MAT 22B: PROBLEM SET 1

DUE TO FRIDAY OCT 4 AT 10:00AM

ABSTRACT. This is the first problem set for the Differential Equations Course in the Fall Quarter 2019. It was posted online on Friday Sep 27 and is due Friday Oct 4 at 10:00am via online submission.

Purpose: The goal of this assignment is to practice the basic techniques for solving first-order differential equations. In particular, we would like to become familiar with method of **integrating factors** and **separation of variables**.

Task: Solve Problems 1 through 7 below. The first 2 problems will not be graded but I trust that you will work on them. Problems 3 to 7 will be graded.

Instructions: It is perfectly good to consult with other students and collaborate when working on the problems. However, you should write the solutions on your own, using your own words and thought process. List any collaborators in the upper-left corner of the first page.

Grade: Each graded Problem is worth 20 points, the total grade of the Problem Set is the sum of the number of points. The maximum possible grade is 100 points.

Textbook: We will use "Elementary Differential Equations and Boundary Value Problems" by W.E. Boyce, R.C. DiPrima and D.B. Meade (11th Edition). Please contact me *immediately* if you have not been able to get a copy of any edition.

Writing: Solutions should be presented in a balanced form, combining words and sentences which explain the line of reasoning, and also precise mathematical expressions, formulas and references justifying the steps you are taking are correct.

Problem 1. Let $t \in \mathbb{R}$ be a real number and $y(t)$ a differentiable real valued function. Find all the solutions to the following differential equations:

(a) $y'(t) = \frac{1}{1+t^2}$,

(b) $y'(t) = 3t^2 - 4 + \frac{1}{t} - \frac{7}{t^2}$, for $t > 0$,

(c) $y'(t) = 5y(t)$,

(d) $y''(t) = -y(t)$.

Problem 2. Let $t \in \mathbb{R}$, and consider the function

$$y(t) = \arccos(t) + 4t^2.$$

(a) Find a first order differential equation which has $y(t)$ as a solution,

(b) Find a second order differential equation which has $y(t)$ as a solution.

Problem 3. (20 pts) Consider the following differential equation:

$$y'(t) - \frac{3y(t)}{t+1} = (1+t)^4, \quad t > 0.$$

(a) Find all the solutions to the differential equation above.

(b) Find all the solutions which satisfy $y(0) = 3$.

(c) Are there any solutions $y(t)$ such that $y(0) = 3$ and $y(1) = 100$?

(d) Are there any solutions $y(t)$ such that $y(0) = 3$ and $y(1) = 36$?

Problem 4. (20 pts) Consider the following differential equation:

$$y'(t) + 2ty(t) = t, \quad t > 0.$$

(a) Find all the solutions to the differential equation above.

(b) Let $y_1(t)$ be the unique solution such that $y_1(2) = 0.5$, $y_2(t)$ the unique solution such that $y(0) = 1$, and $y_3(t)$ the unique solution such that $y(0) = 2$. Compute the long term behaviours

$$\lim_{t\to\infty} y_1(t), \quad \lim_{t\to\infty} y_2(t), \quad \lim_{t\to\infty} y_3(t)$$

for these three solutions.

(c) Plot the graph of the functions $y_1(t), y_2(t)$ and $y_3(t)$.

(d) Is there any solution $y(t)$ which tends to ∞ in its long term behaviour ?

Problem 5. (20 pts) Consider the following differential equation:

$$y'(t) = y^2 \cdot \frac{t-3}{t^3}, \quad t > 0.$$

(a) Find infinitely many the solutions to the differential equation above.

(b) Solve the initial value problem given by the above differential equation and the initial condition $y(1) = -1$.

(c) Is the constant function $y(t) \equiv 0$ a solution to the above differential equation ?

Problem 6. (20 pts) Consider the following differential equation:

$$y'(t) = 2t(1-y)^2, \quad t \in \mathbb{R}$$

(a) Find all solutions of the differential equation above.

(b) Does there exist a solution $y_1(t)$ such that $y_1(2) = 1$ and $y_1(3) = 1$?

(c) Plot qualitatively the graph of six different solutions of the differential equation.

(d) Find the long-term behaviour of all solutions for the differential equation.

Problem 7. (20 pts) Consider the following differential equation:

$$y'(t) = y(t)(1-y(t)), \quad t \in \mathbb{R}^+,$$

(a) Find all solutions of the differential equation above.

(b) Describe the possible long-term behaviour of a solution $y(t)$ in terms of its initial value $y(0)$.

(c) Compare the differential equation above with the differential equation

$$y'(t) = y(t), \quad t \in \mathbb{R}^+,$$

as if they were two models for population growth, i.e. $y(t)$ is the size of a population at time $t \in \mathbb{R}$. Which of the two models is more accurate ?

(d) Consider the differential equation

$$y'(t) = \lambda \cdot y(t)(1-y(t)), \quad t \in \mathbb{R}^+, \lambda \in \mathbb{R}^+.$$

How do the solutions differ for distinct values of $\lambda \in \mathbb{R}$?