t_n) \in S^{n-1}$ belong to the unit sphere.

4. Lagrangian disk surgery is an involutive operation, in the sense that

$$\mu_{\mu_{\mathscr{D}}(D)}(\mu_{\mathscr{D}}(L)) \simeq L$$

are Hamiltonian isotopic. This follows from the fact that, in the local model above, Lagrangian disk surgery applied to Γ^- along $\mu_{\mathscr{D}}(\mathscr{D})$ results in a Lagrangian which is Hamiltonian isotopic to Γ^+.

5. It is natural to wonder whether there is a third way to resolve an immersed point, not just with $\Gamma^\pm$. At least for $n = 2$, it follows from a result of B. Thomson, that $\Gamma^\pm$ are the only embedded exact Lagrangian fillings of the Hopf link, up to Hamiltonian isotopy. Thus $\Gamma^\pm$ are the only two possible Lagrangian handles we could have constructed, up to Hamiltonian isotopy. I conjecture that the higher-dimensional Hopf link has only $\Gamma^\pm$ as embedded Lagrangian fillings, up to Hamiltonian isotopy.

Suppose that a Lagrangian filling L is given, in order to be able to apply Lagrangian disk surgeries, it is crucial to find the Lagrangian disks $\mathscr{D}$. For the purpose of these lectures, I will focus the discussion on the $n = 2$ case.

2.6.1 $\mathbb{L}$-compressing systems

Let $\Lambda \subseteq (T^\infty\mathbb{R}^2, \xi_{\text{st}})$ be a Legendrian link and $L \subseteq (T^*\mathbb{R}^2, \lambda_{\text{st}})$ a Lagrangian filling. We can try to apply Lagrangian disk surgeries to L, as discussed in Section 2.6, to produce Lagrangian fillings that are potentially new, i.e. not Hamiltonian isotopic to L. The input of a Lagrangian disk surgery is L and an embedded Lagrangian disk $\mathscr{D}$ with $\partial\overline{D} \subseteq L$. The next question thus becomes: how do we find such embedded Lagrangian disks?

The short answer is that this is not known for a general situation. We will nevertheless present many interesting cases, some connected to singularity theory and some to algebraic combinatorics, in which these disks appear naturally. In order to have a precise language to talk about these situations, we introduce the following notion:

Definition 2.6.2 ($\mathbb{L}$-compressing systems)**.** Let $L \subseteq (T^*\mathbb{R}^2, \lambda_{\text{st}})$ be a Lagrangian filling of $\partial L \subseteq (T^\infty\mathbb{R}^2, \xi_{\text{st}})$. By definition, an $\mathbb{L}$-compressing system $\mathscr{D}(L) := \{\mathscr{D}_1, \dots, \mathscr{D}_s\}$ of rank s is a collection of disks $\mathscr{D}_i \subseteq T^*\mathbb{R}^2 \setminus L$, $i \in [1, s]$, such that

1. $\mathscr{D}_i$ is an embedded Lagrangian disk with $\partial\overline{\mathscr{D}_i} \subseteq L$, for all $i \in [1, s]$,
2. The collection of $\{\partial\overline{\mathscr{D}_1}, \dots, \partial\overline{\mathscr{D}_s}\}$ is linearly independent in $H_1(L)$.

By definition, an $\mathbb{L}$-compressing system $\mathscr{D}(L) := \{\mathscr{D}_1, \dots, \mathscr{D}_s\}$ is said to be maximal if the boundaries of the disks $\mathscr{D}_i \in \mathscr{D}(L)$ span $H_1(L, \mathbb{Z})$.

A given Lagrangian $L \subseteq (T^*\mathbb{R}^2, \lambda_{\text{st}})$ might not have a maximal $\mathbb{L}$-compressing system. To wit, the Legendrian representative of $m(5_2)$ in Figure 2.2(Left) has two known distinct Lagrangian fillings, both once-punctured tori: both of them admit an $\mathbb{L}$-compressing system of rank 1, but none of them admits an $\mathbb{L}$-compressing system of rank 2. There are relaxations of Definition 2.6.2, allowing for *immersed* Lagrangian disks, cf. [CW24, Section 3], but we focus on maximal $\mathbb{L}$-compressing systems in these notes.

Remark 2.6.3. (1) Definition 2.6.2 naturally generalizes to embedded exact Lagrangians $L \subseteq (W^{2n}, \lambda)$ in a Weinstein manifold by using the concept of Lagrangian skeleta. For instance, a maximal $\mathbb{L}$-compressing system for a given $L \subseteq (W, \lambda)$ is a collection $\mathscr{D}(L)$ of embedded Lagrangian disks $\mathscr{D}_i \subseteq W \setminus L$, $i \in [1, s]$, with $\partial\overline{D_i} \subseteq L$ such that

$$L \cup (\mathscr{D}_1 \cup \dots \mathscr{D}_s)$$

is a Lagrangian skeleton of (W, λ). In the case of $(W, \lambda) \cong (T^*\mathbb{R}^2, \lambda_{\text{st}})$, this generalizes the condition of $\{\partial\mathscr{D}_1, \dots, \partial\mathscr{D}_s\}$ being a basis of $H_1(L, \mathbb{Z})$.

(2) As examples from toric varieties illustrate, and especially in higher dimensions, it is also useful to relax the condition that $\mathscr{D}_i$ is a disk, and rather allow more general Lagrangian submanifolds, see e.g. [RSTZ14, Section 3]. □

Example 2.6.4. *(1) The Legendrian links $\Lambda(\mathbb{G})$ associated to plabic graphs $\mathbb{G}$, discussed in Section 2.5.1, come endowed with both a Lagrangian filling and an $\mathbb{L}$-compressing system. Intuitively, the $\mathbb{L}$-compressing system is given by the bounded faces of $\mathbb{G}$, understood as Lagrangian disks in the zero section. See e.g. [CG24, Section 4.3] and references therein. These $\mathbb{L}$-compressing systems are maximal in many interesting cases, e.g. for reduced plabic graphs.*

(2) Given an isolated plane curve singularity $f : \mathbb{C}^2 \longrightarrow \mathbb{C}$ with a real Morsification, the Lagrangian fillings from Example 2.3.3.(3) come endowed with a maximal $\mathbb{L}$-compressing system. The Lagrangian disks are precisely the Lefschetz thimbles, and their boundary curves in the Lagrangian Milnor fiber are the vanishing cycles, cf. [Cas22, Section 2]. □

2.7. Exercises

1. (Front computation of rotation class) Let $\Lambda \subseteq (\mathbb{R}^3, \xi_{\text{st}})$ be an oriented Legendrian knot and $\pi(\Lambda) \subseteq \mathbb{R}^2_{x,z}$ be its front projection. Consider the quantity

$$r(\Lambda) = \frac{1}{2}(u - d),$$

where u is the number of up cusps in $\pi(\Lambda)$ and d is the number of down cusps in $\pi(\Lambda)$.

(i) Show that $r(\Lambda)$ is a Legendrian isotopy invariant.

(ii) Suppose that L is a Lagrangian filling of a knot $\Lambda \subseteq (\mathbb{R}^3, \xi)$ obtained a sequence of moves involving solely $S0$, the oriented version of $S1$, and Legendrian isotopies possibly in between.[6] Prove that $r(\Lambda) = 0$.

2. (Front computation of Thurston-Bennequin) Let $\Lambda \subseteq (\mathbb{R}^3, \xi_{\text{st}})$ be an oriented Legendrian knot and $\pi(\Lambda) \subseteq \mathbb{R}^2_{x,z}$ be its front projection. Consider the quantity

$$tb(\Lambda) = \operatorname{lk}(\Lambda, \Lambda_\varepsilon),$$

where Λ_ε is an ε-Reeb pushoff in the vertical direction, with $\varepsilon \in \mathbb{R}^+$ arbitrarily small, see e.g. Figure 2.15.

(i) Show that $tb(\Lambda)$ is a Legendrian isotopy invariant.

(ii) Prove the $tb(\Lambda) = w(\pi(\Lambda)) - c_r$, where $w(\pi(\Lambda))$ is the writhe of $\pi(\Lambda)$, understood as a knot diagram (so every crossing is an overcrossing), and c_r is the number of right cusps of $\pi(\Lambda)$.

3. (Examples of rotation and tb) Perform the following computations:

(i) Compute the pair $(r(\Lambda), tb(\Lambda))$ for the Legendrian representatives of $m(5_2)$ in Figure 2.2.

(ii) For the Legendrian knots of the form $\Lambda = \Lambda_\beta$ in Section 2.5.3, show that $r(\Lambda_\beta) = 0$ and find a formula for $tb(\Lambda)$ in terms of the number of crossings and strands of the positive braid word β.

(iii) Show that $r(\Lambda(\mathbb{G})) = 0$ vanishes for Legendrian knots of the form $\Lambda = \Lambda(\mathbb{G})$, as in Section 2.5.1.

Remark 2.7.1. For more examples of *formal* Legendrian invariants, beyond Exercises 1, 2 & 3 and also in higher dimensions, see [CE12, Appendix B]. □

4. (A Legendrian Hopf link) Consider the 2-component oriented Legendrian link $\Lambda_H \subseteq (\mathbb{R}^3, \xi_{\text{st}})$ given by the front in Figure 2.15(left).

(i) Show that Λ_H admits an embedded oriented Lagrangian filling L in $(\mathbb{R}^4, \lambda_{\text{st}})$.

(ii) Prove that the Lagrangian filling L you built in (i) has a maximal $\mathbb{L}$-compressing system.

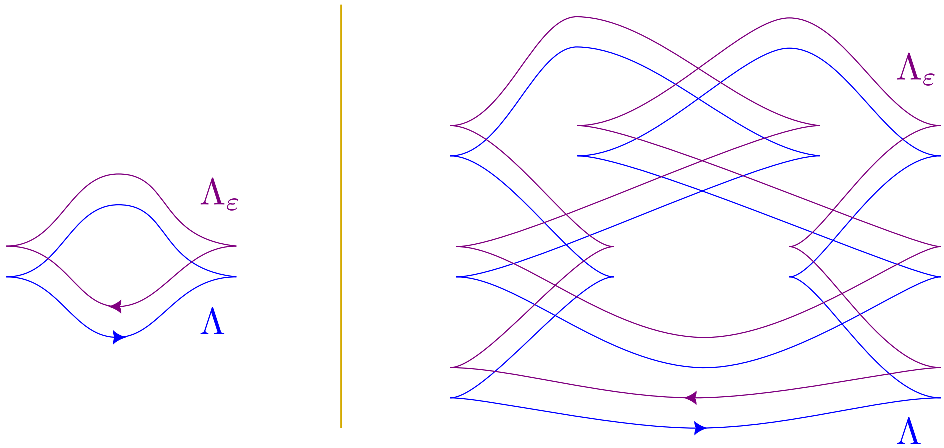


Figure 2.15: Two instances of 2-component Legendrian links formed by taking a Legendrian knot $\Lambda \subseteq (\mathbb{R}^3, \xi_{\mathrm{st}})$ union its ε-Reeb push off, for $\varepsilon \in \mathbb{R}^+$ small enough. (Left) Here Λ is the standard Legendrian unknot. Note the choices of orientations: Λ_ε has reversed orientation than Λ.

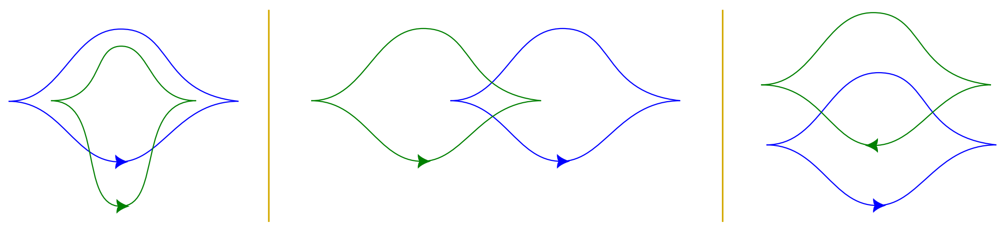


Figure 2.16: Three instances of fronts for different Legendrian Hopf links, Legendrian isotopic to each other.

(iii) Describe the result L' of performing Lagrangian disk surgery on L using the $\mathbb{L}$-compressing system in (ii) as a filling obtained via the moves $S0, S1$ and Legendrian isotopies.

(iv) Show that the three 2-component Legendrian links associated to the fronts in Figure 2.16 are pairwise Legendrian isotopic.

5. (A Lagrangian filling for Reeb push-offs) Let $\Lambda \subseteq (\mathbb{R}^{2n+1}, \xi_{\mathrm{st}})$ be a Legendrian submanifold and consider the 2-component oriented Legendrian submanifold $\Lambda \cup \Lambda_\varepsilon \subseteq (\mathbb{R}^{2n+1}, \xi_{\mathrm{st}})$ given by Λ union its Reeb push-off Λ_ε. Show that $\Lambda \cup \Lambda_\varepsilon$ admits an embedded exact Lagrangian filling and compute its diffeomorphism type.

6. (Properties of Λ_β) Let $\beta \in \mathrm{Br}_n^+$ be a positive braid word and consider its associated Legendrian link $\Lambda_\beta \subseteq (\mathbb{R}^3, \xi_{\mathrm{st}})$, as in Section 2.5.3.

(i) Show that Λ_β admits an embedded exact orientable Lagrangian filling.

[6] These are known as *decomposable* Lagrangian fillings.

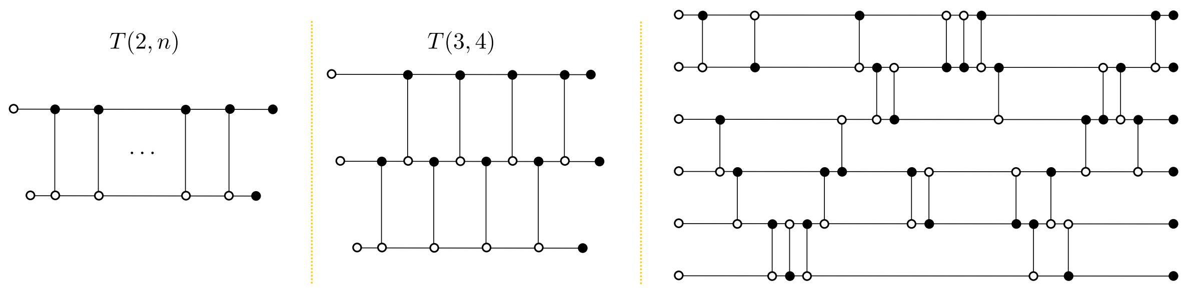


Figure 2.17: (Left) The type of plabic graph associated to a braid word $\beta = \sigma_1^n$. (Center) The plabic graph associated to the braid word $\beta = (\sigma_1\sigma_2)^4$. (Right) The plabic graph associated to a tuple of words (β, δ), where β is written with vertical edges with a black dot on top and δ is written with vertical edges with a white vertex on top. In this case $\beta = \sigma_5\sigma_1\sigma_3\sigma_4\sigma_3\sigma_5^2\sigma_2\sigma_1\sigma_4$ and $\delta = \sigma_1\sigma_3\sigma_4\sigma_5^2\sigma_4\sigma_1\sigma_2\sigma_4\sigma_1\sigma_2\sigma_5\sigma_4\sigma_3\sigma_2\sigma_3\sigma_1$.

(ii) Find the genus of the filling you showed exists in (i) in terms of the length of β and n.

(iii) Consider the plabic graph $\mathbb{G} = \mathbb{G}_\beta$ obtained from β as in Figure 2.17, writing the generators of β as vertical edges of the plabic graph.[7] Show that $\Lambda(\mathbb{G}_\beta)$ is Legendrian isotopic to a Legendrian link whose front is as in Figure 6.1(right).[8]

Remark 2.7.2. Following Exercise 6.(iii), see [CW24, Section 2.5] for more details on these types of plabic graphs and associated Legendrians. □

7. (Cone singularity) Solve the following parts related to Section 2.5.4:

(i) Prove that the smooth embedding in Equation (2.5.1) yields a Legendrian submanifold.

(ii) For $n = 1, 2$, describe the front in $\mathbb{R}^{n+1}$ of a generic Legendrian perturbation applied to the Legendrian submanifold $C_n \subseteq (\mathbb{R}^{2n+1}, \xi_{\mathrm{st}})$.

(iii) Show that the two Legendrians Λ_{C} and $\mathrm{Spun}(\Lambda_u)$ in Figure 2.11 are diffeomorphic to $S^{n-1} \times S^1$ and, in fact, smoothly isotopic to each other.

(iv) If familiar with formal Legendrian, show that Λ_{C} and $\mathrm{Spun}(\Lambda_u)$ are formally Legendrian isotopic to each other if $n \geq 2$.

Remark 2.7.3. Beautifully, Λ_{C} and $\mathrm{Spun}(\Lambda_u)$ are *not* Legendrian isotopic to each other. For $n = 1$, their tb differ, and $n = 2$ this was proven Floer-theoretically in [DR11, Section 4.4]. For $n \geq 3$, this is a result of A. Wong, as part of her (in-progress) thesis. □

8. Verify that the Lagrangian disk surgery described in Section 2.6 is indeed involutive. That is, show that $\mu_{D'}(\mu_D(L))$ is Hamiltonian isotopic to L, where D' is the Lagrangian disk resulting from applying a Lagrangian disk surgery to L along a disk D.

9. Understand Example 2.6.4 in detail. In particular, show that the max-tb Legendrian representative of an algebraic link admits an embedded exact Lagrangian filling with a maximal $\mathbb{L}$-compressing system.

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[7] These are known as *plabic fences* in the literature.

[8] The β_{fig} in that figure is different to the β in this exercise. Specifically, the β_{fig} in the figure is $w_0\beta w_0$.

Chapter 3: Sheaves and singular support

This chapter is to introduces the core ingredients and examples in the microlocal theory of sheaves that we employ to study Legendrian submanifolds. The key concept is that of the *singular support* of a sheaf, a notion first introduced in [KS83], in part inspired by M. Sato's ideas on hyperfunctions and algebraic analysis, and whose foundational properties were crystallized in [KS90]. Decades later, this concept is now fruitfully integrated with the study of modern contact and symplectic topology.

3.1. A motivating construction

Generating families provide a first motivation to study sheaves, and microlocally so. The basic construction, as discussed in [Vit10, Section 9.1.2], is as follows. Let $\mathfrak{f} : M \times \mathbb{R}^m_s \longrightarrow \mathbb{R}$ be a generating family $\mathfrak{f} = \{f_x\}_{x\in M}$ with Cerf diagram $\Pi(\Lambda_\mathfrak{f})$ and $\Lambda_f \subseteq (J^1M, \xi_{\text{st}})$, cf. Section 2.5.2. For instance, if $M = \mathbb{R}$, then $\pi(F) \subseteq \mathbb{R}^2_{x,z}$ is a front diagram that lifts to a Legendrian link $\Lambda(F) \subseteq (\mathbb{R}^3, \xi_{\text{st}})$.

The important space is the subspace of sublevel sets

$$S_\mathfrak{f} := \{(x, q, z) \in M \times \mathbb{R}^m \times \mathbb{R} : \mathfrak{f}_x(q) \leq z\} \subseteq M \times \mathbb{R}^m \times \mathbb{R}, \tag{3.1.1}$$

whose inclusion we denote by $i : S_\mathfrak{f} \longrightarrow M \times \mathbb{R}^m \times \mathbb{R}$. A sheaf associated to it is the extension $k_{S_\mathfrak{f}} := i_*k \in \text{Sh}(M \times \mathbb{R}^m \times \mathbb{R}, \mathcal{K})$ of its constant sheaf by zero, for some object $k \in \mathcal{K}$. This sheaf can be pushforwarded via the projection $p : M \times \mathbb{R}^m \times \mathbb{R} \longrightarrow M \times \mathbb{R}$, whose codomain we understand as a front for (J^1M, ξ_{st}), where the Cerf diagrams live. The pushforwarded sheaf

$$\mathscr{F}_\mathfrak{f} := Rp_*(k_{S_\mathfrak{f}}) \in \text{Sh}(M \times \mathbb{R}, \mathcal{K}).$$

As we will see in Section 3.2.1, such direct image $\mathscr{F}_\mathfrak{f}$ is the sheaf that assigns singular cochains of the preimage $p^{-1}(U)$ to an open subset U. In particular, the stalk of $(\mathscr{F}_\mathfrak{f})_{x,z}$ at a point $(x, z) \in M \times \mathbb{R}$ is simply recording the singular cochain complex of the sublevel set $\{q \in \mathbb{R}^m : f_x(q) \leq z\}$.

The key link between generating functions and the microlocal theory of sheaves is that

$$\boxed{\text{SS}(\mathscr{F}_\mathfrak{f}) \subseteq \Lambda_\mathfrak{f}} \tag{3.1.2}$$

where $\text{SS}(\mathscr{F}_\mathfrak{f})$ is the *singular support* of the sheaf, discussed in Section 3.2.2. This is rigourously capturing the idea that the singular cohomology of the sublevel sets $\{q \in \mathbb{R}^m : f_x(q) \leq z\}$ change *precisely* at the critical points of f_x. Thus, intuitively, the sheaf $\mathscr{F}_\mathfrak{f}$ only changes when crossing a point in the Cerf diagram $\Pi(\Lambda_\mathfrak{f})$ in the vertical direction.

Example 3.1.1. Consider the case $M = \{pt\}$ is a point and $m = 1$, so there is a unique Morse function $f = f_{pt} : \mathbb{R}_q \longrightarrow \mathbb{R}_z$, as depicted in Figure 3.1. Since the cohomology of the sublevel sets does not change past a regular value, the sheaf $\mathscr{F}_\mathfrak{f}$ remains locally constant at the regular values. Intuitively, the sheaf changes exactly at the critical values, drawn as red dots in Figure 3.1(right). □

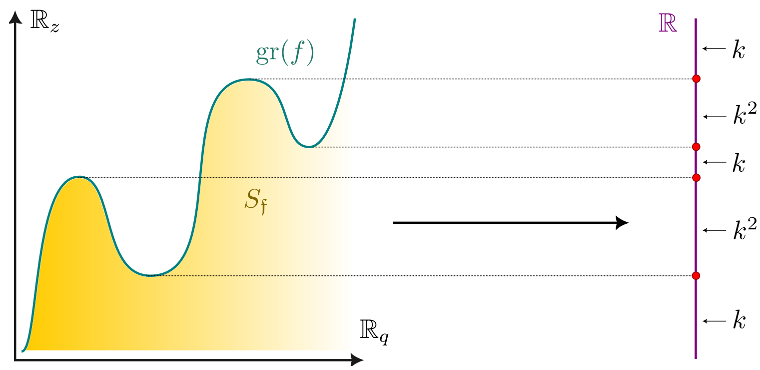

Figure 3.1: (Left) The graph of a Morse function $f : \mathbb{R}_q \longrightarrow \mathbb{R}_z$ and the total sublevel set S_f. (Right) The pushforward of the constant sheaf k on S_f, which is a sheaf on the real line $\mathbb{R}$, depicted vertically. Intuitively, the points on $\mathbb{R}$ where the are changes to the (sections of the) sheaf are drawn in red, and to the right of $\mathbb{R}$ we have indicted the stalks of the sheaf on the complement of these points.

Remark 3.1.2. Technicalities notwithstanding, I would advocate that the reader will benefit from thinking that *all* sheaves with singular support on a Legendrian are, appropriately understood, of the form above and try to understand them from that generating family viewpoint. Many of the (homological) algebraic manipulations with sheaves are also rather natural from the generating family viewpoint. This is a subject of current research, but the expectation is that the theory of generating functions effectively matches (or at least recovers) the microlocal theory of sheaves with Legendrian singular support. □

3.2. Sheaves: Key Concepts

Sheaves are meant to capture geometric objects[1] associated to M that behave in a local-to-global manner, i.e. geometric objects that can be determined by studying them on a basis of open sets *and* can be glued together in a unique manner.

Let $\mathcal{O}p(M)$ denote the poset category of open sets in M, with objects given by open sets and morphism given by open inclusions. By construction, a functor $\mathscr{F} : \mathcal{O}p(M) \longrightarrow \mathcal{K}$ is given by the data of

1. An object $\mathscr{F}(U) \in \mathcal{K}$ for each open set $U \in \mathcal{O}p(M)$,
2. A morphism $\rho_{UV}(\mathscr{F}) : \mathscr{F}(U) \longrightarrow \mathscr{F}(V)$ for each open inclusion $V \subseteq U$ such that

$$\rho_{UU} = \mathrm{Id}, \text{ and } \rho_{WV} \circ \rho_{UV} = \rho_{UW}$$

   if $W \subseteq V \subseteq U$ are open inclusions, where we abbreviated $\rho_{UV} := \rho_{UV}(\mathscr{F})$.

A general functor $\mathscr{F} : \mathcal{O}p(M) \longrightarrow \mathcal{K}$ assigns geometric objects to an open set U via $\mathscr{F}(U)$ and the morphisms $\rho_{UV}(\mathscr{F})$ provide a notion of "restriction to an open subset". That said, such general functor might not adhere to a local-to-global behavior. In order to rigorously capture the local-to-global behavior, and thus define sheaves, we study how such functors interact with open covers, as follows.

Let $U \subseteq M$ be an open set and $U = \cup_{i\in I} U_i$ an open cover. By definition, the diagram $C(\{U_i\}, \mathscr{F})$ associated to a sheaf $\mathscr{F}$ and such an open cover is

$$\boxed{C(\{U_i\}, \mathscr{F}) := \Pi_{i\in I}\mathscr{F}(U_i) \longrightarrow \Pi_{i,j\in I}\mathscr{F}(U_{ij}) \longrightarrow \Pi_{i,j,k\in I}\mathscr{F}(U_{ijk}) \longrightarrow \cdots} \tag{3.2.1}$$

[1] For instance, capturing sections of bundles on M, e.g. certain class of functions from M to another manifold, or tensors on bundles associated to M (such as differential forms).

where we abbreviated $U_{ij} := U_i \cap U_j$, $U_{ijk} := U_i \cap U_j \cap U_k$ and so on. The morphisms in this diagram are in the coefficient category $\mathcal{K}$ and are given by the maps ρ_{VW} where $V = U_{i_1} \cap \ldots \cap U_{i_\ell}$ is an intersection of open sets in the cover and W is $V \cap U_j$, where $j \neq i_k$, i.e. a further intersection with a different open set in the cover. In particular, each arrow $\longrightarrow$ in 3.2.1 contains many arrows in $\mathcal{K}$. Informally, we refer to $C(\{U_i\}, \mathscr{F})$ as the Čech diagram of $\mathscr{F}$ and $\{U_i\}$.

The diagram $C(\{U_i\}, \mathscr{F})$ in Equation (3.2.1) has two purposes: defining sheaves and computing derived global sections of sheaves. For the first purpose, note that $C(\{U_i\}, \mathscr{F})$ is a diagram in the category $\mathcal{K}$ and, by assumption, there exists a limit $\lim C(\{U_i\}, \mathscr{F})$ in $\mathcal{K}$. Since $\mathscr{F}(U)$ maps into the diagram $C(\{U_i\}, \mathscr{F})$ by the restrictions ρ_{U,U_i}, the universal property of limits yields a unique natural map

$$\mathscr{F}(U) \longrightarrow \lim_{i \in I} C(\{U_i\}, \mathscr{F}). \tag{3.2.2}$$

The intuition behind Equation (3.2.1) is that the domain $\mathscr{F}(U)$ is seen as the global object, capturing sections $\mathscr{F}(U)$ on U as is, whereas the codomain $C(\{U_i\}, \mathscr{F})$ encodes the data of local sections (local according to the open over $\{U_i\}$) and how these sections compare on intersections, e.g. whether they agree or not. The notion of behaving in a local-to-global manner is then captured by requiring that all global sections can be uniquely expressed as a collection of local sections, for all covers. This leads to the following definition:

Definition 3.2.1 (Sheaves)**.** Let M be a smooth manifold and $\mathcal{K}$ a coefficient category. A sheaf on M with coefficients in $\mathcal{K}$ is a functor $\mathscr{F} : \mathcal{O}p(M) \longrightarrow \mathcal{K}$ such that for any open cover $U = \cup_{i \in I} U_i$ the natural map

$$\mathscr{F}(U) \longrightarrow \lim_{i \in I} C(\{U_i\}, \mathscr{F}) \tag{3.2.3}$$

is an isomorphism.

For coefficients $\mathcal{K}$ in a 1-category, such as $\mathcal{K} = \text{Sets}$ or $\mathcal{K} = \text{Mod}(k)$, Equation (3.2.5) is that all diagrams

$$\mathscr{F}(U) \longrightarrow \Pi_{i \in I} \mathscr{F}(U_i) \longrightarrow \Pi_{i,j \in I} \mathscr{F}(U_i \cap U_j) \tag{3.2.4}$$

are equalizer diagrams. In words, for any open cover $U = \cup_{i \in I} U_i$, we have that:

(i) Given local sections $s_i \in \mathscr{F}(U)$ such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$, there exists a section $s \in \mathscr{F}(U)$ with $s|_{U_i} = s_i$.

(ii) Two sections $s_1, s_2 \in \mathscr{F}(U)$ are equal iff their restrictions to each U_i are equal.

Informally, (*i*) states that we can glue local sections if they coincide in double intersections, and (*ii*) implies that such gluing is unique. This is the classical definition of a sheaf, e.g. as in [Bre97, Chapter 1.1]. For most of the contents of these notes, the reader might take Equation (3.2.4) as the defining condition of a sheaf. As per usual, if the coefficient category $\mathcal{K}$ has certain operations, e.g. directs sums (in general, coproducts) or tensor products, then it is possible to perform such operation with sheaves, namely, consider the direct sum of sheaves or their tensor product.

Example 3.2.2. (1) The functor $\mathscr{F} : \mathcal{O}p(M) \longrightarrow \text{Mod}(k)$ determined by

$$\mathscr{F}(U) = k \text{ for all } U \in \mathcal{O}p(M), \quad \rho_{UV}(\mathscr{F}) = \text{Id for all } V \subseteq U,$$

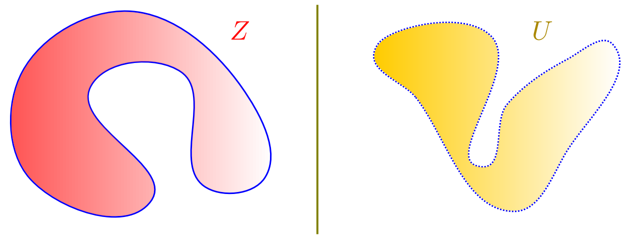

Figure 3.2: (Left) A depiction of a closed subset $j : Z \longrightarrow \mathbb{R}^2$. (Right) A depiction of an open subset $i : U \longrightarrow \mathbb{R}^2$. In such drawings, we indicate the part of the boundary in the set by solid lines, and the part of the boundary *not* in the set by *dashed* lines.

is a sheaf if M is connected. It is denoted by k_M. Intuitively, this sheaf gives the empty Legendrian submanifold in $(T^\infty M, \xi_{st})$.

(2) Let $f : N \longrightarrow M$ be a continuous map and $\mathscr{F} \in \mathrm{Sh}(N, \mathcal{K})$. Its direct image $f_*(\mathscr{F}) \in \mathrm{Sh}(M, \mathcal{K})$ is the sheaf determined by

$$f_*(\mathscr{F}(V)) := \mathscr{F}(f^{-1}(U)), \text{ for all open } U \subseteq M,$$

$$\rho_{UV}(f_*\mathscr{F}) := \rho_{f^{-1}(U)f^{-1}(V)}, \text{ for all opens } V \subseteq U.$$

Let $j : Z \longrightarrow M$ be a closed inclusion of a closed subset $Z \subseteq M$ with smooth boundary $\partial Z \subseteq M$, as depicted in Figure 3.2(left). Intuitively, $j_* Z \in \mathrm{Sh}(M, \mathcal{K})$ is a sheaf that represents the Legendrian $\Lambda^{in}(Z) \subseteq (T^\infty M, \xi_{st})$ given by the inward conormal lift of ∂Z.

(3) The construction in Section 3.1 is a (derived) version of the direct image of (2). Namely, if $\mathfrak{f} : M \times \mathbb{R}^q \longrightarrow \mathbb{R}$ is a generating family quadratic at infinity, the sheaf $\mathscr{F}_{\mathfrak{f}} \in \mathrm{Sh}(M \times \mathbb{R}, \mathcal{K})$ is given by the (derived) direct image $R\pi_*(k_{\mathbb{S}_{\mathfrak{f}}})$. Intuitively, the sheaf $R\pi_*(k_{Z_S})$ represents the Legendrian $\Lambda(F) \subseteq (T^\infty M, \xi_{st})$ generated by the generating family $\mathfrak{f}$.

(4) Let $i : U \longrightarrow M$ be an open inclusion of an open set $U \subseteq M$ with smooth boundary, as in Figure 3.2(right). Given $\mathscr{F} \in \mathrm{Sh}(U, \mathcal{K})$, there exists a sheaf $i_! \mathscr{F} \in \mathrm{Sh}(M, \mathcal{K})$ which is a subfunctor of $i_* \mathscr{F}$ and it is characterized by

$$i_! \mathscr{F}(U) := \{ s \in \mathscr{F}(f^{-1}(U)) : i|_{\mathrm{supp}(s)} : \mathrm{supp}(s) \longrightarrow U \text{ is proper} \},$$

that is, those sections of $i_* \mathscr{F}(U)$ that map properly under i. The sheaf $i_! \mathscr{F} \in \mathrm{Sh}(M, \mathcal{K})$ is said to be the extension of $\mathscr{F}$ by 0 and, intuitively, it represents the Legendrian $\Lambda^{out}(U) \subseteq (T^\infty M, \xi_{st})$ given by the outward conormal lift of ∂U.

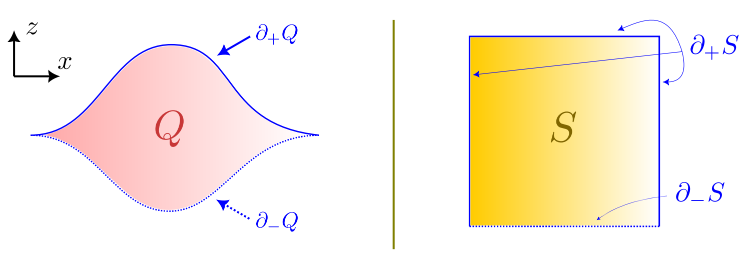

Figure 3.3: Two instances of direct images of sheaves with subsets that have singular boundaries. The boundaries ∂_+ are contained in the set, whereas the boundaries ∂_-, drawn in dashed lines, are not contained in the set.

(5) It is possible to combine the direct image j_* and $i_!$ by choosing codimension-0 subsets $Q \subseteq M$, with not necessarily smooth boundary. For instance, consider the subset

$$S := \{(x, y) \in \mathbb{R}^2 : \max(x, y) \leq 1\}.$$

as depicted in Figure 3.3(right). Let $f : S \longrightarrow \mathbb{R}^2$ be its inclusion in the plane and $k \in \mathrm{Sh}(S)$ the constant, then $f_! k \in \mathrm{Sh}(\mathbb{R}^2)$ is a sheaf. In the study of Legendrian submanifolds, using such sheaves is common, as fronts are typically singular, see e.g. Figure 3.3(left).

(6) Any continuous map $f : N \longrightarrow M$ defines a functor $f^* : \mathrm{Sh}(M) \longrightarrow \mathrm{Sh}(N)$ via

$$(f^*\mathscr{F})(U), \quad U \subseteq U \text{ open}$$

for any $\mathscr{F} \in \mathrm{Sh}(M)$. This is known as the *pull-back* of sheaves. It is particularly useful when f is an inclusion, so f^* rigorously incarnates the intuitive notion of restriction of sheaves from M to a subset N. In this case, the notation $\mathscr{F}|_N := f^*\mathscr{F}$ is sometimes used. □

Given a closed point $x \in M$, a sheaf $\mathscr{F} \in \mathrm{Sh}(M, \mathcal{K})$ does not know what to assign to x, as x is not an open set. Nevertheless, we can gain intuition of what the sheaf $\mathscr{F}$ looks around $x \in M$ by studying smaller and smaller open neighborhoods $U \subseteq M$ containing x and the directed system $\{\mathscr{F}(U), \rho(\mathscr{F})\}$ in $\mathcal{K}$ that $\mathscr{F}$ assigns to them. This leads to:

Definition 3.2.3 (Stalk)**.** Let $x \in M$ be a point and $\mathscr{F} \in \mathrm{Sh}(M, \mathcal{K})$ a sheaf. By definition, the stalk $\mathscr{F}_x$ of $\mathscr{F}$ at x is the colimit

$$\mathscr{F}_x := \mathrm{colim}_{x \in U} \mathscr{F}(U), \tag{3.2.5}$$

where the colimit runs over all open sets $U \subseteq M$ with $x \in U$.

Example 3.2.4. The stalks of some of the sheaves in Example 3.2.2 are as follows:

1. If $\mathscr{F} = k_M$ is the constant sheaf, then $\mathscr{F}_x = k$.

2. If $j : Z \longrightarrow M$ is a closed inclusion, then the stalks of the direct image $\mathscr{F} = j_* k_Z$ are

$$\mathscr{F}_x = \begin{cases} (k_Z)_x = k & \text{if } x \in Z \\ 0 & \text{if } x \notin Z. \end{cases}$$

3. For the derived direct image, and thus the construction in Section 3.1 relating generating functions and sheaves, the stalk at a point $(x, z) \in M \times \mathbb{R}$ is given by the cochain complex

$$R\pi_*(k_{Z_S})_{(x,z)} = C^*(M \times \mathbb{R}^q \times \mathbb{R}, k_{Z_S}) \simeq \mathrm{Sing}(\{s \in \mathbb{R}^q : S_x(s) \leq z\}, k).$$

   That is, the singular cochain complex of the sublevel set $\{S_x(s) \leq z\} \subseteq \mathbb{R}^q$.

4. If $i : U \longrightarrow M$ is a open inclusion, then the stalks of the direct image $\mathscr{F} = j_* k_U$ are

$$\mathscr{F}_x = \begin{cases} k & \text{if } x \in \overline{U} = U \cup \partial U \\ 0 & \text{if } x \notin \overline{U}. \end{cases}$$

   In contrast, the stalks of the extension by zero $\mathscr{F} = i_! k_U$ are

$$\mathscr{F}_x = \begin{cases} (k_U)_x = k & \text{if } x \in U \\ 0 & \text{if } x \notin U. \end{cases}$$

5. For the examples related to Figure 3.3, the stalks are k at the points contained in the set and 0 otherwise. In particular, the stalks are k at the points in the boundary piece ∂_+ and zero at the point in the boundary piece ∂_-.

6. For a pull-back sheaf $f^*\mathscr{F}$ under $f : N \longrightarrow M$, the stalks are $(f^*\mathscr{F})_y \simeq \mathscr{F}_{f(y)}$ for any $y \in N$. In particular, restricting a sheaf preserves its stalks. $\square$

3.2.1 Derived sections

Given a sheaf $\mathscr{F} \in \mathrm{Sh}(M, \mathcal{K})$ and an open set $U \subseteq M$, the object $\mathscr{F}(U) \in \mathcal{K}$ is referred to as the sections of $\mathscr{F}$ on U. There is a more refined notion of sections that we need, that of *derived* sections on U. Derived sections are needed for many of the key definitions in the microlocal theory of sheaves, including the notion of singular support, which is what connects sheaves to Legendrian submanifolds, and the morphisms in category $\mathscr{F} \in \mathrm{Sh}(M, \mathcal{K})$. Intuitively, the Legendrian $\Lambda(\mathscr{F}) \subseteq (T^\infty M, \xi_{\mathrm{st}})$ associated to a (constructible) sheaf $\mathscr{F} \in \mathrm{Sh}(M, \mathcal{K})$ records the set of points and directions in which the derived sections of the sheaf $\mathscr{F}$ change.

The definition of derived sections is part of the general machinery of derived functors, cf. [Bre97, Chapter II.2], [KS90, Section 1.8] or [Wei94, Chapter 2]. Derived sections of $\mathscr{F}$ on U are a certain cochain complex in $\mathcal{K}$ extracted from $\mathscr{F}$ and U. As we focus on Legendrian submanifolds, it suffices to use the Čech diagram $C(\{U_i\}, \mathscr{F})$ from Equation (3.2.1) to obtain a quasi-isomorphic cochain complex. For that, note that $C(\{U_i\}, \mathscr{F})$ can be seen as a cochain complex if $\mathcal{K}$ is an Abelian category, i.e. the composition of two arrows in (3.2.1) is the zero object. We use the following result to compute derived sections:

Proposition 3.2.5 (Cech complex computes derived sections)**.** *Let M be a smooth manifold of dimension n and $\mathscr{F} \in \mathrm{Sh}(M, \mathcal{K})$ a sheaf. Suppose that $U = \cup_{i \in I} U_i$ is an open cover such that all non-empty finite intersections of its open sets are diffeomorphic to $\mathbb{R}^n$. Then the derived sections of $\mathscr{F}$ on U is the quasi-isomorphic to the Čech complex $C(\{U_i\}, \mathscr{F})$.* $\square$

For a proof of Proposition 3.2.5 that suffices for our goals, see [BT82, Theorem 8.9], [Bre97, Theorem 4.13] or [KS90, Prop. 2.8.4]. To summarize, we used the Čech diagram $C(\{U_i\}, \mathscr{F})$ from Equation (3.2.1)in two ways:

1. First, its limit $\lim C(\{U_i\}, \mathscr{F})$, seen as a diagram in $\mathcal{K}$, receives a map from $\mathscr{F}(U)$, and the defining property of a sheaf is that this map is an isomorphism.

2. Second, considering $C(\{U_i\}, \mathscr{F})$ as a complex in $\mathcal{K}$, it is quasi-isomorphic to the derived sections of $\mathscr{F}$ on U.

Notation: For now onward, we denote the global sections of $\mathscr{F}$ on U by $\Gamma(U)$, that is we implicitly derive the functor Γ, denoting $R\Gamma$ by Γ. $\square$

Example 3.2.6. (1) Consider the constant sheaf k_M as in Example 3.2.2.(1). Then its derived sections are

$$\Gamma(M, k_M) \simeq \mathrm{Sing}(M),$$

where $\mathrm{Sing}(M)$ is the singular cochain complex of M, as defined in e.g. [Hat02, Chapter 3.1]. This quasi-isomorphism follows, for instance, from [Bre97, Chapter III].

(2) Consider an open inclusion $i : U \longrightarrow M$ and the sheaves $i_* k_U$ and $i_! k_U$ given by the direct image and the extension by zero of the constant sheaf k_U. Then derived sections are

$$\Gamma(M, i_* k_U) \simeq \mathrm{Sing}(U), \quad \Gamma(M, i_! k_U) \simeq \mathrm{Sing}^c(U),$$

where $\mathrm{Sing}^c(U)$ is the singular cochain complex of U of cochains with compact support, see e.g. [KS90, Section 2.3]. Similarly, if $j : Z \longrightarrow M$ is a closed inclusion, then

$$\Gamma(M, j_* k_Z) \simeq \mathrm{Sing}(Z).$$

In practice, one uses a computationally-friendly model for singular cochains, such as simplicial or cellular cochains, and tries to use that the homotopy type of such complexes is a homotopy invariant of the topological space. □

3.2.2 Singular support

The singular support of a sheaf $\mathscr{F} \in \mathrm{Sh}(M, \mathcal{K})$ is a subset of $T^\infty M$. In a nutshell, thinking of points in $T^\infty M$ as a point $x \in M$ and a (co)direction $\xi \in T^\infty_x M$ at that point, the singular support $\mathrm{SS}(\mathscr{F}) \subseteq T^\infty M$ consists of those points (x, ξ) such that the derived sections near x change as we move in the codirection ξ.

Definition 3.2.7 (Microstalk relative to f)**.** Let $\mathscr{F} \in \mathrm{Sh}(M, \mathcal{K})$, $(x, \xi) \in T^\infty M$ and $f : M \longrightarrow \mathbb{R}$ a smooth function such that $f(x) = 0$ and $df_x = \xi$. By definition, the microstalk ${}_f\mathscr{F}_{(x,\xi)}$ of $\mathscr{F}$ at x relative to f is

$${}_f\mathscr{F}_{(x,\xi)} := \lim_{U} \left(\mathrm{cone}(\Gamma(U; \mathscr{F}) \longrightarrow \Gamma(\{f < 0\} \cap U; \mathscr{F}))[-1]\right), \tag{3.2.6}$$

where the limit is that of the directed systems of open sets containing x and the morphisms being coned are the restriction maps $\rho_{U, U \cap \{f<0\}}(\mathscr{F})$ of $\mathscr{F}$.

To study Legendrians microlocally, our focus is on whether a given microstalk ${}_f\mathscr{F}_{(x,\xi)} \in \mathcal{K}$ vanishes or not. A central notion in the microlocal theory of sheaves is the following:

Definition 3.2.8 (Singular support)**.** Let $\mathscr{F} \in \mathrm{Sh}(M, \mathcal{K})$ be a sheaf on a smooth manifold M. By definition, the singular support $\mathrm{SS}(\mathscr{F})$ of $\mathscr{F}$ is

$$\mathrm{SS}(\mathscr{F}) := \overline{\{(x, \xi) \in T^\infty M \text{ s.t. } \exists f : M \longrightarrow \mathbb{R} \text{ with } {}_f\mathscr{F}_{(x,\xi)} \not\simeq 0\}} \subseteq T^\infty M,$$

where $f : M \longrightarrow \mathbb{R}$ is smooth with $f(x) = 0$ and $df_x = \xi$.

In our case, the singular support in Definition 3.2.8 can be used to study Legendrians because:

1. Given a Legendrian $\Lambda \subseteq (T^\infty M, \xi_{\mathrm{st}})$, one can study sheaves with $\mathrm{SS}(\mathscr{F}) \subseteq \Lambda$. The category of such sheaves will turn out to be a Legendrian invariant of Λ. In many cases, this category is non-empty, and has rich structure, e.g. it is smooth and it has a natural relative Calabi-Yau structure.

2. If the sheaf $\mathscr{F}$ satisfies certain properties, then $\mathrm{SS}(\mathscr{F}) \subseteq (T^\infty M, \xi_{\mathrm{st}})$ is Legendrian, and we can readily construct many interesting Legendrian submanifolds in this manner. Such property on $\mathscr{F}$ is that of being constructible, discussed in Section 4.3 below.

About Definition 3.2.8: arguing that $(x, \xi) \in T^\infty M$ is in the singular support of $\mathscr{F}$ is a relatively simple task: it suffices to find *one* such function f with non-vanishing microstalk ${}_f\mathscr{F}_{(x,\xi)}$. In contrast, arguing that $(x, \xi) \in T^\infty M$ is *not* in the singular support requires showing that all microstalks ${}_f\mathscr{F}_{(x,\xi)}$, for all such functions f, indeed vanish.

The following result, proven in [KS90, Prop. 7.5.3], facilitates the task of proving that a given points is *not* in the singular support:

Proposition 3.2.9 (Test for non-characteristic propagation)**.** *Let $(x,\xi) \in T^\infty M$ be contained in a Legendrian $\mathcal{P} \subseteq (T^\infty M, \xi_{st})$, and $\mathscr{F} \in \mathrm{Sh}(M, \mathcal{K})$ a sheaf such that $\mathrm{SS}(\mathscr{F}) \subseteq \mathcal{P}$. Suppose that $f : M \longrightarrow \mathbb{R}$ is a smooth function with $f(x) = 0$ such that df intersects $\mathcal{P}$ transversely at (x,ξ). Then ${}_f\mathscr{F}_{(x,\xi)}$ is independent of f. In particular, if ${}_f\mathscr{F}_{(x,\xi)}$ vanishes for such an f, then $(x,\xi) \notin \mathrm{SS}(\mathscr{F})$.* □

A function $f : M \longrightarrow \mathbb{R}$ as in Proposition 3.2.9, transversely intersecting a Legendrian containing the singular support, is sometimes called a test function. Based on Proposition 3.2.9, the following definition is often used:

Definition 3.2.10 (Microstalk)**.** Let $\mathscr{F} \in \mathrm{Sh}(M, \mathcal{K})$, $(x,\xi) \in T^\infty M$. By definition, the microstalk $\mathscr{F}_{(x,\xi)}$ of $\mathscr{F}$ at x is

$$\mathscr{F}_{(x,\xi)} :=_f \mathscr{F}_{(x,\xi)}$$

where f is any test function, which is well-defined up to a shift.

In particular, singular support can then be described as

$$\mathrm{SS}(\mathscr{F}) = \overline{\{(x,\xi) \in T^\infty M \text{ s.t. } \mathscr{F}_{(x,\xi)} \not\simeq 0\}}. \tag{3.2.7}$$

Remark 3.2.11. For the reader familiar with the following notions:

1. The microstalk ${}_f\mathscr{F}_{(x,\xi)}$ is the stalk of the sheaf $\Gamma_{\{f\geq 0\}}(\mathscr{F})$ at $x \in M$, where Γ_Z denotes derived sections with compact support relative to Z.

2. More conceptually, ${}_f\mathscr{F}_{(x,\xi)}$ corepresents the vanishing cycle functor associated to f, cf. e.g. [NT23, Section 2.1].

3. A foundational result in the study of singular support is that $\mathrm{SS}(\mathscr{F}) \subseteq (T^\infty M, \xi_{\mathrm{st}})$ is a coisotropic subset. This is proven in [KS90, Theorem 6.5.4]. Since a given Legendrian $\Lambda \subseteq (T^\infty M, \xi_{\mathrm{st}})$ is an example of a coisotropic subset, it thus makes to discuss sheaves $\mathscr{F}$ with singular supported $\mathrm{SS}(\mathscr{F}) \subseteq \Lambda$ contained in such given Λ, cf. the teal comment after Definition 3.2.8. □

We conclude this subsection with the tenet relating sheaves $\mathscr{F} \in \mathrm{Sh}(M, \mathcal{K})$ and Legendrian submanifolds $\Lambda \subseteq (T^\infty M, \xi_{\mathrm{st}})$, inspired by Remark 3.2.11.(3):

If $\Lambda \subseteq (T^\infty M, \xi_{\mathrm{st}})$ is Legendrian and $\mathscr{F}$ has $\mathrm{SS}(\mathscr{F}) = \Lambda$, then $\mathscr{F}$ **represents** Λ. (3.2.8)

There will be more technical jargon for Tenet 3.2.8, such as "$\mathscr{F}$ is a quantization of Λ", but for now this language should be accessible enough to guide us until further details are discussed.

3.2.3 Examples of singular support

For the purposes of studying Legendrian submanifolds, a first intuition when determinig the singular support of a sheaf is given by the behavior of its stalks. For instance, if a point $x \in M$ has a given stalk $\mathscr{F}_x$ but there is a set $V \subseteq M$ of points with $x \in \overline{V} \setminus V$ such that the stalks at $\mathscr{F}$ are not isomorphic to $\mathscr{F}_x$, then the codirection (x,ξ), where ξ is any codirection at x "pointing toward" V, is most likely in the singular support. Let us describe the singular support for the sheaves in Example 3.2.2, cf. also Examples 3.2.4 and 3.2.6.

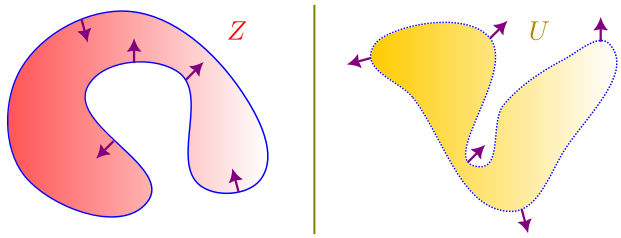

Figure 3.4: (Left) Directions in the singular support of $j_* k_Z$, $j : Z \longrightarrow \mathbb{R}^2$ a closed inclusion, drawn in purple arrows. (Right) Directions in the singular support of $i_! k_U$, $i : U \longrightarrow \mathbb{R}^2$ an open inclusion.

(1) The singular support of the constant sheaf $\mathscr{F} = k_M$ is empty:

$$\mathrm{SS}(k_M) = \emptyset \subseteq (T^\infty M, \xi_{\mathrm{st}}).$$

In stark contrast, the sheaf $\mathscr{F} = C^\infty_M$ of $\mathbb{R}$-valued smooth functions on M, which assigns $C^\infty_M(U) := C^\infty(U, \mathbb{R})$ and has $\rho(\mathscr{F})$ given by restriction of functions, has singular support

$$\mathrm{SS}(C^\infty_M) = (T^\infty M, \xi_{\mathrm{st}}).$$

Intuitively, for any point $x \in \mathbb{R}^n$ and any codirection $(x, \xi) \in T^\infty \mathbb{R}^n$ at that point, we can find a function which is smooth in the half-space $H := \{v \in \mathbb{R}^n : \xi(v) < 0\}$ but *not* smooth in an open neighborhood of the origin. Thus there is a section of $\mathscr{F} = C^\infty_M$ that does not extend as we move from H near the origin towards (and past) the origin in the codirection ξ, e.g. in a direction ortoghonal to the hyperplane $\{v \in \mathbb{R}^n : \xi(v) = 0\}$, pointing outwards from H.

These two examples of sheaves, k_M and C^∞_M are, in a sense, at opposite extremes to what the singular support $\mathrm{SS}(\mathscr{F})$ of a sheaf $\mathscr{F}$ can be. Note that both subsets are coisotropic: $\mathrm{SS}(C^\infty_M)$ being maximal and of dimension $2n - 1$, whereas $\mathrm{SS}(k_M)$ is as minimal as possible (it is empty), and should be interpreted as n-dimensional.[2] In the spirit of Tenet 3.2.8, k_M represents the empty Legendrian in $(T^\infty M, \xi_{\mathrm{st}})$. In the study of Legendrians via the microlocal theory of sheaves, the relevant sheaves are much closer to k_M than to C^∞_M.

(2) Let $j : Z \longrightarrow M$ be a closed inclusion with smooth boundary. Then the singular support of $\mathscr{F} = j_* k_Z$ is given by the set of points at the boundary ∂Z and their inward-pointing codirections. That is, if ν_- denotes the inward pointing covector along ∂Z, pointing towards Z, then

$$\mathrm{SS}(j_* k_Z) = \{(x, \xi) \in (T^\infty M, \xi_{\mathrm{st}}) : x \in \partial Z, \quad \xi = \nu_-(x)\}.$$

These direction are depicted in Figure 3.4(left). Following Tenet 3.2.8, we would say $j_* k_Z$ represents the negative Legendrian conormal lift of $\partial Z \subseteq M$.

(3) Let $i : U \longrightarrow M$ be an open inclusion with $\overline{i(U)}$ having smooth boundary. Then the singular support of the extension by zero $\mathscr{F} = i_! k_U$ is given by the set of points at the boundary ∂Z and their outward-pointing codirections, as drawn in Figure 3.4(right). That is

$$\mathrm{SS}(i_! k_U) = \{(x, \xi) \in (T^\infty M, \xi_{\mathrm{st}}) : x \in \partial \overline{U}, \quad \xi = \nu_+(x)\},$$

where ν_+ denotes the outward pointing covector along $\partial \overline{U}$, pointing away from U. As in Tenet 3.2.8, we would say $i_! k_U$ represents the positive Legendrian conormal lift of $\partial \overline{U} \subseteq M$.

[2]This is admittedly a bit clearer if one consider singular support to be a conic subset of $T^* M$. In that case, the singular support of k_M is the zero section $M \subseteq T^* M$, which is n-dimensional.

3.2.4 The categories of sheaves

Sheaves talk to each other, i.e. there is a well-defined notion of morphisms $\mathrm{Hom}(\mathscr{F},\mathscr{G})$ between two sheaves $\mathscr{F},\mathscr{G} \in \mathrm{Sh}(M,\mathcal{K})$ and compositions of such morphisms. Said formally, $\mathrm{Sh}(M,\mathcal{K})$ arises as the class of objects of a category. At a basic level, the set of functors $\mathrm{Sh}(M,\mathcal{K})$ is a category simply because the coefficients $\mathcal{K}$ are a category. The morphisms between sheaves are then the morphisms between such functors.[3] Defining such morphisms between sheaves does *not* use the local-to-global property characterizing sheaves among all functors $\mathcal{O}p(M) \longrightarrow \mathcal{K}$. Rather, the sheaf property is seen in that such morphisms $\mathrm{Hom}(\mathscr{F},\mathscr{G})$ can be computed in a local-to-global manner. We provide a brief account of such morphisms, cf. e.g. [Bre97, Section 1.2] or [KS90, Section 2.2] for details.

Definition 3.2.12 (Sheaf morphisms)**.** Let $\mathscr{F},\mathscr{G} \in \mathrm{Sh}(M,\mathcal{K})$ be two sheaves. By definition, the Hom sheaf $\mathscr{H}(\mathscr{F},\mathscr{G}) \in \mathrm{Sh}(M,\mathcal{K})$ is given by

$$\mathscr{H}(\mathscr{F},\mathscr{G})(U) := \mathrm{Hom}_{\mathcal{K}}(\mathscr{F}(U),\mathscr{G}(U)), \quad U \subseteq M \text{ open},$$

and its restriction morphisms are inherited from those of $\mathscr{F}$ and $\mathscr{G}$. By definition, the morphisms $\mathrm{Hom}(\mathscr{F},\mathscr{G})$ between $\mathscr{F},\mathscr{G} \in \mathrm{Sh}(M,\mathcal{K})$ are given by the derived sections of $\mathscr{H}(\mathscr{F},\mathscr{G})$, that is

$$\mathrm{Hom}(\mathscr{F},\mathscr{G}) := \Gamma(M,\mathscr{H}(\mathscr{F},\mathscr{G})).$$

The composition of morphisms

$$\mathrm{Hom}(\mathscr{F}_1,\mathscr{F}_2) \otimes \mathrm{Hom}(\mathscr{F}_2,\mathscr{F}_3) \longrightarrow \mathrm{Hom}(\mathscr{F}_1,\mathscr{F}_3)$$

is defined by the morphism induced by the compositions

$$\mathrm{Hom}_{\mathcal{K}}(\mathscr{F}_1(U),\mathscr{F}_2(U)) \otimes \mathrm{Hom}_{\mathcal{K}}(\mathscr{F}_2(U),\mathscr{F}_3(U)) \longrightarrow \mathrm{Hom}_{\mathcal{K}}(\mathscr{F}_1(U),\mathscr{F}_3(U)), \quad U \subseteq M \text{ open},$$

in the coefficient category $\mathcal{K}$, after taking derived sections. To ease notation, we eponymously denote the category with objects $\mathrm{Sh}(M,\mathcal{K})$ by $\mathrm{Sh}(M,\mathcal{K})$:

Definition 3.2.13 (Sheaf category)**.** The category $\mathrm{Sh}(M,\mathcal{K})$ of sheaves on M with values in $\mathcal{K}$ is the category with:

1. **Objects** are sheaves $\mathscr{F} \in \mathrm{Sh}(M,\mathcal{K})$.
2. **Morphisms** between sheaves $\mathscr{F},\mathscr{G}$ are $\mathrm{Hom}(\mathscr{F},\mathscr{G})$. □

The category $\mathrm{Sh}(M,\mathcal{K})$ in Definition 3.2.13 contains rather wild objects, e.g. there are sheaves with stalks given by arbitrary objects in $\mathcal{K}$. A first step is to consider the subcategory $\mathrm{Sh}^c(M,\mathcal{K}) \subseteq \mathrm{Sh}(M,\mathcal{K})$ of compact objects, cf. e.g. [Lur09, Section 5.3.4]. Intuitively, $\mathrm{Sh}^c(M,\mathcal{K})$ are objects with some finiteness properties, in a sense they are "finitely presented".

Example 3.2.14. For specificity, consider the category $\mathcal{K} = \mathrm{Mod}(k)$, understood as sheaves over a point $\mathcal{K} = \mathrm{Sh}(\{*\},\mathcal{K})$. Then the stalk of a sheaf $\mathrm{Sh}(\{*\},\mathcal{K})$ can be any object in $\mathcal{K}$, such as an unbounded complex of possibly infinity rank k-modules in each degree. Considering compact objects cuts out a subcategory $\mathrm{Mod}(k)^c \subseteq \mathrm{Mod}(k)$ which can be shown to be equivalent to perfect complexes, i.e. complexes in $\mathrm{Mod}(k)$ which are quasi-isomorphic to bounded complexes of finite projective k-modules. The latter is a more manageable category, as its objects can be presented by "finite data": in each degree of the complex, the k-module is of finite rank, and the complex is non-zero only in finitely many degrees. □

[3] I.e. the inclusion of sheaves into presheaves is fully faithful.

3.3. The category of sheaves of a Legendrian Λ

Our focus is to study Legendrian submanifolds $\Lambda \subseteq (T^\infty M, \xi_{\mathrm{st}})$. The notion of singular support, in Definition 3.2.8, provides the key link to associate a category to such a Λ:

Definition 3.3.1 (Sheaf category of Λ)**.** Let M be a real analytic manifold, $\Lambda \subseteq (T^\infty M, \xi_{\mathrm{st}})$ a Legendrian and $\mathcal{K}$ a coefficient category. By definition, the category $\mathrm{Sh}^c_\Lambda(M, \mathcal{K})$ of compact sheaves on M with values in $\mathcal{K}$ and singular support in Λ is the subcategory of $\mathrm{Sh}^c(M, \mathcal{K})$ given by the objects $\mathscr{F} \in \mathrm{Sh}^c(M, \mathcal{K})$ that satisfy $\mathrm{SS}(\mathscr{F}) \subseteq \Lambda$. Informally,

$$\mathrm{Sh}^c_\Lambda(M, \mathcal{K}) := \{\mathscr{F} \in \mathrm{Sh}^c(M, \mathcal{K}) \text{ s.t. } \mathrm{SS}(\mathscr{F}) \subseteq \Lambda\}.$$

The category $\mathrm{Sh}^c_\Lambda(M, \mathcal{K})$ in Definition 3.3.1, of sheaves that represent (a part of) Λ, will be used to study a given Legendrian $\Lambda \in (T^\infty M, \xi_{\mathrm{st}})$. The following properties of $\mathrm{Sh}^c_\Lambda(M, \mathcal{K})$ are useful in that task:

1. $\mathrm{Sh}^c_\Lambda(M, \mathcal{K})$ is a Legendrian invariant of Λ. This is discussed in Chapter 5.

2. The category $\mathrm{Sh}^c_\Lambda(M, \mathcal{K})$ is smooth and, in fact, finite type. In consequence, its derived stack $\mathfrak{M}(\Lambda)$ of pseudoperfect objects, as defined in [TV07, Def. 3.2], is a reasonably behaved algebraic geometric object, cf. [TV07, Theorem 3.6]. It can occasionally be more manageable to work with $\mathfrak{M}(\Lambda)$, rather than $\mathrm{Sh}^c_\Lambda(M, \mathcal{K})$ directly, as we can apply techniques from algebraic geometry.[4] These geometric spaces are discussed in more detail in Section 3.3.1 below.

   An important feature of $\mathfrak{M}(\Lambda)$ is that the tangent complex

   $$T_{\mathscr{F}}\mathfrak{M}(\Lambda) \simeq \mathrm{Hom}(\mathscr{F}, \mathscr{F})$$

   records the endomorphisms of any pseudoproper object $\mathscr{F} \in \mathrm{Sh}^{pp}_\Lambda(M, \mathcal{K})$. In Chapter 6 we will describe in detail a connected component of $\mathfrak{M}(\Lambda)$ for a class of Legendrian links $\Lambda \subseteq (T^\infty \mathbb{R}^2, \xi_{\mathrm{st}})$.

3. For $\Lambda \subseteq (T^\infty \mathbb{R}^2, \xi_{\mathrm{st}})$ a Legendrian link, there exists a natural functor

   $$\mathfrak{m}_\Lambda : \mathrm{Sh}^c(M, \mathcal{K}) \longrightarrow \mathrm{Loc}^c(\Lambda, \mathcal{K})$$

   from this category to the category of compact local systems on Λ with coefficients in $\mathcal{K}$. This functor $\mathfrak{m}_\Lambda$ is known as the microlocalization functor, and it is an instance of the theory of $\mu\mathrm{Hom}$ and $\mu\mathrm{Sh}_\Lambda$, cf. [KS90, Chapter VI], [Gui23, Parts 10& 11] or [Nad16, Section 3.4]. In practice, $\mathfrak{m}_\Lambda(\mathscr{F})$ is computed by studying the microstalks of $\mathscr{F}$, which are the stalks of the local system $\mathfrak{m}_\Lambda(\mathscr{F})$.

   Note that the codomain category can be neatly described as $\mathrm{Perf}(C_{-*}(\Omega\Lambda))$, the category of perfect modules over the algebra of chains in the based loop space of Λ. For instance, if Λ is a Legendrian link with c components, then $\mathrm{Loc}^c(\Lambda, \mathcal{K})$ is equivalent to perfect modules over the ring $k\langle t_1^{\pm 1}, \ldots, t_c^{\pm 1}\rangle$.

Remark 3.3.2. The microstalk functors

$$\mu_{(x,\xi)} : \mathrm{Sh}^c(M, \mathcal{K}) \longrightarrow \mathrm{Perf}(k), \quad \mathscr{F} \longmapsto \mathscr{F}_{(x,\xi)}$$

are corepresentable, i.e. of the form $\mu_{(x,\xi)} = \mathrm{Hom}(M_{(x,\xi)}, \cdot)$ for some $M_{(x,\xi)} \in \mathrm{Sh}^c(M, \mathcal{K})$. These corepresentatives of the microstalk functors generate $\mathrm{Sh}^c(M, \mathcal{K})$, cf. [Nad16, Lemma 3.15] or [GPS24, Prop. 4.17]. □

[4] Intuitively, $\mathrm{Sh}^c_\Lambda(M, \mathcal{K})$ is fundamentally a non-commutative being and $\mathfrak{M}_\Lambda$ is a commutative being.

3.3.1 The moduli of sheaves of a Legendrian Λ

It can be useful to extract a geometric space out of the category $\mathrm{Sh}^c_\Lambda(M, \mathcal{K})$. For that we use results from derived algebraic geometry, working within the framework of D^--stacks, cf. [TV05, TV08], and specifically using the concept of the derived moduli stack of pseudo-perfect objects of a category, as introduced in [TV07, Section 3.1].

In a nutshell, given a dg-category $\mathcal{C}$ of finite type, there is a geometric space $\mathfrak{M}_\mathcal{C}$ associated to it. The geometric space is rigorously defined as the D^--stack associated to the functor of points

$$\mathfrak{M}_\mathcal{C} : \mathrm{sCAlg}_k \to \mathrm{sSet}, \quad \mathfrak{M}_\mathcal{C}(A) = \mathrm{Map}_{\text{dg-cat}_k}(\mathcal{C}^{op}, \mathrm{Perf}(A)).$$

Here sCAlg_k is the category of simplicial commutative k-algebras (denoted sk-CAlg in [TV07, Section 2.3]), sSet the category of simplicial sets, $\mathrm{Perf}(A)$ is the dg-category of perfect A-modules (denoted $\hat{A}_{pe}$ in [TV07, Section 2.4]), and $\mathrm{Map}_{\text{dg-cat}_k}$ denotes the mapping space of a model structure for the category of small dg-categories. See [TV07, Section 3] for the necessary details.[5] The D^--stack $\mathfrak{M}_\mathcal{C}$ is said to be the moduli of pseudo-perfect objects of $\mathcal{C}$.

This is a functorial construction, in that the assignment $\mathcal{C} \mapsto \mathfrak{M}_\mathcal{C}$ defines a functor

$$\mathfrak{M} : \mathrm{Ho}(\text{dg-cat}_k)^{op} \longrightarrow D^-\mathrm{St}(k) \tag{3.3.1}$$

between the opposite of the homotopy category $\mathrm{Ho}(\text{dg-cat}_k)$ of dg-categories and the category $D^-\mathrm{St}(k)$ of D^--stacks, the functor being enriched over the homotopy category of sSet. In particular, given a dg-functor $f : \mathcal{C} \longrightarrow \mathcal{D}$, there is a map $\mathfrak{M}(f) : \mathfrak{M}(\mathcal{D}) \longrightarrow \mathfrak{M}(\mathcal{C})$, which sends a pseudo-perfect object $\mathcal{D}^{op} \longrightarrow \mathrm{Perf}(k)$ to its pull-back via $\mathcal{C}^{op} \longrightarrow \mathcal{D}^{op} \longrightarrow \mathrm{Perf}(k)$.

By [TV07, Theorem 3.6], $\mathfrak{M}_\mathcal{C}$ is a reasonably geometric space, e.g. locally geometric and locally of finite presentation, if $\mathcal{C}$ is of finite type. In the case of $\Lambda \subseteq (T^\infty M, \xi_{\mathrm{st}})$ a Legendrian link, $\mathrm{Sh}^c_\Lambda(M, \mathcal{K})$ is of finite type by the results from [Nad17, Sta18].[6] We are thus lead to the following:

Definition 3.3.3 (Moduli of sheaves with singular support on Λ)**.** Let $\Lambda \subseteq (T^\infty M, \xi_{\mathrm{st}})$ be a Legendrian submanifold. By definition, the moduli $\mathfrak{M}(\Lambda)$ of $\mathcal{K}$-valued sheaves with singular support on Λ is the derived stack of pseudo-perfect objects of $\mathrm{Sh}^c_\Lambda(M, \mathcal{K})$.

The space $\mathfrak{M}_\Lambda$ typically has many interesting components and substacks. To wit, if $\mathcal{K} = \mathrm{Mod}(k)$, we can already focus on sheaves whose microstalk is concentrated in one degree and, for instance, also specify the isomorphism type of the k-module in that degree. Throughout these notes, we often focus on substacks of this form, cf. Section 6.2.

3.3.2 Singular support and Maslov potential

In this brief technical aside, we remark that there are two pieces of data often attached to sheaves and Legendrians:

1. For a sheaf $\mathscr{F} \in \mathrm{Sh}(M, \mathcal{K})$ and a point $(x, \xi) \in SS(\mathscr{F})$, there exists a half-integer d, called the *shift* of $\mathscr{F}$ at (x, ξ). Intuitively, it captures the homological degree of the microstalk at (x, ξ). This number is defined in [KS90, Definition 7.5.4], and see also [Gui23, Section 1.4].

[5] In zero characteristic zero, sCAlg is equivalent to $\mathrm{cdga}_{\leq 0}$, via the appropriate version of the Dold-Kan correspondence. Therefore, in zero characteristic, the inputs for (functor of points of) the derived stack $\mathfrak{M}_\mathcal{C}$ can be taken to be non-positively graded commutative dg-algebras.

[6] In generality, it is known that these sheaves categories are smooth, cf. e.g. [GPS24, Corollary 4.26].

2. For a Legendrian $\Lambda \subseteq (T^\infty M, \xi_{st})$, there is the notion of Maslov class $\vartheta_\Lambda \in H^1(\Lambda, \mathbb{Z})$. Intuitively, it measures the failure of the Lagrangian projection of Λ to being graphical over the zero section.

For instance, for a Legendrian knot $\Lambda \subseteq (\mathbb{R}^3, \xi_{st})$, the Maslov potential assigns an integer number to each segment of the front diagram connecting two cusps, with the rule that it must increase or decrease one unit when passing through a cusp upwards or downwards. Such Maslov potential exists modulo twice the rotation number $2r(\Lambda)$, e.g. it does if $r(\Lambda) = 0$.

In these notes, we shall often focus on sheaves which are *pure*, as defined in [KS90, Section 7.5]: the microstalk at a point in Λ is concentrated in the degree essentially given by the Maslov potential at the segments containing that point.

3.4. Exercises

1. (Legendrian unknot) Consider the inclusion $i : Q \longrightarrow \mathbb{R}^2_{x,z}$ of the set Q in $\mathbb{R}^2$ as depicted in Figure 3.5(left). Specifically, its upper boundary $\partial_+ Q$ and lower boundary $\partial_- Q$ are parametrized by

$$\partial_\pm Q = \left(x, \mp \left(\frac{1-x^2}{3}\right)^{3/2} \mp (x^2 - 1) \left(\frac{1-x^2}{3}\right)^{1/2} \right), \quad x \in [-1, 1].$$

The (image of the) set Q includes $\partial_+ Q$, depicted in solid blue, but does not include $\partial_- Q$, depicted in dashed blue.

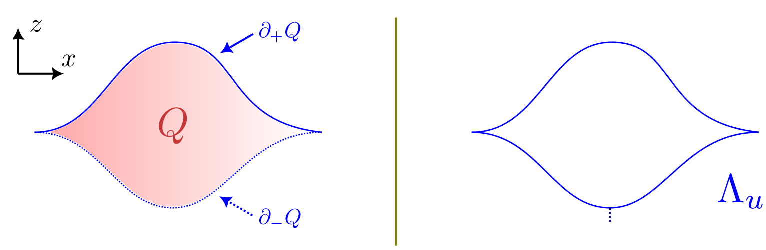

Figure 3.5: (Left) The set $Q \subseteq \mathbb{R}^2_{x,z}$ in Exercise 1, here $\partial_+ Q \subseteq Q$ is depicted in solid blue and $\partial_- Q \subseteq \overline{Q}$, which is not considered part of Q, in dashed blue. (Right) A front for the standard Legendrian unknot $\Lambda_u \subseteq (\mathbb{R}^3, \xi_{st})$.

(i) Consider the sheaf $\mathscr{F} := i_! k \in \mathrm{Sh}^c(\mathbb{R}^2, \mathrm{Mod}(k))$. Show that $\mathrm{SS}(\mathscr{F}) \subseteq \Lambda_u$, where $\Lambda_u \subseteq (\mathbb{R}^3, \xi_{st})$ is the Legendrian lift of the front in Figure 3.5 (right).

(ii) By (i), $\mathscr{F} \in \mathrm{Sh}^c_{\Lambda_u}(\mathbb{R}^2, \mathrm{Mod}(k))$. Compute its endomorphisms $\mathrm{End}(\mathscr{F})$ in this category.

2. (Legendrian Hopf link) Let $\Lambda_h \subseteq (\mathbb{R}^3, \xi_{st})$ be the Legendrian Hopf link whose front is as depicted in Figure 3.6, one component in blue and the other in green, cf. also Figure 2.16(Right). Consider the two inclusions $i_1 : Q_1 \longrightarrow \mathbb{R}^2$ and $i_2 : Q_2 \longrightarrow \mathbb{R}^2$ of the sets as depicted in the figure, cf. Exercise 1 above.

(i) Consider the sheaves $\mathscr{F}_1 := (i_1)_! k$ and $\mathscr{F}_2 := (i_2)_! k$ in $\mathrm{Sh}^c(\mathbb{R}^2, \mathrm{Mod}(k))$ and their direct sum $\mathscr{F}_\oplus = \mathscr{F}_1 \oplus \mathscr{F}_2$. Show that $\mathrm{SS}(\mathscr{F}_\oplus) \subseteq \Lambda_h$.

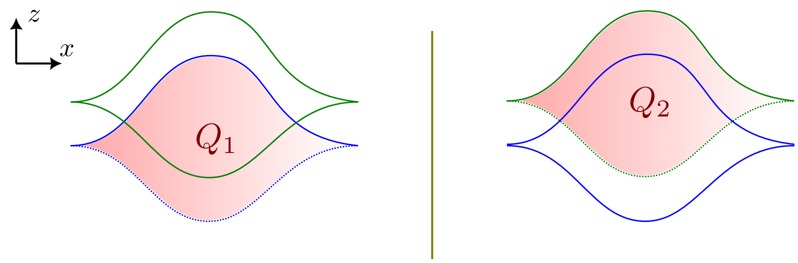

Figure 3.6: The sets $Q_1, Q_2 \subseteq \mathbb{R}^2_{x,z}$ in Exercise 2.

(ii) Find a sheaf $\mathscr{G} \in \mathrm{Sh}^c_{\Lambda_h}(\mathbb{R}^2, \mathrm{Mod}(k))$ which sits in a short exact sequence between $\mathscr{F}_1$ and $\mathscr{F}_2$ but it is *not* isomorphic to $\mathscr{F}_\oplus$.

(iii) Formulate and solve analogs of parts (i) and (ii) for the other two fronts of Λ_h in Figure 2.16.

3. (Examples from Legendrian knots to sheaves) For each of the Legendrian lifts Λ of the fronts $\pi(\Lambda) \subseteq (\mathbb{R}^3, \xi_{\mathrm{st}})$ depicted in Figure 3.7, find a sheaf $\mathscr{F} \in \mathrm{Sh}^c(\mathbb{R}^2, \mathrm{Mod}(k))$ such that $\mathrm{SS}(\mathscr{F}) = \Lambda$. (You may select any Maslov potential.)

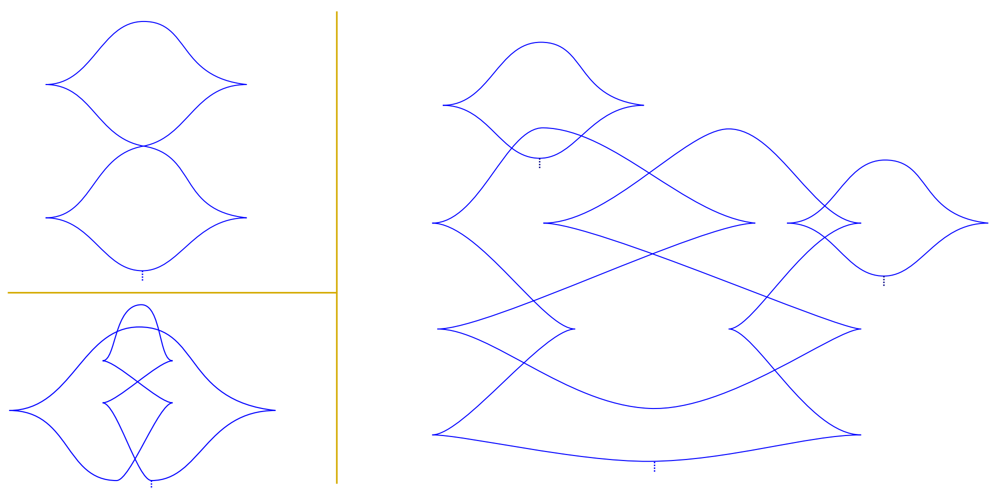

Figure 3.7: A series of fronts for Legendrian links in $\Lambda \subseteq (\mathbb{R}^3, \xi_{\mathrm{st}})$ for Exercise 3. (Upper left) A front for the Legendrian unknot $\Lambda_u \subseteq (\mathbb{R}^3, \xi_{\mathrm{st}})$. (Bottom left) A front for the max-tb Legendrian right-handed trefoil. (Right) A Legendrian link with components given by a max-tb Legendrian $m(5_2)$ and two Legendrian unknots, non-trivially linked to it.

4. Consider the front $\pi \subseteq \mathbb{R}^2$ in Figure 3.7(Upper left) and denote by $Q_u, Q_b \subseteq \mathbb{R}^2$ the upper and bottom bounded connected components of $\mathbb{R}^2 \setminus \pi$, as in Exercise 1 above, with inclusions $i_u : Q_u \longrightarrow \mathbb{R}^2$ and $i_b : Q_b \longrightarrow \mathbb{R}^2$.[7] Let $\mathscr{F}_u = i_! k_{Q_u}$ and $\mathscr{F}_b = i_! k_{Q_b}$ be their associated characteristic sheaves. Compute the singular support of $\mathscr{F}_u \oplus \mathscr{F}_b$.

5. For the examples in Section 2.5, given by Legendrian links of the form $\Lambda = \Lambda_\beta$, as in Section 2.5.3, and the Legendrian links $\Lambda = \Lambda(\mathbb{G})$ associated to plabic graph, as in Section 2.5.1:

[7] So Q_u, Q_b contain their the upper boundaries ∂_+ but not their lower boundaries ∂_-.

1. Show that there always exists a sheaf $\mathscr{F} \in \mathrm{Sh}(\mathbb{R}^2, \mathrm{Mod}(k))$ with $\mathrm{SS}(\mathscr{F}) = \Lambda$.
2. For the sheaf you found in (1), compute its algebra of endomorphisms.

6. Consider a Legendrian $\Lambda \subseteq (\mathbb{R}^{2n+1}, \xi_{\mathrm{st}})$ with a front $\pi(\Lambda) \subseteq \mathbb{R}^{n+1}$, Q a smooth q-dimensional submanifold, and $\Lambda \times Q \subseteq \mathbb{R}^{2(n+q)+1}$ its trivial front spinning, cf. Section 2.5.4. Describe $\mathrm{Sh}_{\Lambda\times Q}(\mathbb{R}^{n+q+1}, \mathrm{Mod}(k))$ in terms of $\mathrm{Sh}_{\Lambda}(\mathbb{R}^{n+1}, \mathrm{Mod}(k))$.

7. For any $n \in \mathbb{N}$, provide an example of a Legendrian submanifold $\Lambda \subseteq (T^*\mathbb{R}^n, \xi_{\mathrm{st}})$ such that there exists no sheaf $\mathscr{F} \in \mathrm{Sh}_{\Lambda}(\mathbb{R}^n, \mathrm{Mod}(k))$.

8. Let $f : M \longrightarrow \mathbb{R}$ be a Morse function on a closed smooth manifold M and $k_M \in \mathrm{Sh}(M, \mathrm{Mod}(k))$ the constant sheaf. Compute the singular support of Rf_*k_M.

9. Let $\Lambda = \emptyset \subseteq (T^\infty S^n, \xi_{\mathrm{st}})$ be the empty Legendrian submanifold, $n \geq 3$. Show that $\mathrm{Sh}_{\emptyset}(S^n, \mathrm{Mod}(k))$ is equivalent to the category of modules over the polynomial algebra $k[x]$, where $|x| = n - 1$.

10. Show that the stalk and microstalk functors, as described in Definition 3.2.3 and Definition 3.2.10, are both corepresentable functors. In addition, try to describe, at least intuitively, their corepresentatives.

Hint: left adjoints can be described via left Kan extensions.

3.5. Key references

These resources might be helpful to complement and expand the content of this chapter:

1. The foundational textbook [KS90] contains a thorough treatment of the definition and properties of singular support. Though written in the framework of (classically) derived categories, many of the results extend with less restrictive technical hypothesis, e.g. working with unbounded complexes, or dg-categories and ∞-categories. See for instance [RS18], [CL23, Appendix A], [CL24, Appendix A] and references therein. The authors of [KS90] have also written a number of accounts autour the theme of [KS90], see e.g. [KS01, KS85a].

2. As a rule of thumb, many of the research articles and monographs by the authors of [KS90] and, more recently, S. Guillermou and C. Viterbo, tend to be rigorous and enlightening contributions to the microlocal theory of sheaves, often in relation to symplectic topology. To wit, the notes [Vit10] provide a helpful introductory account explaining the relation between modern contact and symplectic topology and the microlocal theory of sheaves. The monograph [Gui23] is a wonderful follow-up to C. Viterbo's notes, with an emphasis on the applications of sheaf theory to contact and symplectic topology.

3. Beyond textbooks and monographs, the school "Symplectic topology, sheaves and mirror symmetry" took place in Paris in Summer 2016 at the IMJ-PRG. Its website contains a number of lecture notes that can be of use. Prior to that event, a group of mathematicians in France ran a *groupe de travail* in 2013-14 on [GKS12], with some of the materials available online.[8]

4. For the derived stacks appearing in Section 3.3.1, [TV07] applies to (finite type) dg-categories. Experts assert that the results can nevertheless be generalized to ∞-categories. This is implicitly used already in some articles in the literature, see e.g. [PT25a, PT25b].

[8] Active urls: "school2016.imj-prg.fr" and "imo.universite-paris-saclay.fr/$\sim$patrick.massot/en/gdtSS/gdt2013.html".

Chapter 4: Exodromy description of singular support

Let $\Lambda \subseteq (T^\infty M, \xi_{\text{st}})$ be a Legendrian submanifold: the present goal is to describe the category $\text{Sh}^c_\Lambda(M, \mathcal{K})$ in a reasonably explicit manner. For instance, we will be able to describe objects in $\text{Sh}^c_\Lambda(M, \mathcal{K})$ in terms of linear algebra, understood as the study of functors from posets to $\text{Mod}(k)$. The overarching strategy to describe $\text{Sh}^c_\Lambda(M, \mathcal{K})$ explicitly is:

1. Introduce the category $\text{Sh}^c_{cons}(M, \mathcal{K}) \subseteq \text{Sh}^c(M, \mathcal{K})$ of constructible sheaves, which has the important property that
$$\text{Sh}^c_\Lambda(M, \mathcal{K}) \subseteq \text{Sh}^c_{cons}(M, \mathcal{K}).$$
2. Describe the category $\text{Sh}^c_{cons}(M, \mathcal{K})$ in more combinatorial terms. In this case, in terms of $\mathcal{K}$-valued modules over certain posets.
3. Given a Legendrian Λ, translate the condition $\text{SS}(\mathscr{F}) \subseteq \Lambda$ for a sheaf $\mathscr{F} \in \text{Sh}^c_{cons}(M, \mathcal{K})$ in these combinatorial terms. This yields a combinatorial description of the subcategory $\text{Sh}^c_\Lambda(M, \mathcal{K}) \subseteq \text{Sh}^c_{cons}(M, \mathcal{K})$ we are studying.

The present chapter implements this strategy. An additional advantage of the notion of singular support is that it crystallizes some of the classical concepts in sheaf theory and their relation to each other, see Chapter 4.

Classical concept	**Microlocal viewpoint**
Local systems $\text{Loc}(M, \mathcal{K})$	$\text{Sh}_\emptyset(M; \mathcal{K})$, i.e. empty singular support
$\mathcal{S}$-constructible sheaves $\text{Sh}^{\mathcal{S}}(M; \mathcal{K})$	$\text{Sh}_{\nu^*\mathcal{S}}(M; \mathcal{K})$, i.e. singular support in conormal
Stalk functor $\text{Sh}(M, \mathcal{K}) \longrightarrow \mathcal{K}$	Microstalk $\text{Sh}_\Lambda(M, \mathcal{K}) \longrightarrow \mathcal{K}$ with $\xi = \emptyset$.

Table 4.1: Table summary of classical notions in sheaf theory expressed in terms of singular support, i.e. from the microlocal viewpoint.

In a nutshell, $\text{Sh}_\Lambda(M, \mathcal{K})$ lies in between
$$\text{Loc}(M, \mathcal{K}) \subseteq \text{Sh}_\Lambda(M, \mathcal{K}) \subseteq \text{Sh}^{\mathcal{S}}(M, \mathcal{K})$$
where $\mathcal{S}$ is a stratification given by any front projection of Λ and $\text{Sh}^{\mathcal{S}}(M, \mathcal{K}) \subseteq \text{Sh}_{cons}(M, \mathcal{K})$ is the category of constructible sheaves with respect to that specific stratification $\mathcal{S}$. From the viewpoint of Chapter 4, these inclusions are simply inclusions of singular support:
$$\boxed{\text{Sh}_\emptyset(M, \mathcal{K}) \subseteq \text{Sh}_\Lambda(M, \mathcal{K}) \subseteq \text{Sh}_{\nu^*\mathcal{S}}(M, \mathcal{K})} \tag{4.0.1}$$

4.1. A motivation for constructibility

Let $\Lambda \subseteq (T^\infty M, \xi_{\text{st}})$ be a Legendrian submanifold. The category $\text{Sh}^c(M, \mathcal{K})$ can be difficult to work with: for instance, it contains sheaves whose stalks vary from point to point, and can do so in a rather uncontrolled manner. Even if the stalks of a sheaf $\mathscr{F} \in \text{Sh}^c(M, \mathcal{K})$ do not vary, the associated morphisms ρ might be challenging to understand. Thus, trying to describe the subcategory $\mathrm{Sh}^c_\Lambda(M,\mathcal{K}) \subseteq \mathrm{Sh}^c(M,\mathcal{K})$ by starting with the ambient category $\mathrm{Sh}^c(M,\mathcal{K})$ can require a fair amount of work.

The first important fact is that there exists a better understood subcategory

$$\mathrm{Sh}^c_{cons}(M,\mathcal{K}) \subseteq \mathrm{Sh}^c(M,\mathcal{K})$$

which contains $\mathrm{Sh}^c_\Lambda(M,\mathcal{K})$. Such category $\mathrm{Sh}^c_{cons}(M,\mathcal{K})$ is known as the category of constructible sheaves. The result we use is [KS90, Theorem 8.4.2], which states:

Theorem 3 (Microlocal characterization of constructibility)**.** Let M be a real analytic manifold and $\mathscr{F} \in \mathrm{Sh}^c(M,\mathcal{K})$ a sheaf. Then

$$\mathscr{F} \in \mathrm{Sh}^c_{cons}(M,\mathcal{K}) \Longleftrightarrow \mathrm{SS}(\mathscr{F}) \subseteq (T^\infty M, \xi_{\mathrm{st}}) \text{ is Legendrian.}$$

In particular, $\mathrm{Sh}^c_\Lambda(M,\mathcal{K}) \subseteq \mathrm{Sh}^c_{cons}(M,\mathcal{K})$ for any Legendrian $\Lambda \subseteq (T^\infty M, \xi_{\mathrm{st}})$.

Constructible sheaves and their categories are rather classical objects, preceding the microlocal theory of sheaves, cf. e.g. [Sch03, Chapter 2] or [Gor20, Sections 5.7&5.8]. For its connection to the microlocal theory of sheaves, cf. [KS90, Chapter VIII]. From Theorem 3, we have the faithful inclusions

$$\mathrm{Sh}^c_\Lambda(M,\mathcal{K}) \subseteq \mathrm{Sh}^c_{cons}(M,\mathcal{K}) \subseteq \mathrm{Sh}^c(M,\mathcal{K}) \tag{4.1.1}$$

of categories of sheaves, for any given Legendrian $\Lambda \subseteq (T^\infty M, \xi_{\mathrm{st}})$. Since we are focused on $\mathrm{Sh}^c_\Lambda(M,\mathcal{K})$, once we understand $\mathrm{Sh}^c_{cons}(M,\mathcal{K})$, we use Equation (4.1.1) to effectively forget about $\mathrm{Sh}^c(M,\mathcal{K})$. Note that only $\mathrm{Sh}^c_\Lambda(M,\mathcal{K})$ in Equation (4.1.1) depends on the given Legendrian Λ.

This chapter will now discuss the category of constructible sheaves $\mathrm{Sh}^c_{cons}(M,\mathcal{K})$, with an eye towards studying Legendrian submanifolds via the first inclusion of Equation (4.1.1). The main result will be the Exodromy Equivalence (Theorem 5) describing constructible sheaves in a reasonably down-to-Earth manner. As a toy case, which is logically needed, we will discuss local systems and the Monodromy Equivalence (Theorem 4).

4.2. Locally constant sheaves and the Monodromy Equivalence

For the purposes of studying Legendrian submanifolds, Theorem 3 implies that we can focus on a particular class of sheaves on M, known as *constructible sheaves*. These are sheaves for which there exists a stratification of M such that the sheaves are *locally constant* along each stratum. Thus, we first need to understand locally constant sheaves:

Definition 4.2.1 (Locally constant sheaves)**.** A sheaf $\mathscr{F}$ on M is said to be locally constant if for every point $x \in M$ there exists a neighborhood $U \subseteq M$ such that the restriction $\mathscr{F}|_U$ is isomorphic to the constant sheaf k_U.

Locally constant sheaves are also known as local systems in the literature. The constant sheaf k_M is an example of a locally constant sheaf but, typically, there exist many locally constant sheaves which are not *globally* constant.

Example 4.2.2. Consider the circle $S^1 = \mathbb{R}/\mathbb{Z}$ and the covering map

$$f_\ell : S^1 \longrightarrow S^1, \quad \theta \mapsto \ell\theta, \quad \ell \in \mathbb{Z}$$

and the sheaf $(f_\ell)_* k_{S^1} \in \mathrm{Sh}(S^1, \mathcal{K})$, as in Example 3.2.2.(2). Then $(f_\ell)_* k_{S^1}$ is locally constant but not isomorphic to k_{S^1}. Intuitively, when we study the local sections of $(f_\ell)_* k_{S^1}$ for small open sets going around S^1 and back to where we start, the resulting map, from the space of sections on the initial open set to itself, is non-trivial. This is formalized by the notion of parallel transport and monodromy. □

Let $\mathrm{Loc}(M, \mathcal{K}) \subseteq \mathrm{Sh}(M, \mathcal{K})$ denote the subcategory of locally constant sheaves on M. The following fact allows us to work with locally constant sheaves in a more manageable manner than with an arbitrary sheaf:

Theorem 4 (Monodromy Equivalence)**.** Let M be a connected smooth manifold, $\Pi_\infty(M)$ is fundamental ∞-groupoid, and $\mathrm{Loc}(M, \mathcal{K}) \subseteq \mathrm{Sh}(M, \mathcal{K})$ the subcategory of locally constant sheaves. Then there exists an equivalence of categories

$$\mathrm{Loc}(M, \mathcal{K}) \cong \mathrm{Fun}(\Pi_\infty(M), \mathcal{K}). \tag{4.2.1}$$

In particular, $\mathrm{Loc}(M, \mathcal{K})$ only depends on the homotopy type of M.

There are a number of incarnations of Theorem 4 in the literature, cf. e.g. [Lur17, Theorem A.4.19]. The following are consequences of Theorem 4:

1. Since $\Pi_\infty(M)$ is generated by the endomorphisms of a point, which are isomorphic to $C_{-*}(\Omega M)$, there exists an equivalence of categories

$$\boxed{\mathrm{Loc}(M, \mathcal{K}) \cong C_{-*}(\Omega M)\text{-mod}} \tag{4.2.2}$$

   between the category of locally constant sheaves and modules over the algebra $C_{-*}(\Omega M)$ of chains on the based loop space ΩM, endowed with the Pontryagin product. Similarly, the category of compact objects $\mathrm{Loc}^c(M, \mathcal{K}) \subseteq \mathrm{Loc}(M, \mathcal{K})$ is given by

$$\boxed{\mathrm{Loc}^c(M, \mathcal{K}) \cong \mathrm{Perf}(C_{-*}(\Omega M))} \tag{4.2.3}$$

2. Importantly, Equation (4.2.3) implies that if M is contractible then k_M is the unique locally constant sheaf on M, up to isomorphism.

   Another relevant instance is $M = S^1$. Then Equation (4.2.3) is the statement that

$$\mathrm{Loc}(S^1) \cong k[\pi_1(S^1)]\text{-mod} \cong k[\mathbb{Z}]\text{-mod} \cong k[t, t^{-1}]\text{-mod}$$

   since $\Omega S^1 \simeq \mathbb{Z}$ and $\pi_0(\Omega S^1) \cong \pi_1(S^1) \cong \mathbb{Z}$. Therefore, a locally constant sheaf of S^1 is given by the data of a k-module V and an automorphism $T \in \mathrm{Aut}_k(V)$. Then, $t \in k[t, t^{-1}]$ is the algebraic incarnation of the monodromy and acts on V via T.

   Similarly, if M is a surface with boundary – in our case Lagrangian fillings of Legendrian links – then $M \simeq K(\pi, 1)$, as M is homotopic to a wedge of circles. In particular, the objects of $\mathrm{Loc}(M, \mathcal{K})$ are simply representations of $\pi_1(M)$, i.e. modules over the non-commutative ring of Laurent polynomials in several variables.

3. In practice, Theorem 4 is used to describe a locally constant sheaf $\mathscr{F}$ by giving the data of any of its stalks[1] $\mathscr{F}_x$, which is simply giving an object of $\mathcal{K}$, and the $\mathrm{Aut}(\mathcal{K})$-valued monodromies based at that point $x \in M$, i.e. the subgroup of automorphisms of $\mathrm{Hom}(\mathscr{F}_x, \mathscr{F}_x)$ obtained by parallel transporting $\mathscr{F}_x$, using the ρ-morphisms, along chains in M based at x.

[1] It follows from Definition 4.2.1 that all the stalks of a locally constant sheaf in a connected manifold are isomorphic. Therefore, for locally constant sheaves, it suffices to give the stalk at one point to determine the isomorphism type of the stalks at any other.

To understand how locally constant sheaves relate to Legendrian submanifolds, in the way of Tenet 3.2.8, computing the singular support of a locally constant sheaf is a first step. In fact, the following stronger characterization holds:

Lemma 4.2.3. *Let $\mathscr{F} \in \mathrm{Sh}(M, \mathcal{K})$ be a sheaf. Then*

$$\mathscr{F} \text{ is locally constant} \Longleftrightarrow \mathrm{SS}(\mathscr{F}) = \emptyset.$$

Proof. ($\Longrightarrow$) By Definition 4.2.1, all restriction maps of a locally constant sheaf are isomorphisms. Therefore any microstalk ${}_f\mathscr{F}_{(x,\xi)}$, as in Definition 3.2.10, must vanish. By choosing f to be a test function and applying Proposition 3.2.9, we conclude that no codirection can be in the singular support.

($\Longleftarrow$) Similarly, $\mathrm{SS}(\mathscr{F}) = \emptyset$ implies that all restriction maps must be isomorphisms, as their cones are acyclic. Thus $\mathscr{F}$ is a locally constant sheaf. □

In the sense of Tenet 3.2.8, Lemma 4.2.3 implies that a locally constant sheaf on M represents the empty Legendrian in $(T^\infty M, \xi_{\mathrm{st}})$. That is the reason why, in many cases, one actually quotients by $\mathrm{Loc}(M, \mathcal{K})$ in the microlocal theory of sheaves, as they are essentially trivial objects from the microlocal perspective.

4.3. Constructible sheaves and the Exodromy Equivalence

A poset-stratified smooth manifold $(M, \mathcal{S})$ is a continuous surjection $\mathcal{S} : M \longrightarrow \mathcal{P}$ from M to a poset $\mathcal{P}$, the latter endowed with the Alexandroff topology. The ith dimensional strata $\mathcal{S}_i$ of $\mathcal{S}$ is defined to be:

$$\mathcal{S}_i := \{\mathcal{S}^{-1}(i) \subseteq M : i \in \mathcal{P} \text{ and } \mathcal{S}^{-1}(i) \neq 0\}.$$

Such $\{\mathcal{S}_i\}$ is a stratification of M in the classical sense. In particular, the closure of each stratum is a union of strata and the poset structure $i \leq j$ in $\mathcal{P}$ captures the inclusions $\mathcal{S}_i \subseteq \overline{\mathcal{S}_j}$. See e.g. [BGH20, Part I] or [Sch03, Chapter 4].

Example 4.3.1 (Stratifications from subsets)**.** Consider an open inclusion $i : U \longrightarrow M$, the associated closed inclusion $j : Z \longrightarrow M$ with $Z = M \setminus U$, and assume that $\partial\overline{U} = \partial Z$ is a smooth submanifold of M. Then we can consider the stratification $\mathcal{S}$ given by the three strata

$$\mathcal{S}_1 := M \setminus \overline{U}, \quad \mathcal{S}_2 := U, \quad \mathcal{S}_3 := \partial\overline{U} = \partial Z.$$

Since the only relevant inclusions are $\mathcal{S}_3 \subseteq \overline{\mathcal{S}_2}$ and $\mathcal{S}_3 \subseteq \overline{\mathcal{S}_1} = \mathcal{S}_1$, the associated poset is $\mathcal{P} = (\mathcal{S}_1 \leftarrow \mathcal{S}_3 \rightarrow \mathcal{S}_2)$, which is notation to indicate that the poset $\mathcal{P} = (\{\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3\}, <)$ has partial order $\mathcal{S}_3 < \mathcal{S}_1$ and $\mathcal{S}_3 < \mathcal{S}_2$. □

Example 4.3.2 (Stratifications of $\mathbb{R}^2$ from fronts)**.** The local fronts described in Section 2.4, cf. Figure 2.1, stratify the base of any Legendrian fibration. In particular, for a Legendrian link $\Lambda \subseteq (T^\infty\mathbb{R}^2, \xi_{\mathrm{st}})$, its front stratifies $\mathbb{R}^2$, with strata consisting of single points, single segments and open subsets of $\mathbb{R}^2$. □

Definition 4.3.3 (Constructible sheaves)**.** Let M be a smooth manifold and $\mathcal{S} : M \longrightarrow \mathcal{P}$ a stratification. A sheaf $\mathscr{F}$ on M is said to be $\mathcal{S}$-constructible if the restriction of $\mathscr{F}$ each stratum $\mathcal{S}_i \subseteq M$ is locally constant.

The subcategory of $\mathrm{Sh}(M, \mathcal{K})$ consisting of $\mathcal{S}$-constructible sheaves is denoted by $\mathrm{Sh}^{\mathcal{S}}(M, \mathcal{K})$. The subcategory of $\mathrm{Sh}(M, \mathcal{K})$ consisting of sheaves which are $\mathcal{S}$-constructible for *some* stratification $\mathcal{S}$ is denoted by $\mathrm{Sh}_{cons}(M, \mathcal{K})$.

Example 4.3.4. (1) A locally constant sheaf $\mathscr{F} \in \mathrm{Loc}(M, \mathcal{K})$ is $\mathcal{S}$-constructible with respect to the trivial stratification $\mathcal{S} : M \longrightarrow \{*\}$, where $\{*\}$ is the one-element poset. In fact, $\mathrm{Loc}(M, \mathcal{K}) \cong \mathrm{Sh}^{\mathcal{S}}(M, \mathcal{K})$ for this trivial stratification $\mathcal{S}$.

(2) Following Example 4.3.1, both $j_* k_Z$ and $i_! k_U$ are $\mathcal{S}$-constructible with respect to the stratification given by those three strata

$$\mathcal{S}_1 := M \setminus \overline{U}, \quad \mathcal{S}_2 := U, \quad \mathcal{S}_3 := \partial\overline{U} = \partial Z.$$

Note that $Z = \mathcal{S}_1 \cup \mathcal{S}_3$ and the stalks of $j_* k_Z$ in both $\mathcal{S}_1$ and $\mathcal{S}_3$ are isomorphic to k, while the stalks of $j_* k_Z$ in $\mathcal{S}_2$ vanish. In contrast, the stalks of $i_! k_U$ vanish in $M \setminus U = \mathcal{S}_1 \cup \mathcal{S}_3$ and are isomorphic to k in $\mathcal{S}_2$. $\square$

Example 4.3.4.(2) shows that a constructible sheaf $\mathscr{F} \in \mathrm{Sh}_{cons}(M, \mathcal{K})$ typically has non-empty singular support $\mathrm{SS}(\mathscr{F})$.

Remark 4.3.5. Example 4.3.4.(2) illustrates that the category $\mathrm{Loc}(M, \mathcal{K}) \subseteq \mathrm{Sh}(M, \mathcal{K})$ of locally constant sheaves is not closed under operations such as taking (direct or exceptional images) and related functors. The larger category $\mathrm{Sh}_{cons}(M, \mathcal{K})$ has the desirable property of being closed under the 6-functors, see e.g. [Ver76, Section 2] or [BH22, Theorem 1.6] for instances of the 6-functors formalism in the constructible context. Along with Theorem 3, this is a conceptual reason why we can focus on $\mathrm{Sh}_{cons}(M, \mathcal{K})$ and forget the larger category $\mathrm{Sh}(M, \mathcal{K})$. $\square$

Let $\mathcal{S} : M \longrightarrow \mathcal{P}$ be a stratification and $\mathrm{Exit}^{\mathcal{S}}(M)$ the category of $\mathcal{S}$-exit paths, as in [Lur17, A.6] or [BGH20, Example 8.4.2].[2] The objects of $\mathrm{Exit}^{\mathcal{S}}(M)$ are points in M and, intuitively, the morphisms are paths from any stratum $\mathcal{S}_i$ to any stratum $\mathcal{S}_j$ with $i < j$.[3] Coarsely speaking, $\mathrm{Exit}^{\mathcal{S}}(M)$ is a category that records the fundamental groupoid of each stratum $\Pi_\infty(\mathcal{S}_i)$ and the ways these strata are glued together. We use $\mathrm{Exit}^{\mathcal{S}}(M)$ to study $\mathcal{S}$-constructible sheaves $\mathrm{Sh}^{\mathcal{S}}(M, \mathcal{K})$ via the following result:

Theorem 5 (Exodromy Equivalence)**.** Let M be a connected smooth manifold, $\mathcal{S} : M \longrightarrow \mathcal{P}$ a Whitney stratification, $\mathrm{Exit}^{\mathcal{S}}(M)$ its exit-path category and $\mathrm{Sh}^{\mathcal{S}}(M, \mathcal{K}) \subseteq \mathrm{Sh}(M, \mathcal{K})$ the subcategory of constructible sheaves. Then there exists an equivalence of categories

$$\mathrm{Sh}^{\mathcal{S}}(M, \mathcal{K}) \cong \mathrm{Fun}(\mathrm{Exit}^{\mathcal{S}}(M), \mathcal{K}). \tag{4.3.1}$$

This is proven in [Lur17, Theorem A.9.3], cf. also [Tre09, Theorems 1.2 & 6.10] and [BGH20, Chapters 0&1]. Theorem 5 generalizes Theorem 4: as in Example 4.3.4.(1), $\mathrm{Loc}(M, \mathcal{K}) \cong \mathrm{Sh}^{\mathcal{S}}(M, \mathcal{K})$ if $\mathcal{S} : M \longrightarrow \{*\}$ is chosen as the trivial stratification, and for this trivial stratification all paths are exit-paths, and so there is also the equivalence $\mathrm{Exit}^{\mathcal{S}}(M) \cong \Pi_\infty(M)$. The nomenclature of *exodromy*, by the authors of [BGH20], is a portmanteau of *exit paths* and *monodromy*: it hints at the fact that the Exodromy Equivalence in Theorem 5 generalizes the Monodromy Equivalence in Theorem 4, in the same manner that $\mathrm{Exit}^{\mathcal{S}}(M)$ and the $\mathcal{K}$-valued representations $\mathrm{Fun}(\mathrm{Exit}^{\mathcal{S}}(M), \mathcal{K})$ of its exit paths generalize $\Pi_\infty(M)$ and its monodromies.

In practice, when studying Legendrian submanifolds, we often use Theorem 5 for stratifications $\mathcal{S}$ with particularly well-behaved. To precisely described "well-behaved", note that a sheaf does often *not* have sections on a stratum $\mathcal{S}_i$ of a stratification $\mathcal{S}$, as typically $\mathcal{S}_i$ is

[2]This category is denoted $\mathrm{Sing}^{\mathcal{S}}(M)$ in [Lur17], while $\Pi_\infty(M)$ is denoted by $\mathrm{Sing}(M)$ there.

[3]Thus the use of the word *exit*: the path must exit $\mathcal{S}_i$ into a "larger" stratum $\mathcal{S}_j$, larger in that $\mathcal{S}_i \subseteq \overline{\mathcal{S}_j}$.

not open. Instead, one can construct an open set in M which contains a given stratum $\mathcal{S}_i$ by adding other strata to it. By definition, $\mathcal{S}_i^\star$ the star of a stratum $\mathcal{S}_i$ is

$$\mathcal{S}_i^\star := \cup_{i\leq j}\mathcal{S}_j$$

i.e. the union of strata in $\mathcal{S}$ that contain the given stratum $\mathcal{S}_i$ in their closure. The type of well-behaved stratifications we mentioned above are:

Definition 4.3.6 (Contractible stratifications)**.** Let $\mathcal{S} : M \longrightarrow \mathcal{P}$ be a stratification. By definition, $\mathcal{S}$ is said to be contractible if, for each stratum $\mathcal{S}_i$ of $\mathcal{S}$, both the stratum $\mathcal{S}_i$ and its star $\mathcal{S}_i^\star$ are contractible.

In the following consequence, we interpret $\mathcal{P}$ as a category whose objects are the elements of the (partially ordered) set $\{str_i\}_{i\in\mathcal{P}}$ and its morphisms are given by the partial order. That is $\mathrm{Hom}(\mathcal{S}_i, \mathcal{S}_j)$ contains a unique element if $i \leq j$ and it is empty otherwise. The category $\mathrm{Fun}(\mathcal{P}, \mathcal{K})$ of $\mathcal{K}$-valued functors on $\mathcal{P}$ is often referred to as the category of representation or modules over the poset. For contractible stratifications, Theorem 5 implies:

Corollary 4.3.7. *Let $\mathcal{S} :\longrightarrow \mathcal{P}$ be a contractible stratification of M, with $\mathcal{P} = \{\mathcal{S}_i\}_{i\in I}$. Then*

$$\mathrm{Sh}^{\mathcal{S}}(M, \mathcal{K}) \cong \mathit{Fun}(\mathcal{P}, \mathcal{K}). \tag{4.3.2}$$

Specifically, an equivalence is given by mapping $\mathscr{F} \in \mathrm{Sh}^{\mathcal{S}}(M, \mathcal{K})$ to the functor

$$F(\mathscr{F}) : \mathcal{P} \longrightarrow \mathcal{K}, \quad F(\mathscr{F})(\mathcal{S}_i) := \Gamma(\mathcal{S}_i^\star, \mathscr{F})$$

that assigns the derived sections of a sheaf $\mathscr{F}$ to each stratum. □

Proof. For a contractible stratification, the stratification $\mathcal{S} : M \longrightarrow |\mathcal{P}|$ induces an equivalence of categories $\mathrm{Exit}^{\mathcal{S}}(M) \cong \mathcal{P}$ and the result follows from Theorem 5. □

See also [GPS24, Lemma 4.5] or [**?**, Prop. 3.9]. For completeness, the functors $F(\mathscr{F})$ in Corollary 4.3.7 are determined on morphisms via

$$F(\mathscr{F})(i \xrightarrow{\leq} j) := \{f_{ij} : \Gamma(\mathcal{S}_i^\star, \mathscr{F}) \longrightarrow \Gamma(\mathcal{S}_j^\star, \mathscr{F})\},$$

and zero if $i \not\leq j$. Here we have used that an inclusion $\mathcal{S}_i \subseteq \overline{\mathcal{S}_j}$ induces an inclusion $\mathcal{S}_i^\star \subseteq \mathcal{S}_j^\star$ of stars, and thus a (unique) map $f_{ij} : \Gamma(\mathcal{S}_i^\star, \mathscr{F}) \longrightarrow \Gamma(\mathcal{S}_j^\star, \mathscr{F})$ between the corresponding derived sections of $\mathscr{F}$. For a contractible stratification, we also have the following useful property:

$$\mathscr{F}_x \cong \Gamma(\mathcal{S}_i^\star, \mathscr{F}) \text{ if } x \in \mathcal{S}_i. \tag{4.3.3}$$

From this viewpoint, the representation of the poset $\mathcal{P}$ from $\mathscr{F}$ is readily described on objects by assigning $i \longrightarrow \mathscr{F}_x$, where $i \in \mathcal{P}$ and $x \in \mathcal{S}_i$. The isomorphism type of $\mathscr{F}_x$ is independent of $x \in \mathcal{S}_i$ since the sheaf $\mathscr{F}$ is $\mathcal{S}$-constructible.

Example 4.3.8 (Contractible stratifications of $\mathbb{R}^2$ from fronts)**.** The three stratifications of Example 4.3.2 are contractible and thus . □

Chapter 5: Invariance and sheaf quantization

The first goal of this section is to show that the category $\mathrm{Sh}^c_\Lambda(M, \mathcal{K})$ is a Legendrian invariant of Λ. Such invariance follows from a more general procedure, often known as *sheaf quantization* in the literature. The second goal of this section will be to discuss this sheaf quantization is a more general setting, with a view towards studying Lagrangian fillings of Legendrian submanifolds.

5.1. Contact invariance of $\mathrm{Sh}^c_\Lambda(M, \mathcal{K})$

The invariance of $\mathrm{Sh}^c_\Lambda(M, \mathcal{K})$ under contact isotopies is established in [GKS12, Section 3], see also e.g. [STZ17, Section 4]. The precise statement is Theorem 6 below. There are two important aspects to the result:

1. In practice, if Λ_0 and Λ_1 are related by an explicit Legendrian isotopy, we want the equivalence between their corresponding categories $\mathrm{Sh}^c_{\Lambda_0}(M, \mathcal{K})$ and $\mathrm{Sh}^c_{\Lambda_1}(M, \mathcal{K})$ to be as computable and explicit as possible. In Section 5.1.1 we gain intuition for the invariance in Theorem 6 by explicitly studying how sheaves change under Reidemeister moves for Legendrian links. (This provides a hands-on proof of Theorem 6 for Legendrian links.)
2. The proof of Theorem 6, even is reasonably simple, is as important as its statement. It starts to highlight the principle of sheaf quantization which, at core, addresses the following problem: given a Legendrian $\Lambda \subseteq (T^\infty N, \xi_{\mathrm{st}})$, construct a sheaf $\mathscr{F} \in \mathrm{Sh}^c_\Lambda(N, \mathcal{K})$, i.e. a sheaf $\mathscr{F} \in \mathrm{Sh}^c(N, \mathcal{K})$ with $\mathrm{SS}(F) \subseteq \Lambda$. In the proof of Theorem 6, this Legendrian Λ will be the Legendrian graph of a contact isotopy.

Since a Legendrian isotopy in a contact manifold can be extended to an ambient contact isotopy, see e.g. [Gei08, Section 2.6], we focus on studying invariance under contact isotopies to deduce invariance under Legendrian isotopies.

5.1.1 A motivation: Legendrian Reidemeister moves

To develop intuition on the Legendrian invariance of $\mathrm{Sh}^c_\Lambda(M, \mathcal{K})$, we treat in detail the cases of $\dim(\Lambda) = 0$ and $\dim(\Lambda) = 1$.

An exercise in 1-dimensional contact topology

Consider coordinates $(x, \xi) \in (T^\infty \mathbb{R}, \xi_{\mathrm{st}})$, where $x \in \mathbb{R}$ is the coordinate in the zero section and $|\xi| \in S^0$ is the fiber coordinate in the cotangent fiber, so $\xi = \pm 1$. Let us focus on the Legendrian isotopy

$$\Lambda_t := \{(t, 1)\} \cup \{(-t, -1)\}, \quad t \in [-1, 1],$$

where $\Lambda_t \cong S^0$ is a Legendrian in $(T^\infty \mathbb{R}, \xi_{\mathrm{st}})$ for each t. The isotopy starts at the 0-sphere $\Lambda_{-1} = \{(-1, 1)\} \cup \{(1, -1)\}$ and ends at $\Lambda_1 = \{(1, 1)\} \cup \{(-1, -1)\}$.From the perspective of sheaves on $\mathbb{R}$, $\Lambda_{-1} = \mathrm{SS}(k_{[-1,1]})$ where $k_{[-1,1]} := j_* k$, $j : [-1, 1] \longrightarrow \mathbb{R}$ and $k \in \mathrm{Sh}([-1, 1], \mathcal{K})$ is the constant sheaf on $Z = [-1, 1]$. We want to understand how $\mathrm{Sh}_{\Lambda_t}(\mathbb{R}, \mathcal{K})$ varies as we go from Λ_{-1} to Λ_1:

1. For $t \in [-1,0)$, it should be apparent that not much qualitatively changes. In this case, the sheaf $k_{[-1,1]} \in \mathrm{Sh}_{\Lambda_{-1}}(\mathbb{R},\mathcal{K})$ with $\Lambda_{-1} = \mathrm{SS}(k_{[-1,1]})$ is mapped to $k_{[-t,t]} \in \mathrm{Sh}_{\Lambda_t}(\mathbb{R},\mathcal{K})$, where now $\Lambda_t = \mathrm{SS}(k_{[-t,t]})$.

2. For $t = \{0\}$, it is not as immediate what sheaf $\mathscr{F} \in \mathrm{Sh}_{\Lambda_0}(\mathbb{R},\mathcal{K})$ should be the continuation of $k_{[-t,t]}$ at $t \to 0_-$. The correct answer is $k_{\{0\}}$, the skyscraper sheaf at the origin $\{0\}$, which represents the Legendrian boundary of the cotangent fiber $T_0^*\mathbb{R}$. Note that, indeed, $\mathrm{SS}(k_{\{0\}}) = \{(0,1)\} \cup \{(0,-1)\}$.

3. For $t \in (0,1)$, one might guess that the isotopy sends $k_{[-t,t]} \to k_{\{0\}} \to k_{(-t,t)}$, as $\mathrm{SS}(k_{(-t,t)}) = \Lambda_t$ for $t \in (0,1]$. This is almost correct, except for the presence of homological grading: the correct sheaf is $k_{(-t,t)}[1]$, which still has $\mathrm{SS}(k_{(-t,t)}[1]) = \Lambda_t$ for $t \in (0,1]$. Through the isotopy Λ_t, the sheaves vary as $k_{[-t,t]} \to k_{\{0\}} \to k_{(-t,t)}[1]$.

The above is an illustrative example of the importance of the grading shift in Section 3.3.2. In particular, of the importance of the choice of Maslov potential on the Legendrian Λ and the dependence of $\mathrm{Sh}^c_\Lambda(M,\mathcal{K})$ on that Maslov potential.

5.1.2 The invariance result

The main step in showing that $\mathrm{Sh}^c_\Lambda(M,\mathcal{K})$ is invariant under contact isotopies is the construction of a sheaf kernel from a contact isotopy. More precisely:

Definition 5.1.1 (Sheaf kernel of a contact isotopy)**.** Let $\{\varphi_t\} \in \mathrm{Cont}(T^\infty M, \xi_{\mathrm{st}})$, $t \in I = [0,1]$, be a contact isotopy. By definition, a sheaf

$$K(\{\varphi_t\}) \in \mathrm{Sh}(M \times M \times I) \tag{5.1.1}$$

is said to be a kernel for $\{\varphi_t\}$ if

1. $\mathrm{SS}(K(\{\varphi_t\})) \subseteq \Lambda(\{\varphi_t\})$,
2. $K(\{\varphi_t\})|_{t=0} \simeq k_\Delta$,

where $\Lambda(\{\varphi_t\}) \subseteq T^\infty(M \times M \times I)$ is the Legendrian graph of $\{\varphi_t\}$ and $\Delta \subseteq M \times M$ the diagonal submanifold.

We often abbreviate $K_\varphi := K(\{\varphi_t\}))$ to ease notation. For background on general sheaf kernels, see [KS90, Section 3.6] or [GKS12, Section 1.6]. The use of the word kernel in Definition 5.1.1 is in analogy with the notion of an *integral kernel* in analysis, see e.g. [Hö3, Chapter V], or Fourier-Mukai kernels in algebraic geometry, cf. e.g. [BZNP17, Section 1].

Remark 5.1.2. The point of this notion of *kernel* is that one can sometimes define a map

$$\Phi_K : \mathcal{C}(X) \longrightarrow \mathcal{C}(Y)$$

by constructing an object $K \in \mathcal{C}(X \times Y)$, called the kernel, where X, Y are geometric objects of some type, and $\mathcal{C}$ is a type of space or category associated to them. In many cases, one can characterize maps $\mathcal{C}(X) \longrightarrow \mathcal{C}(Y)$ that are of this form, and some times such maps are abundant.[1] Intuitively, the map Φ_K is obtained by pulling back an object from $\mathcal{C}(X)$ to

[1] For instance, the Schwarz Kernel Theorem [Hö3, Theorem 5.2.1] states that *any* linear map from test functions to distributions is of this form. See [BZNP17, Theorem 1.1.3] for the algebraic geometric context.

$\mathcal{C}(X \times Y)$ via the first projection $p_1 : X \times Y \longrightarrow X$, multiplying it with K and then projecting it down to Y via the second projection $p_2 : X \times Y \longrightarrow Y$. What "pulling back", "multiplying" and "projecting" exactly mean depends on the type of mathematical objects being used. □

Definition 5.1.1.(1) states, via Tenet 3.2.8, that the kernel $K(\{\varphi_t\})$ represents the Legendrian graph of the contact isotopy $\{\varphi_t\}$. Definition 5.1.1.(2) is stating that $K_\varphi|_{t=0}$ represents the graph of the identity (whose graph is $\Delta \subseteq M \times M$) and does so in the stronger sense that $K_\varphi|_{t=0}$ is gives the constant sheaf k_Δ on Δ, and not any other *locally* constant sheaf on Δ. It is proven in [GKS12, Prop. 3.2] that if the kernel K_φ of a contact isotopy exists, then K_φ is unique, uniqueness being appropriately understood, cf. [GKS12, Lemma 3.3].

A kernel as in Definition 5.1.1 defines a collection of functors

$$\Phi_{K_{\varphi,s}} : \mathrm{Sh}(M) \longrightarrow \mathrm{Sh}(M), \quad \Phi_{K_{\varphi,s}}(\mathscr{F}) := (p_1)_!(K_\varphi|_{t=s} \otimes p_2^{-1}(\mathscr{F})). \tag{5.1.2}$$

for each $s \in I$, where $p_i : M \times M \longrightarrow M$ denotes the projection onto the ith factor. This corresponds to setting $X = Y = M$, $\mathcal{C}(M) = \mathrm{Sh}(M)$ in Remark 5.1.2 and considering the construction in a 1-parametric family depending on $t \in I$. In the case of sheaves, we are pull-backing $\mathscr{F} \in \mathrm{Sh}(M, \mathcal{K})$ to a sheaf in the product $M \times M$ via sheaf pull-back by p_1, as in Example 3.2.2.(6), multiplying means taking the tensor product in $M \times M$, and projecting down to the second component M of $M \times M$ means taking the direct image sheaf under the second projection p_2, as in Example 3.2.2.(2). The functor in Equation (5.1.2) is often referred to as the integral transform of the sheaf kernel $K_{\varphi,s} := K_\varphi|_{t=s} \in \mathrm{Sh}(M \times M)$, or said to be the convolution with $K_{\varphi,s}$.

The functor in Equation (5.1.2), built from a sheaf kernel for $\{\varphi_t\}$, is precisely the one giving the necessary isomorphism induced by the contact isotopy $\{\varphi_t\}$:

Theorem 6 (Legendrian invariance of $\mathrm{Sh}^c_\Lambda(M, \mathcal{K})$)**.** Let $\varphi_t \in \mathrm{Cont}_0(T^\infty M, \xi_{\mathrm{st}})$ be a contact isotopy, $t \in [0,1]$, and $\Lambda \subseteq (T^\infty M, \xi_{\mathrm{st}})$ a Legendrian. Then there exists a sheaf kernel $K_\varphi \in \mathrm{Sh}(M \times M \times I)$ for $\{\varphi_t\}$ such that the convolution functor

$$\Phi_{K_{\varphi,1}} : \mathrm{Sh}^c_\Lambda(M, \mathcal{K}) \longrightarrow \mathrm{Sh}^c_{\varphi_1(\Lambda)}(M, \mathcal{K}), \quad \Phi_{K_{\varphi,1}}(\mathscr{F}) := (p_1)_!(K_\varphi|_{t=1} \otimes p_2^{-1}(\mathscr{F})), \tag{5.1.3}$$

is an equivalence. In particular, the isomorphism type of $\mathrm{Sh}^c_\Lambda(M, \mathcal{K})$ is a Legendrian invariant of $\Lambda \subseteq (T^\infty M, \xi_{\mathrm{st}})$.

5.2. The microlocalization functor

In Section 3.3, we assigned a sheaf category $\mathrm{Sh}^c_\Lambda(M, \mathcal{K})$ to a Legendrian submanifold $\Lambda \subseteq (T^\infty M, \xi_{\mathrm{st}})$ by using the notion of singular support, cf. Definition 3.2.8.

The singular support of a sheaf can be used to define a *different* sheaf category $\mu\mathrm{Sh}_\Lambda(\Lambda, \mathcal{K})$, also associated to Λ. In fact, one can define a sheaf $\mu\mathrm{Sh}_\Lambda$ of (sheaf) categories on $(T^\infty M, \xi_{\mathrm{st}})$, which ends up being support along Λ, and $\mu\mathrm{Sh}_\Lambda(\Lambda, \mathcal{K})$ is its category of global sections. The construction of $\mu\mathrm{Sh}_\Lambda$ is presented in [KS90, Part 10], and see also [Nad16, Section 3.4], following key results from [KS90, Chapters IV & VI]. The sheaf of categories obtained from this construction receives different monikers, including the Kashiwara-Schapira stack or microlocal sheaves.

Intuitively, $\mu\mathrm{Sh}_\Lambda(\Lambda, \mathcal{K})$ is given by the quotient of $\mathrm{Sh}^c(M, \mathcal{K})$ by the subcategory of sheaves with singular support away from a neighborhood of Λ. Lack of details notwithstanding, here are useful properties of $\mu\mathrm{Sh}_\Lambda(\Lambda, \mathcal{K})$:

1. $\mu\mathrm{Sh}_\Lambda(\Lambda, \mathcal{K})$ only depends on Λ as an abstract smooth manifold. In particular, it is

independent of the Legendrian isotopy type of $\Lambda \subseteq (T^\infty M, \xi_{\mathrm{st}})$. This is in stark contrast with $\mathrm{Sh}^c_\Lambda(M, \mathcal{K})$, which depends crucially on the Legendrian isotopy type of Λ.

For instance, if $M := \mathbb{R}^n$ and $\Lambda := st(\Lambda_0) \subseteq (T^\infty \mathbb{R}^n, \xi_{\mathrm{st}})$ denotes the Legendrian sphere given by a stabilization of the standard Legendrian unknot which preserves its formal Legendrian class, then $\mathrm{Sh}^c_\Lambda(M, \mathcal{K})$ is the zero category, whereas there is an equivalence $\mu\mathrm{Sh}_\Lambda(\Lambda, \mathcal{K}) \cong \mathrm{Loc}^c(S^{n-1}, \mathcal{K})$, and thus $\mu\mathrm{Sh}_\Lambda(\Lambda, \mathcal{K})$ is not the zero category.[2]

2. The two categories $\mathrm{Sh}^c_\Lambda(M, \mathcal{K})$ and $\mu\mathrm{Sh}_\Lambda(\Lambda, \mathcal{K})$ are related. Precisely, there is a type of restriction functor

$$\boxed{\mathfrak{m}_\Lambda : \mathrm{Sh}^c_\Lambda(M, \mathcal{K}) \longrightarrow \mu\mathrm{Sh}_\Lambda(\Lambda, \mathcal{K})} \tag{5.2.1}$$

known as the microlocalization functor. A useful property of the functor $\mathfrak{m}_\Lambda$ in Equation (5.2.1) is that it admits a smooth relative Calabi-Yau structure, cf. [KL24b, Theorem 1.1].

3. In a number of interesting cases, there is a global equivalence

$$\boxed{\mu\mathrm{Sh}_\Lambda(\Lambda, \mathcal{K}) \cong \mathrm{Loc}(\Lambda, \mathcal{K}).} \tag{5.2.2}$$

For instance, if $\mathcal{K} = \mathrm{Mod}(k)$ then there are two obstructions to the existence of a non-trivial object in $\mu\mathrm{Sh}_\Lambda(\Lambda, \mathrm{Mod}(k))$, cf. [Gui23, Section 10.3]. In general, there are different obstructions for other coefficients $\mathcal{K}$, see e.g. the brane obstructions from [JT24, Sections 1.6-1.8] or [Jin24]. Technically, in more generality, the right hand side of Equation (5.2.2) should be understood as a category of twisted coefficients. At core, these obstructions classes come standard obstruction theory, as in [Hat02, Section 4.3], from trying to trivialize the Gauss map along Λ post-composed with maps to the delooping of a Picard group.[3] Note that, in general, $\mu\mathrm{Sh}_\Lambda$ is *locally* equivalent to the category of local systems on Λ, cf. [NS20, Corollary 6.4], so these obstructions measure a local-to-global passage for such an equivalence to hold globally.

Remark 5.2.1. Historically, Sato's microlocalization and μhom allowed to compute morphisms in $\mu\mathrm{Sh}_\Lambda(\Lambda)$, among other uses, see [KS90, Theorem 6.1.2] or [Gui23, Section 10.1]). This preceded the systematic study of $\mu\mathrm{Sh}_\Lambda(\Lambda)$ and its relation to $\mathrm{Sh}^c_\Lambda(M, \mathcal{K})$ via $\mathfrak{m}_\Lambda$. Indeed, μhom was already studied in [KS90, Section 4.4] whereas properties of the functor $\mathfrak{m}_\Lambda$, such as being relative Calabi-Yau, and the latter categories, are strictly recent developments. □

Since these notes focus on cases where the equivalence in Equation (5.2.2) holds, e.g. any Legendrian links $\Lambda \subseteq (T^\infty \mathbb{R}^2, \xi_{\mathrm{st}})$ with vanishing rotation, we often refer to the microlocalization functor as a functor

$$\mathfrak{m}_\Lambda : \mathrm{Sh}^c_\Lambda(M, \mathcal{K}) \longrightarrow \mathrm{Loc}(\Lambda, \mathcal{K}).$$

Instead of resorting to the original and more general definition of $\mathfrak{m}_\Lambda$, we directly define this functor as follows:

Definition 5.2.2 (Microlocalization functor)**.** Let $\Lambda \subseteq (T^\infty M, \xi_{\mathrm{st}})$ be a Legendrian submanifold whose $\mathcal{K}$-brane obstructions vanish. By definition, the microlocalization functor is the functor

$$\mathfrak{m}_\Lambda : \mathrm{Sh}^c_\Lambda(M, \mathcal{K}) \longrightarrow \mathrm{Loc}(\Lambda, \mathcal{K})$$

that sends to a sheaf $\mathscr{F}$ to the local system given by its microlocal stalks $\mathscr{F}_{(x,\xi)}$, where $\xi \in T_x\Lambda$ is the conormal direction of the coorientation of Λ.

[2] This also works for $\dim(\Lambda) = 1$ with a stabilization of the standard Legendrian unknot with rotation zero.

[3] Intuitively, the Picard group $\mathrm{Pic}(\mathcal{C})$ of a category $\mathcal{C}$ is parametrizing homological automorphisms of $\mathcal{C}$, of the form $T \otimes (\cdot)$ for invertible objects T.

Definition 5.2.2 has the advantage of making $\mathfrak{m}_\Lambda$ amenable to direct computations, whereas some of the more abstract properties of $\mathfrak{m}_\Lambda$, such as invariance of the relative Calabi-Yau structure, are obscured by this choice. It is not perhaps an immediate fact that the microlocal stalks form a local system on Λ if the $\mathcal{K}$-brane obstructions vanish: this can be verified rather directly in the case of Legendrian links in $(T^\infty\mathbb{R}^2, \xi_{\mathrm{st}})$ via front computations.

Remark 5.2.3. We implicitly assume from now onwards that, whenever $\mathfrak{m}_\Lambda$ is used as in Definition 5.2.2, all Legendrians Λ have $\mathcal{K}$-brane obstructions vanishing. □

5.3. Quantization of Lagrangian Fillings

Let $\Lambda \subseteq (T^\infty M, \xi_{\mathrm{st}})$ be a Legendrian submanifold. In Definition 3.3.1 we introduced the category $\mathrm{Sh}^c_\Lambda(M, \mathcal{K})$ and in Definition 5.2.2 the functor $\mathfrak{m}_\Lambda$. Two questions:

1. Does $\mathrm{Sh}^c_\Lambda(M, \mathcal{K})$ have any object $\mathscr{F}$?

2. A refinement of the question in (1): given a fixed local system $\mathscr{L} \in \mathrm{Loc}(\Lambda, \mathcal{K})$, does $\mathrm{Sh}^c_\Lambda(M, \mathcal{K})$ have any object $\mathscr{F}$ with $\mathfrak{m}_\Lambda(\mathscr{F}) = \mathscr{L}$?

In the literature, this is known as the problem of sheaf quantization. Here the term quantization, already appearing in [SKK73, Section 3.3] and [KS85b, Definition 11.4.10], is used in the sense of canonical quantization of transformations.[4] In [NS20, Section 7] the term antimicrolocalization is also employed as it is, in a sense, aiming to find an inverse of $\mathfrak{m}_\Lambda$.

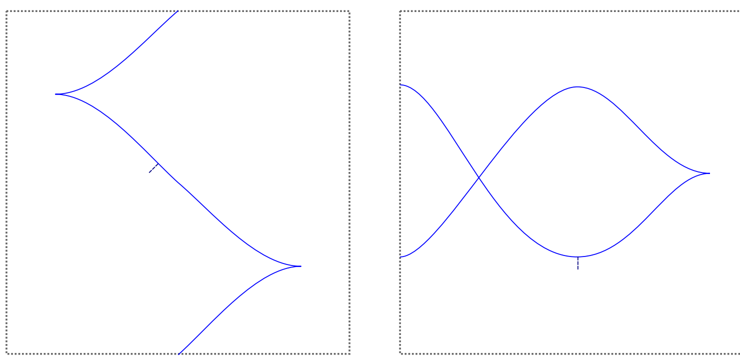

Figure 5.1: Two fronts for two Legendrian tangles in $(T^\infty\mathbb{R}^2, \xi_{\mathrm{st}})$. These are local examples of Legendrians such that no sheaf in $\mathbb{R}^2$ has singular support contained in them. Thus, if a front for a Legendrian Λ contains either of these two models, the associated category $\mathrm{Sh}^c_\Lambda(M, \mathcal{K})$ has no objects.

Example 5.3.1. *(1) If a Legendrian Λ is stabilized, in the sense of [EES05, Section 4.3], then $Sh^c_\Lambda(M, \mathcal{K})$ does not contain any object. In particular, even locally, for either of the Legendrian arcs associated to the fronts in Figure 5.1 , there are no sheaves in $\mathbb{R}^2$ with these singular supports.*

(2) For many Legendrian knots $\Lambda \subseteq (\mathbb{R}^3, \xi_{st}) \subseteq (T^\infty\mathbb{R}^2, \xi_{st})$, e.g. the max-tb Legendrian unknot or the Legendrian knot in Figure 2.2(left), there are non-trivial objects in $Sh^c_\Lambda(M, \mathcal{K})$, with $M = \mathbb{R}^2$ and $\mathcal{K} = Mod(k)$. That said, the local system $\mathfrak{m}_\Lambda(\mathscr{F}) \in \mathrm{Loc}(S^1, Mod(k))$ is always the trivial local system on the circle, for any $\mathscr{F} \in Sh^c_\Lambda(M, \mathcal{K})$. □

[4]Lecture 6 of I. Dolgachev's notes "A Brief Introduction To Physics For Mathematicians", and T. Tao's blog titled "Lars Hormander" on 11/30/2012 are good introductions to this use of the term quantization.

Example 5.3.1 illustrates that the answers to (1) and (2) above might be negative. In general, there is currently no known verifiable criterion that given a Legendrian submanifold $\Lambda \subseteq (T^\infty M, \xi_{st})$ can decide whether $\mathrm{Sh}^c_\Lambda(M, \mathcal{K})$ is empty or non-empty, even for Legendrian knots. In these notes we shall be studying Legendrian submanifolds for which $\mathrm{Sh}^c_\Lambda(M, \mathcal{K})$ is non-empty and in fact the answer to the sheaf quantization questions above is affirmative. To illustrate a first instance where a general sheaf quantization is known, we highlight the following result:

Theorem 7 (Sheaf quantization of nearby Lagrangians)**.** Let $\Lambda \subseteq (J^1 M, \xi_{st})$ be a Legendrian submanifold with no Reeb chords. Then there exists a sheaf $\mathscr{F} \in \mathrm{Sh}^c_\Lambda(M, \mathcal{K})$.

Theorem 7 is proven in [Vit19, Theorem 1.5], see also [Gui23, Part 12]. In fact, any sheaf $\mathscr{F}$ as in Theorem 7 has its endomorphism complex isomorphic to $\mathrm{Sing}(\Lambda)$.

5.4. Decorated sheaves

In Section 3.3, we introduced the category $\mathrm{Sh}^c_\Lambda(M, \mathcal{K})$ of sheaves in M with singular support contained in Λ. The microlocalization functor introduced in Section 5.2 allows for a finer concept, as introduced in [CL24, Section 2.2]. Intuitively, we consider subsets $T \subseteq \Lambda$ and study $\mathscr{F} \in \mathrm{Sh}^c_\Lambda(M, \mathcal{K})$ endowed with trivializations of the microlocal stalks along the subset T. A typical situation is a case where $\Lambda \subseteq T$ is contractible, so that the local system $\mathfrak{m}_\Lambda(\mathscr{F})$ can be trivialized on $\Lambda \setminus T$. In the same manner that a Legendrian Λ yields the category $\mathrm{Sh}^c_\Lambda(M, \mathcal{K})$, a pair (Λ, T) yields a refinement of this category $\mathrm{Sh}^c_\Lambda(M, \mathcal{K})$, cf. Definition 5.4.4 below. This subsection introduces this refinement.

Remark 5.4.1. In practice, the moduli space for this refinement is often a bit better behaved than $\mathfrak{M}_\Lambda$. This is a common theme throughout mathematics: adding the information of a framing prevents isotropy groups from being non-trivial, or at least it makes them more tractable. Also, this refinement has proven to be useful in applications, for instance it provides the $\mathcal{A}$-cluster scheme in the Fock-Goncharov theory of cluster ensembles, see e.g. [CW24, Section 2.8.3] and [CGG$^+$22, Section 8], and it is crucial when dealing with frozen variables, cf. [CLSBW23, Section 6.2]. □

5.4.1 Category of decorated sheaves for $(\Lambda, \mathfrak{t})$

Our focus is on Legendrian links $\Lambda \subseteq (T^\infty \mathbb{R}^2, \xi_{st})$, so we discuss such refinement in this case:

Definition 5.4.2 (Pointed Legendrian links)**.** A pointed Legendrian $(\Lambda, \mathfrak{t})$ is a pair given by a Legendrian link $\Lambda \subseteq (T^\infty \mathbb{R}^2, \xi_{st})$ and a set of basepoints $\mathfrak{t} \subseteq \Lambda$, with at least one basepoint from $\mathfrak{t}$ in each component of Λ.

In the same manner that a Legendrian Λ names a category of sheaves $\mathrm{Sh}^c_\Lambda(M, \mathcal{K})$, a pointed Legendrian link will name a category of decorated sheaves $\mathrm{Sh}^c_{\Lambda,\mathfrak{t}}(\mathbb{R}^2, \mathcal{K})$. In Definition 5.4.2, it is alright for $\mathfrak{t}$ to have more than one basepoint per component of Λ, though essentially nothing particularly interesting occurs once one adds more than one basepoint per component. This is explained in [CL24, Section 2.3]. Therefore, unless additional combinatorial data is involved, a first common choice is to have exactly one basepoint per component.

Remark 5.4.3. In the literature, it is often the case that the set of basepoints is denoted by T. Technically, we are choosing a set of basepoints $\mathfrak{t} \subseteq \Lambda$ on the complement of $T \subseteq \Lambda$. To be precise, in the literature one often chooses $T \subseteq \Lambda$ to be a set of basepoints, with at least one basepoint per component of Λ. If that data is given, the set of basepoints

$\mathfrak{t} = \{t_1, \dots, t_{|\mathfrak{t}|}\} \subseteq \Lambda \setminus T$ we select in Definition 5.4.2 is such that each component of $\Lambda \setminus T$ contains a unique basepoint from $\mathfrak{t}$. Since the data of T is equivalent to the data of $\mathfrak{t}$, we also refer to $\mathfrak{t}$ as a set of basepoints: the roles of T and $\mathfrak{t}$ are interchangeable. □

To define the category $\mathrm{Sh}^c_{\Lambda,\mathfrak{t}}(\mathbb{R}^2, \mathcal{K})$ of $\mathfrak{t}$-decorated sheaves, we consider two functors:

1. The first functor is

$$m_{\Lambda,\mathfrak{t}} : \mathrm{Sh}^c_\Lambda(M, \mathcal{K}) \longrightarrow \prod_{i=1}^{|\mathfrak{t}|} \mathcal{K}, \quad \mathscr{F} \longmapsto m_{\Lambda,\mathfrak{t}}(\mathscr{F}) := (\mathscr{F}_{(t_1,\xi_{t_1})}, \dots, \mathscr{F}_{(t_{|\mathfrak{t}|},\xi_{t_{|\mathfrak{t}|}})}), \quad (5.4.1)$$

   which records the microstalks at the basepoints $t_i \in \mathfrak{t}$ in the co-direction $\xi_{t_i} \in T^*_{t_i}\mathbb{R}^2$ of the co-normal lift of Λ. Equivalently, $m_{\Lambda,\mathfrak{t}}$ assigns to an object $\mathscr{F}$ the stalks of the local system $\mathfrak{m}_\Lambda(\mathscr{F}) \in \mathrm{Loc}(\Lambda)$ at the points in $\mathfrak{t}$, where $\mathfrak{m}_\Lambda$ is the microlocalization functor from Definition 5.2.2.[5] In this latter perspective, we are implicitly using an equivalence

$$\prod_{i=1}^{|\mathfrak{t}|} \mathcal{K} \cong \prod_{i=1}^{|\mathfrak{t}|} \mathrm{Loc}(\{t_1, \dots, t_{|\mathfrak{t}|}\}) \cong \mathrm{Loc}(\mathfrak{t}).$$

2. The second functor is the diagonal functor

$$\Delta : \mathcal{K} \longrightarrow \prod_{i=1}^{|\mathfrak{t}|} \mathcal{K}, \quad (5.4.2)$$

   which is effectively independent of Λ and $\mathfrak{t}$.

The microstalk functor $m_{\Lambda,\mathfrak{t}}$, used in (1) above, admits a left adjoint $m^\ell_{\Lambda,\mathfrak{t}}$ because it preserves products, see e.g. [Nad16, Section 3.6] or [GPS24, Lemma 4.13]. Similarly, the diagonal functor Δ in (2) above admits a left adjoint Δ^ℓ given by the coproduct. Both functors $m_{\Lambda,\mathfrak{t}}$ and Δ preserve coproducts, and thus $m^\ell_{\Lambda,\mathfrak{t}}$ and Δ^ℓ preserve compact objects. The refinement of Definition 3.3.1 is:

Definition 5.4.4 (Sheaf category for $(\Lambda, \mathfrak{t})$)**.** Let $(\Lambda, \mathfrak{t}) \subseteq (T^*_\infty\mathbb{R}^2, \xi_{st})$ be a pointed Legendrian link. By definition, the dg-category $\mathrm{Sh}^c_{\Lambda,\mathfrak{t}}(\mathbb{R}^2, \mathcal{K})$ is the homotopy colimit of the black diagram

$$\begin{tikzcd} \prod_{i=1}^{|\mathfrak{t}|} \mathcal{K}^c \arrow[r, "m^\ell_{\Lambda,\mathfrak{t}}"] \arrow[d, "\Delta^\ell"] & \mathrm{Sh}^c_\Lambda(\mathbb{R}^2, \mathcal{K}) \arrow[d, dashed] \\ \mathcal{K}^c \arrow[r, dashed] & \mathrm{Sh}^c_{\Lambda,\mathfrak{t}}(\mathbb{R}^2, \mathcal{K}) \end{tikzcd}$$

The category $\mathrm{Sh}^c_{\Lambda,\mathfrak{t}}(\mathbb{R}^2, \mathcal{K})$ is said to be the category of $\mathcal{K}$-valued sheaves with singular support in $(\Lambda, \mathfrak{t})$, also known as $\mathfrak{t}$-decorated sheaves with singular support in Λ. □

[5] We have implicitly chosen an isomorphism $\mu\mathrm{sh}_\Lambda(\Lambda) \cong \mathrm{Loc}(\Lambda)$ between the global sections of the Kashiwara-Schapira stack $\mu\mathrm{sh}$ and the dg-derived category of local systems $\mathrm{Loc}(\Lambda)$ on Λ. This is possible in the case where $\Lambda = S^1 \sqcup \ldots \sqcup S^1$ by [Gui23, Chapter 10].

The homotopy colimit of the diagram and the universal map associated to it are highlighted in blue in Definition 5.4.4. Here the homotopy colimit is taken in the ∞-category dg-Cat$_k$ of well generated dg-categories over k, cf. [Tab09]. Intuitively, $\mathrm{Sh}^c_{\Lambda,\mathfrak{t}}(\mathbb{R}^2,\mathcal{K})$ captures sheaves with singular support on Λ with the additional data of their microstalks at the basepoints of $\mathfrak{t}$ and a common identification of these microstalks.

5.4.2 Moduli of decorated sheaves for $(\Lambda, \mathfrak{t})$

In Section 3.3.1 we discussed how to associate a geometric space $\mathfrak{M}_\Lambda$ to the category $\mathrm{Sh}^c_\Lambda(M,\mathcal{K})$. Similarly, the category $\mathrm{Sh}^c_{\Lambda,\mathfrak{t}}(\mathbb{R}^2,\mathcal{K})$ in Definition 5.4.4 admits a corresponding geometric space:

Definition 5.4.5 (Moduli of sheaves for $(\Lambda,\mathfrak{t})$)**.** Let $(\Lambda,\mathfrak{t}) \subseteq (T^*_\infty\mathbb{R}^2,\xi_{st})$ be a pointed Legendrian link. By definition, the derived stack $\mathfrak{M}(\Lambda,\mathfrak{t})$ of $\mathcal{K}$-valued sheaves with singular support on $(\Lambda,\mathfrak{t})$ is the derived stack of pseudo-perfect objects of the category $\mathrm{Sh}^c_{\Lambda,\mathfrak{t}}(\mathbb{R}^2,\mathcal{K})$.

By [TV07, Prop. 3.4], the functor $\mathfrak{M}$ admits a left adjoint and therefore $\mathfrak{M}$ preserves homotopy limits. In particular, it sends a homotopy pullback in $\mathrm{Ho}(\text{dg-cat}_k)^{op}$ to a homotopy pullback in $D^-\mathrm{St}(k)$. Since homotopy pullbacks in $\mathrm{Ho}(\text{dg-cat}_k)^{op}$ are homotopy pushouts in $\mathrm{Ho}(\text{dg-cat}_k)$, the functor $\mathfrak{M}$ applied to the homotopy pushout in Definition 5.4.4 yields a homotopy pullback. Therefore, the derived stack in Definition 5.4.5 can equivalently be described as the homotopy limit of the black diagram

$$\begin{tikzcd}
\mathfrak{M}\left(\prod_{i=1}^{|\mathfrak{t}|}\mathcal{K}^c\right) & \mathfrak{M}(\mathrm{Sh}^c_\Lambda(\mathbb{R}^2,\mathcal{K})) \arrow[l, "\mathfrak{M}(m^\ell_{\Lambda,\mathfrak{t}})"'] \\
\mathfrak{M}(\mathcal{K}^c) \arrow[u, "\mathfrak{M}(\Delta^\ell)"] & \mathfrak{M}(\mathrm{Sh}^c_{\Lambda,\mathfrak{t}}(\mathbb{R}^2,\mathcal{K})) := \mathfrak{M}(\Lambda,\mathfrak{t}). \arrow[l, dashed] \arrow[u, dashed]
\end{tikzcd}$$

The homotopy limit and the universal maps are highlighted in blue, for clarity. By [Nad16, Thm. 3.21] or [GPS24, Cor. 4.23], via the Yoneda embedding, the pseudo-perfect objects in $\mathrm{Sh}^c_\Lambda(\mathbb{R}^2,\mathcal{K})$ are those sheaves in $\mathrm{Sh}^c_\Lambda(\mathbb{R}^2,\mathcal{K})$ with stalks in $\mathcal{K}^c$. Thus, the map

$$\mathfrak{M}(\mathrm{Sh}^c_\Lambda(\mathbb{R}^2)_0) \longrightarrow \mathfrak{M}\left(\prod_{i=1}^{|\mathfrak{t}|}\mathcal{K}^c\right)$$

on geometric points is given by the microstalk functor $m_{\Lambda,\mathfrak{t}}$ at the basepoints of $\mathfrak{t}$. Similarly, the map

$$\mathfrak{M}(\mathrm{Mod}(k)^c) \longrightarrow \mathfrak{M}\left(\prod_{i=1}^{|\mathfrak{t}|}\mathcal{K}^c\right)$$

on geometric points is also given by the diagonal map Δ. Therefore, $\mathfrak{M}(\Lambda,\mathfrak{t})$ in Definition 5.4.5 indeed captures the intuition of parametrizing sheaves $\mathscr{F} \in \mathrm{Sh}(M,\mathcal{K})$ with an additional choice of common trivialization of the microstalks along a given subset of Λ.

Chapter 6: A case study

The goal of this section is to study the categories $\mathrm{Sh}^c_\Lambda(M,\mathcal{K})$ and their moduli stack $\mathfrak{M}(\Lambda)$ for a class of Legendrian links $\Lambda_\beta \subseteq (\mathbb{R}^3, \xi_{st})$ indexed by positive braids words β. This is a class of Legendrian links that can be used to illustrate a number of interesting examples and counter-examples in low-dimensional contact topology, and has been instrumental in helping experts gain intuition, and then prove results, in the study of Lagrangian fillings.

6.1. The class of Legendrian links Λ_β

Let $\beta \in \mathrm{Br}_n^+$ be a positive braid word. We associate two Legendrians to such data:

Definition 6.1.1 (Legendrians associated to β)**.** Let $\beta \in \mathrm{Br}_n^+$ be a positive braid word.

1. $\Lambda_\beta \subseteq (\mathbb{R}^3, \xi_{st})$ is the Legendrian defined by the front in Figure 6.1.(Left). It is said to be the (-1)-closure of β.
2. $\Lambda^\circ_\beta \subseteq (J^1S^1, \xi_{st})$ is the Legendrian defined by the front in Figure 6.1.(Right). It is said to be the circular closure of β.

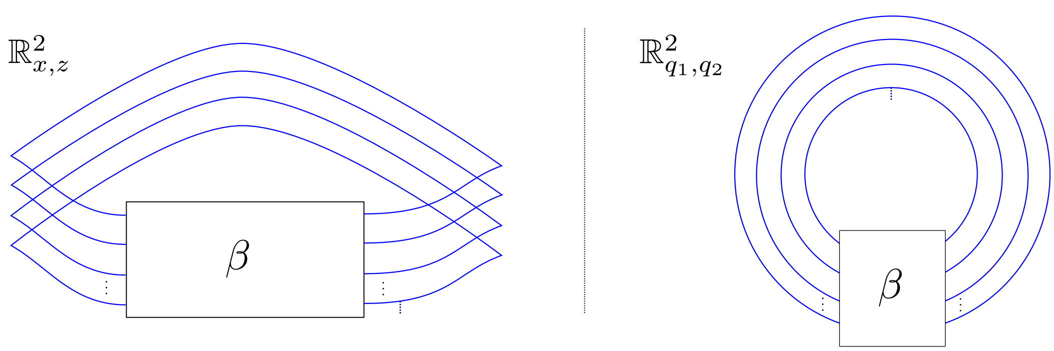

Figure 6.1: (Left) The front in $\mathbb{R}^2_{x,z}$ for $\Lambda_\beta \subseteq (\mathbb{R}^3, \ker\{dz - ydx\})$ referred to as the (-1)-closure of β. (Right) The front of the Legendrian link $\Lambda^\circ_\beta \subseteq (T^\infty\mathbb{R}^2_{x,y}, \xi_{st})$, the circular closure of β, which closely related to Λ_β.

From the viewpoint of smooth knot theory, the class of Legendrians $\Lambda_\beta \subseteq (\mathbb{R}^3, \xi_{st})$ is reasonably broad, as it includes smooth knots in each of the three classes: torus knots, satellite knots and hyperbolic knots. Thurston's results imply that any smooth knot must be in one of these three classes, cf. [Thu82, Cor. 2.5]. In fact, for each of these smooth classes, there exist infinitely many Legendrian knots of the form Λ_β in such class, each not smoothly isotopic to one another. This is proven in [CG22, Section 6.2].

Remark 6.1.2. (1) The two Legendrians $\Lambda_\beta, \Lambda^\circ_\beta$ in Definition 6.1.1 are related: Λ_β is Legendrian isotopic to the image of Λ°_β along a contact embedding of (J^1S^1, ξ_{st}) into $(\mathbb{R}^3, \xi_{st})$ that sends the zero section $S^1 \subseteq (J^1S^1, \xi_{st})$ to the max-tb Legendrian unknot in $(\mathbb{R}^3, \xi_{st})$. In short, Λ_β is the result of satelliting Λ°_β along the Legendrian unknot.

(2) The literature has also featured the rainbow closure of a positive braid word β, as depicted in Figure 6.2 (Right), cf. [STZ17, Section 6.5]. It is an exercise to show that this is an instance of a (-1)-closure, specifically the (-1)-closure of $w_0\beta w_0$, cf. Figure 6.2.

(3) The smooth type of Λ_β is that of the smooth link typically associated to the braid $w_0^{-1}\beta w_0^{-1}$, by closing the braid smoothly without any framing, i.e. the 0-closure, with no twists. Briefly put, the crossings in each of the two sides of the β-box in Figure 6.1(Left) are each smoothly contributing w_0^{-1}. □

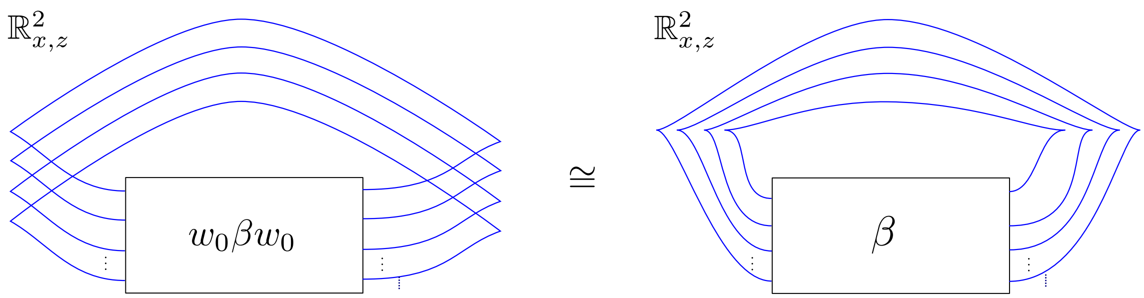

Figure 6.2: (Left) The (-1)-closure of $w_0\beta w_0$, where w_0 is any positive braid word representing the half-twist. (Right) The rainbow closure of β, which is Legendrian isotopic to the (-1)-closure of $w_0\beta w_0$.

Example 6.1.3. *(1) Consider $c := \sigma_1 \ldots \sigma_{n-1} \in \mathrm{Br}_n^+$ and let w_0 be any positive braid word representing the half-twist, e.g. $w_0 = \sigma_1(\sigma_2\sigma_1)\cdots(\sigma_{n-1}\cdots\sigma_2\sigma_1)$. Then the Legendrian Λ_β given by the (-1)-closure of $\beta = w_0c^m w_0$ is a Legendrian representative of the (n,m)-torus link. If $\gcd(n,m) = 1$, then Λ_β is a knot and it is the unique max-tb Legendrian representative of the (n,m)-torus link.*

(2) If $\beta = \sigma_1^2\sigma_2^2\sigma_1^2\sigma_2^2\sigma_1\sigma_2 \in \mathrm{Br}_3^+$, then Λ_β is Legendrian isotopic to the Legendrian representative of $m(5_2)$ depicted in Figure 2.2 (Left). Note that, following Remark 6.1.2.(3), the smooth type of the knot is given by the 0-closure of

$$w_0^{-1}\sigma_1^2\sigma_2^2\sigma_1^2\sigma_2^2\sigma_1\sigma_2)w_0^{-1} = (\sigma_1\sigma_2\sigma_1)^{-1}(\sigma_1^2\sigma_2^2\sigma_1^2\sigma_2^2\sigma_1\sigma_2)(\sigma_1\sigma_2\sigma_1)^{-1} = \sigma_1\sigma_2^3\sigma^{-1}\sigma_2,$$

which is indeed a braid for $m(5_2)$. The Legendrian representative of $m(5_2)$ depicted in Figure 2.2(Right) is not of the form Λ_β for any $n \in \mathbb{N}$ and $\beta \in \mathrm{Br}_n^+$: e.g. it does not admit a binary Maslov index, cf. [STZ17, Section 7.2.2]. □

Remark 6.1.4. Since there is a contactomorphism $(T^\infty\mathbb{R}^2, \xi_{\mathrm{st}}) \cong (J^1S^1, \xi_{\mathrm{st}})$, as in Example 2.1.2, we can obtain a different front for Λ_β° by using $\Pi : (J^1S^1, \xi_{\mathrm{st}}) \longrightarrow S^1 \times \mathbb{R}$, where the resulting front is in $S^1 \times \mathbb{R}$. See Equation (2.4.3).(3) and Figure 2.4. The contactomorphism above can be chosen such that the front for Λ_β° in $S^1 \times \mathbb{R}$ is as in Figure 6.3. □

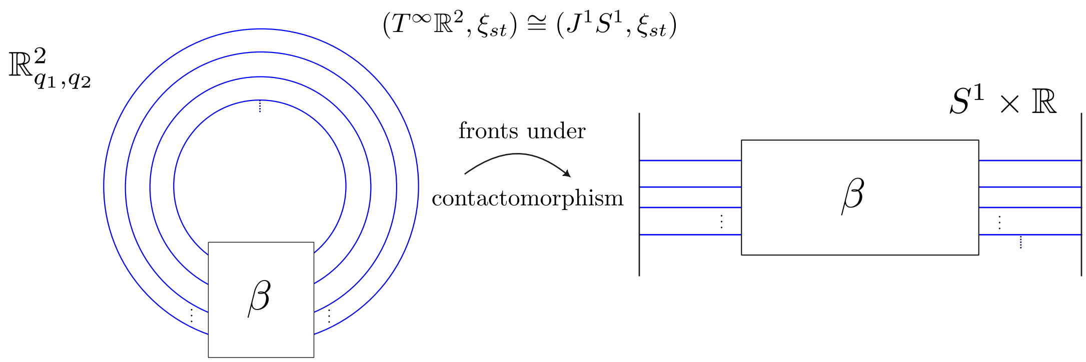

Figure 6.3: (Left) A front for Λ°_β in $R^2_{q_1,q_2}$, under the Legendrina fibration π in Figure 2.4. (Right) A front for Λ°_β under the Legendrian fibration Π in Figure 2.4, having used a contactomorphism $(T^\infty\mathbb{R}^2, \xi_{\rm st}) \cong (J^1S^1, \xi_{\rm st})$.

6.2. Sheaves with singular support on Λ_β

The goal of this section is to describe sheaves with singular support on Legendrian links of the form Λ_β or Λ°_β, as introduced in Section 6.1. For specificity, we focus on sheaves $\mathscr{F} \in \mathrm{Sh}(M, \mathcal{K})$ with the following two properties:

1. The stalk of $\mathscr{F}$ in the unbounded component of $\mathbb{R}^2 \setminus \pi(\Lambda)$ is acyclic, where $\pi(\Lambda)$ is a front for the given Legendrian link.
2. The microstalks of $\mathscr{F}$ are isomorphic to a shift of the ground ring k. These sheaves are known as simple in the literature, cf. [KS90, Def. 7.5.4] or [Gui23, Def. 1.4.2].

For that, we first introduce the following class of algebraic varieties defined over $\mathbb{Z}$, known as braid varieties.

6.2.1 Braid varieties

Let G be a simple algebraic group with Weyl group $W(\mathsf{G})$. We fix a Borel subgroup $\mathsf{B} \subseteq \mathsf{G}$ and a Cartan subgroup $\mathsf{T} \subset \mathsf{B}$. Pairs of flags $\mathsf{B}_1, \mathsf{B}_2 \in \mathsf{G}/\mathsf{B}$ in relative position $w \in W(\mathsf{G})$, i.e. $\mathsf{B}_1 = [g_1\mathsf{B}]$ and $\mathsf{B}_2 = [g_2\mathsf{B}]$ such that $g_1^{-1}g_2 \in \mathsf{B}w\mathsf{B}$, are denoted by $\mathsf{B}_1 \xrightarrow{w} \mathsf{B}_2$. Let $\mathrm{Br}(\mathsf{G})$ be the braid group associated with $W(\mathsf{G})$. The Artin generators of $\mathrm{Br}(\mathsf{G})$ are denoted by σ_i, which lift the Coxeter generators $s_i \in W(\mathsf{G})$, where the index i runs through the simple positive roots of the Lie algebra of G. Our main object of study is defined as follows:

Definition 6.2.1. Let $\beta = \sigma_{i_1} \cdots \sigma_{i_\ell}$ be a positive braid word and $\delta(\beta) \in W(\mathsf{G})$ its Demazure product. The *braid variety* associated with β is

$$X(\beta) := \{(\mathsf{B}_1, \dots, \mathsf{B}_{\ell+1}) \in (\mathsf{G}/\mathsf{B})^{\ell+1} \mid \mathsf{B}_1 = \mathsf{B}, \mathsf{B}_k \xrightarrow{s_{i_k}} \mathsf{B}_{k+1}, \mathsf{B}_{\ell+1} = \delta(\beta)\mathsf{B}\}, \quad (6.2.1)$$

where $\delta(\beta) \in W(\mathsf{G}) \cong N_{\mathsf{G}}(T)/T$ has been lifted to $N_{\mathsf{G}}(T)$; this is well-defined since the flag $\delta(\beta)\mathsf{B}$ does not depend on such a lift.

Consider $\mathrm{Sh}_{\Lambda_\beta}(M; \mathcal{K})$ with $\mathcal{K} = \mathrm{Mod}(k)$ and its moduli stack $\mathfrak{M}(\Lambda_\beta)$. Then

$$\mathfrak{M}_1(\Lambda^\circ_\beta) \cong \mathsf{G}\backslash\{(\mathsf{B}_1, \dots, \mathsf{B}_{\ell+1}; g) \in (\mathsf{G}/\mathsf{B})^{\ell+1} \times \mathsf{G} : \mathsf{B}_1 \xrightarrow{\beta} \mathsf{B}_{\ell+1}, \mathsf{B}_1 = g\mathsf{B}_{\ell+1}\},$$

$$\mathfrak{M}_1(\Lambda_\beta) \cong \mathsf{G}\backslash\{(\mathsf{B}_1, \dots, \mathsf{B}_{\ell+1}) \in (\mathsf{G}/\mathsf{B})^{\ell+1} : \mathsf{B}_1 \xrightarrow{\beta} \mathsf{B}_{\ell+1}, \mathsf{B}_1 = \mathsf{B}_{\ell+1}\}.$$

Here the G-action is given by $(g; \mathsf{B}_1, \dots, \mathsf{B}_{\ell+1}) \longmapsto (g\mathsf{B}_1, \dots, g\mathsf{B}_{\ell+1})$. Given a fixed Borel $\mathsf{B} \subseteq \mathsf{G}$ and an opposite B_-, with $\mathsf{B} \cap \mathsf{B}_- = T$, we can fix the first flag in $\mathfrak{M}_1(\Lambda_\beta)$ by using the action of G. Then we get

$$\mathfrak{M}_1(\Lambda_\beta) \cong \mathsf{B}_-\backslash\{(\mathsf{B}_1, \dots, \mathsf{B}_{\ell+1}) \in (\mathsf{G}/\mathsf{B})^{\ell+1} : \mathsf{B}_1 \xrightarrow{\beta} \mathsf{B}_{\ell+1}, \mathsf{B}_1 = \mathsf{B}_{\ell+1} = \mathsf{B}\}.$$

Suppose that the positive braid is of the form βw_0 and write $\ell' = \ell + \ell(w_0)$. Then we can use the w_0 to slice the action more and get

$$\begin{aligned}
\mathfrak{M}_1(\Lambda_{\beta w_0}) &\cong \mathsf{G}\backslash\{(\mathsf{B}_1, \dots, \mathsf{B}_{\ell'+1}) \in (\mathsf{G}/\mathsf{B})^{\ell'+1} : \mathsf{B}_1 \xrightarrow{\beta w_0} \mathsf{B}_{\ell'+1}, \mathsf{B}_1 = \mathsf{B}_{\ell'+1}\} \\
&\cong \mathsf{B}_-\backslash\{(\mathsf{B}_1, \dots, \mathsf{B}_{\ell'+1}) \in (\mathsf{G}/\mathsf{B})^{\ell'+1} : \mathsf{B}_1 \xrightarrow{\beta w_0} \mathsf{B}_{\ell'+1}, \mathsf{B}_1 = \mathsf{B}_{\ell'+1} = \mathsf{B}\} \\
&\cong \mathsf{B}_-\backslash\{(\mathsf{B}_1, \dots, \mathsf{B}_{\ell'+1}) \in (\mathsf{G}/\mathsf{B})^{\ell'+1} : \mathsf{B}_1 \xrightarrow{\beta} \mathsf{B}_{\ell+1}, \mathsf{B}_{\ell+1} \xrightarrow{w_0} \mathsf{B}_{\ell'+1}, \mathsf{B}_1 = \mathsf{B}_{\ell'+1} = \mathsf{B}\} \\
&\cong \mathsf{B}_-\backslash\{(\mathsf{B}_1, \dots, \mathsf{B}_{\ell+1}) \in (\mathsf{G}/\mathsf{B})^{\ell+1} : \mathsf{B}_1 \xrightarrow{\beta} \mathsf{B}_{\ell+1}, \mathsf{B}_{\ell+1} \xrightarrow{w_0} \mathsf{B}, \mathsf{B}_1 = \mathsf{B}\} \\
&\cong T\backslash\{(\mathsf{B}_1, \dots, \mathsf{B}_{\ell+1}) \in (\mathsf{G}/\mathsf{B})^{\ell+1} : \mathsf{B}_1 \xrightarrow{\beta} \mathsf{B}_{\ell+1}, \mathsf{B}_1 = \mathsf{B}, \mathsf{B}_{\ell+1} = w_0\mathsf{B}\} \\
&\cong T\backslash X(\beta)
\end{aligned}$$

In conclusion

$$\boxed{\mathfrak{M}_1(\Lambda_{\beta w_0}) \cong T\backslash X(\beta)} \tag{6.2.2}$$

The diagonal subgroup $\Delta_T \subseteq T$, isomorphic to $\mathbb{G}_m$, acts trivially on $X(\beta)$. The quotient T/Δ_T acts non-trivally on $X(\beta)$. This (T/Δ_T)-action is free if $\Lambda_{\beta w_0}$ is a knot; it is not free if $\Lambda_{\beta w_0}$ is a link. In particular, if $\Lambda_{\beta w_0}$ is a knot

$$X(\beta) \cong Y(\beta) \times (\mathbb{G}_m)^{n-1}, \quad \text{where } Y(\beta) := (T/\Delta_T)\backslash X(\beta).$$

Here $Y(\beta)$ is a smooth irreducible affine variety and we are identifying $T/\Delta_T \cong (\mathbb{G}_m)^{n-1}$. Then the isomorphism (6.2.2) becomes

$$\mathfrak{M}_1(\Lambda_{\beta w_0}) \cong Y(\beta) \times B\mathbb{G}_m.$$

For $(\Lambda_\beta, \mathfrak{t}_\beta)$ pointed with one marked point per strand, framed version of isomorphism (6.2.2) is

$$\boxed{\mathfrak{M}_1(\Lambda_{\beta w_0}, \mathfrak{t}_\beta) \cong X(\beta) \times B\mathbb{G}_m} \tag{6.2.3}$$

Bibliography

[ACGK25] Mohammed Abouzaid, Sylvain Courte, Stéphane Guillermou, and Thomas Kragh. Twisted generating functions and the nearby Lagrangian conjecture. *Duke Math. J.*, 174(5):–, 2025.

[AD14] Michèle Audin and Mihai Damian. *Morse theory and Floer homology*. Universitext. Springer, London; EDP Sciences, Les Ulis, 2014. Translated from the 2010 French original by Reinie Erné.

[AdGiZV85] V. I. Arnol′ d, S. M. Guse˘ in Zade, and A. N. Varchenko. *Singularities of differentiable maps. Vol. I*, volume 82 of *Monographs in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 1985. The classification of critical points, caustics and wave fronts, Translated from the Russian by Ian Porteous and Mark Reynolds.

[AG01] V. I. Arnol'd and A. B. Givental′. Symplectic geometry [MR0842908 (88b:58044)]. In *Dynamical systems, IV*, volume 4 of *Encyclopaedia Math. Sci.*, pages 1–138. Springer, Berlin, 2001.

[Ali24] Yagub Aliyev. Apollonius problem and caustics of an ellipsoid. *Int. Electron. J. Geom.*, 17(2):402–420, 2024.

[AN94] V. I. Arnol'd and S. P. Novikov, editors. *Dynamical systems. VII*, volume 16 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, 1994. Integrable systems, nonholonomic dynamical systems, A translation of *Current problems in mathematics. Fundamental directions, Vol. 16 (Russian)*, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1987 [MR0922069 (88g:58004)], Translation by A. G. Reyman [A. G. Reĭman] and M. A. Semenov-Tian-Shansky [M. A. Semenov-Tyan-Shanskiĭ].

[Arn89] V. I. Arnol'd. *Mathematical methods of classical mechanics*, volume 60 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1989. Translated from the Russian by K. Vogtmann and A. Weinstein.

[Arn90a] V. I. Arnol'd. Contact geometry: the geometrical method of Gibbs's thermodynamics. In *Proceedings of the Gibbs Symposium (New Haven, CT, 1989)*, pages 163–179. Amer. Math. Soc., Providence, RI, 1990.

[Arn90b] V. I. Arnol'd. *Singularities of caustics and wave fronts*, volume 62 of *Mathematics and its Applications (Soviet Series)*. Kluwer Academic Publishers Group, Dordrecht, 1990.

[Arn95] V. I. Arnol'd. Some remarks on symplectic monodromy of Milnor fibrations. In *The Floer memorial volume*, volume 133 of *Progr. Math.*, pages 99–103. Birkhäuser, Basel, 1995.

[BGH20] Clark Barwick, Saul Glasman, and Peter Haine. Exodromy, 2020.

[BH22] Bhargav Bhatt and David Hansen. The six functors for Zariski-constructible sheaves in rigid geometry. *Compos. Math.*, 158(2):437–482, 2022.

[Bre97] Glen E. Bredon. *Sheaf theory*, volume 170 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1997.

[BST15] Frédéric Bourgeois, Joshua M. Sabloff, and Lisa Traynor. Lagrangian cobordisms via generating families: construction and geography. *Algebr. Geom. Topol.*, 15(4):2439–2477, 2015.

[BT82] Raoul Bott and Loring W. Tu. *Differential forms in algebraic topology*, volume 82 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1982.

[BZNP17] David Ben-Zvi, David Nadler, and Anatoly Preygel. Integral transforms for coherent sheaves. *J. Eur. Math. Soc. (JEMS)*, 19(12):3763–3812, 2017.

[Car01] E. Cartan. Sur quelques quadratures dont l'élément différentiel contient des fonctions arbitraires. *Bull. Soc. Math. France*, 29:118–130, 1901.

[Cas22] Roger Casals. Lagrangian skeleta and plane curve singularities. to appear *J. Fixed Point Theory App. (Viterbo 60 Volume)*, 2022.

[Cay09] Arthur Cayley. *The collected mathematical papers. Volume 2*. Cambridge Library Collection. Cambridge University Press, Cambridge, 2009. Reprint of the 1889 original.

[CdS01] Ana Cannas da Silva. *Lectures on symplectic geometry*, volume 1764 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2001.

[CE12] Kai Cieliebak and Yakov Eliashberg. *From Stein to Weinstein and back*, volume 59 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2012. Symplectic geometry of affine complex manifolds.

[Cer70] Jean Cerf. La stratification naturelle des espaces de fonctions différentiables réelles et le théorème de la pseudo-isotopie. *Inst. Hautes Études Sci. Publ. Math.*, (39):5–173, 1970.

[CG22] Roger Casals and Honghao Gao. Infinitely many Lagrangian fillings. *Ann. of Math. (2)*, 195(1):207–249, 2022.

[CG24] Roger Casals and Honghao Gao. A Lagrangian filling for every cluster seed. *Invent. Math.*, 237(2):809–868, 2024.

[CGG$^+$22] Roger Casals, Eugene Gorsky, Mikhail Gorsky, Ian Le, Linhui Shen, and José Simental. Cluster structures on braid varieties. J. Amer. Math. Soc. (to appear), 2022.

[CGGS20] Roger Casals, Eugene Gorsky, Mikhail Gorsky, and José Simental. Algebraic weaves and braid varieties. Amer. J. Math. (to appear), 2020.

[CGGS21] Roger Casals, Eugene Gorsky, Mikhail Gorsky, and José Simental. Positroid links and braid varieties. arXiv:2105.13948, 2021.

[Cha95] M. Chaperon. On generating families. In *The Floer memorial volume*, volume 133 of *Progr. Math.*, pages 283–296. Birkhäuser, Basel, 1995.

[Cha10] Baptiste Chantraine. Lagrangian concordance of Legendrian knots. *Algebr. Geom. Topol.*, 10(1):63–85, 2010.

[Che96a] Yu.$\tilde{\mathrm{V}}$. Chekanov. Critical points of quasifunctions, and generating families of Legendrian manifolds. *Funktsional. Anal. i Prilozhen.*, 30(2):56–69, 96, 1996.

[Che96b] Yu.$\tilde{\mathrm{V}}$. Chekanov. Lagrangian tori in a symplectic vector space and global symplectomorphisms. *Math. Z.*, 223(4):547–559, 1996.

[Che02] Yuri Chekanov. Differential algebra of Legendrian links. *Invent. Math.*, 150(3):441–483, 2002.

[Chi17] Sheng-Fu Chiu. Nonsqueezing property of contact balls. *Duke Math. J.*, 166(4):605–655, 2017.

[CL23] Roger Casals and Wenyuan Li. Conjugate fillings and Legendrian weaves. pages 1–100, 2023.

[CL24] Roger Casals and Wenyuan Li. Positive microlocal holonomies are globally regular, 2024.

[CLSBW23] Roger Casals, Ian Le, Sherman-Bennett, and Daping Weng. Demazure weaves for reduced plabic graphs (with a proof that Muller-Speyer twist is Donaldson-Thomas). arXiv:2308.06184, 2023.

[CM19] Roger Casals and Emmy Murphy. Legendrian fronts for affine varieties. *Duke Math. J.*, 168(2):225–323, 2019.

[CMP19] Roger Casals, Emmy Murphy, and Francisco Presas. Geometric criteria for overtwistedness. *J. Amer. Math. Soc.*, 32(2):563–604, 2019.

[CN25] Roger Casals and Yoon Jae Nho. Spectral networks and betti lagrangians, 2025.

[CW24] Roger Casals and Daping Weng. Microlocal theory of Legendrian links and cluster algebras. *Geom. Topol.*, 28(2):901–1000, 2024.

[Dar82] G. Darboux. Sur le problème de pfaff. *Bull. Sci. Math.*, 6:14–36, 49–38, 1882.

[DR11] Georgios Dimitroglou Rizell. Knotted Legendrian surfaces with few Reeb chords. *Algebr. Geom. Topol.*, 11(5):2903–2936, 2011.

[EES05] Tobias Ekholm, John Etnyre, and Michael Sullivan. Non-isotopic Legendrian submanifolds in $\mathbb{R}^{2n+1}$. *J. Differential Geom.*, 71(1):85–128, 2005.

[EES09] Tobias Ekholm, John B. Etnyre, and Joshua M. Sabloff. A duality exact sequence for Legendrian contact homology. *Duke Math. J.*, 150(1):1–75, 2009.

[EFM01] Judith Epstein, Dmitry Fuchs, and Maike Meyer. Chekanov-Eliashberg invariants and transverse approximations of Legendrian knots. *Pacific J. Math.*, 201(1):89–106, 2001.

[EH05] John B. Etnyre and Ko Honda. Cabling and transverse simplicity. *Ann. of Math. (2)*, 162(3):1305–1333, 2005.

[EHK16] Tobias Ekholm, Ko Honda, and Tamás Kálmán. Legendrian knots and exact Lagrangian cobordisms. *J. Eur. Math. Soc. (JEMS)*, 18(11):2627–2689, 2016.

[EKP06] Yakov Eliashberg, Sang Seon Kim, and Leonid Polterovich. Geometry of contact transformations and domains: orderability versus squeezing. *Geom. Topol.*, 10:1635–1747, 2006.

[ELT12] John B. Etnyre, Douglas J. LaFountain, and Bülent Tosun. Legendrian and transverse cables of positive torus knots. *Geom. Topol.*, 16(3):1639–1689, 2012.

[EM02] Y. Eliashberg and N. Mishachev. *Introduction to the h-principle*, volume 48 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2002.

[ENV13] John B. Etnyre, Lenhard L. Ng, and Vera Vértesi. Legendrian and transverse twist knots. *J. Eur. Math. Soc. (JEMS)*, 15(3):969–995, 2013.

[EP96] Y. Eliashberg and L. Polterovich. Local Lagrangian 2-knots are trivial. *Ann. of Math. (2)*, 144(1):61–76, 1996.

[EPR25] Michael Entov, Leonid Polterovich, and Lenya Ryzhik. A contact topological glossary for non-equilibrium thermodynamics, 2025.

[Eps60] D. B. A. Epstein. Linking spheres. *Proc. Cambridge Philos. Soc.*, 56:215–219, 1960.

[Etn03] John B. Etnyre. Introductory lectures on contact geometry. In *Topology and geometry of manifolds (Athens, GA, 2001)*, volume 71 of *Proc. Sympos. Pure Math.*, pages 81–107. Amer. Math. Soc., Providence, RI, 2003.

[Etn05] John B. Etnyre. Legendrian and transversal knots. In *Handbook of knot theory*, pages 105–185. Elsevier B. V., Amsterdam, 2005.

[FR11] Dmitry Fuchs and Dan Rutherford. Generating families and Legendrian contact homology in the standard contact space. *J. Topol.*, 4(1):190–226, 2011.

[Fro77] G. Frobenius. Ueber das Pfaffsche Problem. *J. Reine Angew. Math.*, 82:230–315, 1877.

[Gei08] Hansjörg Geiges. *An introduction to contact topology*, volume 109 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2008.

[Gir17] E. Giroux. Ideal Liouville Domains - a cool gadget. *ArXiv e-prints: 1708.08855*, August 2017.

[GKS12] Stéphane Guillermou, Masaki Kashiwara, and Pierre Schapira. Sheaf quantization of Hamiltonian isotopies and applications to nondisplaceability problems. *Duke Math. J.*, 161(2):201–245, 2012.

[GM88] Mark Goresky and Robert MacPherson. *Stratified Morse theory*, volume 14 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1988.

[Gol14] R. Golovko. A note on the front spinning construction. *Bull. Lond. Math. Soc.*, 46(2):258–268, 2014.

[Gor20] Mark Goresky. Morse theory, stratifications and sheaves. In *Handbook of geometry and topology of singularities. I*, pages 275–319. Springer, Cham, [2020] ©2020.

[GP10] Victor Guillemin and Alan Pollack. *Differential topology*. AMS Chelsea Publishing, Providence, RI, 2010. Reprint of the 1974 original.

[GPS24] Sheel Ganatra, John Pardon, and Vivek Shende. Microlocal Morse theory of wrapped Fukaya categories. *Ann. of Math. (2)*, 199(3):943–1042, 2024.

[Gro86] Mikhael Gromov. *Partial differential relations*, volume 9 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3)*. Springer-Verlag, Berlin, 1986.

[Gui23] Stéphane Guillermou. Sheaves and symplectic geometry of cotangent bundles. *Astérisque*, (440):x+274, 2023.

[GV24] Stéphane Guillermou and Claude Viterbo. The singular support of sheaves is γ-coisotropic. *Geom. Funct. Anal.*, 34(4):1052–1113, 2024.

[H03] Lars Hörmander. *The analysis of linear partial differential operators. I.* Classics in Mathematics. Springer-Verlag, Berlin, 2003. Distribution theory and Fourier analysis, Reprint of the second (1990) edition [Springer, Berlin; MR1065993 (91m:35001a)].

[Hat02] Allen Hatcher. *Algebraic topology.* Cambridge University Press, Cambridge, 2002.

[Jin24] Xin Jin. A Hamiltonian $\coprod_n BO(n)$-action, stratified Morse theory and the J-homomorphism. *Compos. Math.*, 160(9):2005–2099, 2024.

[JT24] Xin Jin and David Treumann. Brane structures in microlocal sheaf theory. *Journal of Topology*, 17(1):e12325, 2024.

[KL22] Christopher Kuo and Wenyuan Li. Spherical adjunction and Serre functor from microlocalization. *arXiv preprint arXiv:2210.06643*, 2022.

[KL24a] Christopher Kuo and Wenyuan Li. Duality, Künneth formulae, and integral transforms in microlocal geometry. *arXiv preprint arXiv:2405.15211*, 2024.

[KL24b] Christopher Kuo and Wenyuan Li. Relative Calabi-Yau structure on microlocalization. *arXiv preprint arXiv:2408.04085*, 2024.

[KS83] Masaki Kashiwara and Pierre Schapira. Micro-support d'un faisceau: application aux modules différentiels. In *Conference on complex analysis, Nancy 82 (Nancy, 1982)*, volume 8 of *Inst. Élie Cartan*, pages 204–219. Univ. Nancy, Nancy, 1983.

[KS85a] Masaki Kashiwara and Pierre Schapira. Microlocal study of sheaves. *Astérisque*, (128):235, 1985. Corrections to this article can be found in Astérisque No. 130, p. 209.

[KS85b] Masaki Kashiwara and Pierre Schapira. Microlocal study of sheaves. *Astérisque*, (128):235, 1985. Corrections to this article can be found in Astérisque No. 130, p. 209.

[KS90] Masaki Kashiwara and Pierre Schapira. *Sheaves on manifolds*, volume 292 of *Grundlehren der Mathematischen Wissenschaften.* Springer-Verlag, Berlin, 1990. With a chapter in French by Christian Houzel.

[KS01] Masaki Kashiwara and Pierre Schapira. Ind-sheaves. *Astérisque*, (271):136, 2001.

[Kuo23] Christopher Kuo. Wrapped sheaves. *Adv. Math.*, 415:Paper No. 108882, 71, 2023.

[Kuw24] Tatsuki Kuwagaki. Sheaf quantization from exact WKB analysis. *J. Reine Angew. Math.*, 814:165–203, 2024.

[Li21] Wenyuan Li. Estimating reeb chords using microlocal sheaf theory. *arXiv:2106.04079*, 2021.

[LS91] Francois Lalonde and Jean-Claude Sikorav. Sous-varietes lagrangiennes et lagrangiennes exactes des fibres cotangents. *Comment. Math. Helv.*, 66(1):18–33, 1991.

[Lur09] Jacob Lurie. *Higher topos theory*, volume 170 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2009.

[Lur17] Jacob Lurie. Higher algebra. *Preprint, available at https://www.math.ias.edu/ lurie/*, 2017.

[McD90] Dusa McDuff. The structure of rational and ruled symplectic 4-manifolds. *J. Amer. Math. Soc.*, 3(3):679–712, 1990.

[Mil63] J. Milnor. *Morse theory*, volume No. 51 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1963. Based on lecture notes by M. Spivak and R. Wells.

[Mil68] John Milnor. *Singular points of complex hypersurfaces*. Annals of Mathematics Studies, No. 61. Princeton University Press, Princeton, N.J., 1968.

[MS98] Dusa McDuff and Dietmar Salamon. *Introduction to symplectic topology*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, second edition, 1998.

[Nad09] David Nadler. Microlocal branes are constructible sheaves. *Selecta Math. (N.S.)*, 15(4):563–619, 2009.

[Nad16] David Nadler. Wrapped microlocal sheaves on pairs of pants. arXiv:1604.00114, 2016.

[Nad17] David Nadler. Arboreal singularities. *Geom. Topol.*, 21(2):1231–1274, 2017.

[NS20] David Nadler and Vivek Shende. Sheaf quantization in Weinstein symplectic manifolds. *arXiv preprint arXiv:2007.10154*, 2020.

[NT23] David Nadler and Jeremy Taylor. The whittaker functional is a shifted microstalk, 2023.

[NZ09] David Nadler and Eric Zaslow. Constructible sheaves and the Fukaya category. *J. Amer. Math. Soc.*, 22(1):233–286, 2009.

[Oh15] Yong-Geun Oh. *Symplectic topology and Floer homology. Vol. 1*, volume 28 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2015. Symplectic geometry and pseudoholomorphic curves.

[Pol91] L. Polterovich. The surgery of Lagrange submanifolds. *Geom. Funct. Anal.*, 1(2):198–210, 1991.

[Pos06] Alexander Postnikov. Total positivity, grassmannians, and networks. preprint, 2006.

[PT25a] Mauro Porta and Jean-Baptiste Teyssier. The derived moduli of stokes data, 2025.

[PT25b] Mauro Porta and Jean-Baptiste Teyssier. Homotopy theory of stokes data, 2025.

[RS18] Marco Robalo and Pierre Schapira. A lemma for microlocal sheaf theory in the ∞-categorical setting. *Publ. Res. Inst. Math. Sci.*, 54(2):379–391, 2018.

[RSTZ14] Helge Ruddat, Nicolò Sibilla, David Treumann, and Eric Zaslow. Skeleta of affine hypersurfaces. *Geom. Topol.*, 18(3):1343–1395, 2014.

[Sch03] Jörg Schürmann. *Topology of singular spaces and constructible sheaves*, volume 63 of *Instytut Matematyczny Polskiej Akademii Nauk. Monografie Matematyczne (New Series) [Mathematics Institute of the Polish Academy of Sciences. Mathematical Monographs (New Series)]*. Birkhäuser Verlag, Basel, 2003.

[Sei99] Paul Seidel. Lagrangian two-spheres can be symplectically knotted. *J. Differential Geom.*, 52(1):145–171, 1999.

[Sei01] Paul Seidel. Vanishing cycles and mutation. In *European Congress of Mathematics, Vol. II (Barcelona, 2000)*, volume 202 of *Progr. Math.*, pages 65–85. Birkhäuser, Basel, 2001.

[Sei08] Paul Seidel. *Fukaya categories and Picard-Lefschetz theory*. Zurich Lectures in Advanced Mathematics. European Mathematical Society (EMS), Zürich, 2008.

[SKK73] Mikio Sato, Takahiro Kawai, and Masaki Kashiwara. Microfunctions and pseudo-differential equations. In *Hyperfunctions and pseudo-differential equations (Proc. Conf., Katata, 1971; dedicated to the memory of André Martineau)*, volume Vol. 287 of *Lecture Notes in Math.*, pages 265–529. Springer, Berlin-New York, 1973.

[Sma61] Stephen Smale. Generalized Poincaré's conjecture in dimensions greater than four. *Ann. of Math. (2)*, 74:391–406, 1961.

[Sma62] S. Smale. On the structure of manifolds. *Amer. J. Math.*, 84:387–399, 1962.

[Sta18] Laura Starkston. Arboreal singularities in Weinstein skeleta. *Selecta Math. (N.S.)*, 24(5):4105–4140, 2018.

[STWZ19] Vivek Shende, David Treumann, Harold Williams, and Eric Zaslow. Cluster varieties from Legendrian knots. *Duke Math. J.*, 168(15):2801–2871, 2019.

[STZ17] Vivek Shende, David Treumann, and Eric Zaslow. Legendrian knots and constructible sheaves. *Invent. Math.*, 207(3):1031–1133, 2017.

[Swi92] Jacek Swiatkowski. On the isotopy of Legendrian knots. *Ann. Global Anal. Geom.*, 10(3):195–207, 1992.

[Tab09] Goncalo Tabuada. Homotopy theory of well-generated algebraic triangulated categories. *J. K-Theory*, 3(1):53–75, 2009.

[Tam18] Dmitry Tamarkin. Microlocal condition for non-displaceability. In *Algebraic and analytic microlocal analysis*, volume 269 of *Springer Proc. Math. Stat.*, pages 99–223. Springer, Cham, 2018.

[Thu82] William P. Thurston. Three-dimensional manifolds, Kleinian groups and hyperbolic geometry. *Bull. Amer. Math. Soc. (N.S.)*, 6(3):357–381, 1982.

[Tra01] Lisa Traynor. Generating function polynomials for Legendrian links. *Geom. Topol.*, 5:719–760, 2001.

[Tre09] David Treumann. Exit paths and constructible stacks. *Compos. Math.*, 145(6):1504–1532, 2009.

[TV05] Bertrand Toën and Gabriele Vezzosi. Homotopical algebraic geometry. I. Topos theory. *Adv. Math.*, 193(2):257–372, 2005.

[TV07] Bertrand Toën and Michel Vaquié. Moduli of objects in dg-categories. *Ann. Sci. École Norm. Sup. (4)*, 40(3):387–444, 2007.

[TV08] Bertrand Toën and Gabriele Vezzosi. Homotopical algebraic geometry. II. Geometric stacks and applications. *Mem. Amer. Math. Soc.*, 193(902):x+224, 2008.

[Ver76] Jean-Louis Verdier. Classe d'homologie associée à un cycle. In *Séminaire de géométrie analytique (École Norm. Sup., Paris, 1974–75)*, volume No. 36-37 of *Astérisque*, pages 101–151. Soc. Math. France, Paris, 1976.

[VG88] A. M. Vershik and V. Ya. Gershkovich. Estimation of the functional dimension of the orbit space of germs of distributions in general position. *Mat. Zametki*, 44(5):596–603, 700, 1988.

[Vic13] Nicolas Vichery. Homogénéisation symplectique et Applications de la théorie des faisceaux à la topologie symplectique. 2013.

[Vit10] Claude Viterbo. An Introduction to Symplectic Topology through Sheaf theory. *Lectures Notes at Princeton (2010) and New York (2011)*, 2010.

[Vit19] Claude Viterbo. Sheaf Quantization of Lagrangians and Floer cohomology . 2019.

[Wei94] Charles A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994.

[Zee65] E. C. Zeeman. Twisting spun knots. *Trans. Amer. Math. Soc.*, 115:471–495, 1965.

[Zha24] Bingyu Zhang. Capacities from the Chiu-Tamarkin complex. *J. Symplectic Geom.*, 22(3):441–524, 2024.