SOLUTIONS TO PROBLEM SET 5

MAT 108

Abstract. These are the solutions to Problem Set 5 for MAT 108 in the Fall Quarter 2020. The problems were posted online on Wednesday Nov 11 and due Friday Nov 20.

Problem 1. (Project 10.20 in textbook) Let (x_n) be a sequence of real numbers which is decreasing and bounded below. Show that x_n converges.

Solution. This second half of the Monotone Convergence Theorem follows from the first half (similar in style to Problem 1 of Problem Set 5). Let $(x_n)_{n\in\mathbb{N}}$ be a sequence which is decreasing and bounded below. The first condition means that $x_{n+1} \leq x_n$ for all $n \in \mathbb{N}$, and the second condition means that there exists $R \in \mathbb{R}$ such that $x_n \geq R$ for all $n \in \mathbb{N}$.

Consider a new sequence $(y_n)_{n\in\mathbb{N}}$ defined by

$$y_n = -x_n.$$

Then, for all $n \in \mathbb{N}$, we have

$$-y_{n+1} = x_{n+1} \leq x_n = -y_n,$$

so $y_{n+1} \geq x_n$. Therefore, (y_n) is increasing. Similarly,

$$-y_n = x_n \geq R,$$

so $y_n \leq -R$. Therefore, (y_n) is bounded above. By the Monotone Convergence Thereorem, (y_n) converges to some real number $L \in \mathbb{R}$.

We claim that (x_n) converges to $-L$. Let $\varepsilon > 0$. Then there exists a natural number $n_0 \in \mathbb{N}$ such that

$$|y_n - L| < \varepsilon$$

for all $n \geq n_0$. Therefore, for all $n \geq n_0$, we have

$$|x_n - (-L)| = |-y_n + L| = |y_n - L| < \varepsilon,$$

which completes the proof.

Problem 2. (20 points, 5 each) Consider the following four sequences of real numbers:

$$x_n = \frac{2n+1}{3n-4}, \quad y_n = \frac{1}{n!}, \quad z_n = \frac{n!}{n^n}, \quad w_n = \frac{3n^2-1}{n^2+n}.$$

In this exercise, you must use the ε-definition of the limit (Definition in Section 10.4) to show the following statements.

(a) Show that $\lim_{n\to\infty} x_n = 2/3$,

(b) Show that $\lim_{n\to\infty} y_n = 0$,

(c) Show that $\lim_{n\to\infty} z_n = 0$,

(d) Show that $\lim_{n\to\infty} w_n = 3$.

In each of these four cases above, you must write a complete detailed and self-contained proof that the limit is the one stated. Each can be done directly from the definition.

Be **clear** in the use of ε, the quantifiers and the indices when you write the four proofs above. In particular, write clearly what you are given and what you must prove when writing down the definition of each of the limits.

Solution.

(a) Let $\varepsilon > 0$. Let $N \in \mathbb{N}$ satisfy $N > \frac{11}{3\varepsilon} + \frac{4}{3}$. For $n \geq N$, we have

$$\left|\frac{2n+1}{3n-4} - \frac{2}{3}\right| = \frac{11}{3|3n-4|} < \frac{11}{|3n-4|} \leq \frac{11}{|3N-4|} < \varepsilon.$$

(b) Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that $\frac{1}{N} < \varepsilon$. For $n \geq N$, we have

$$\left|\frac{1}{n!} - 0\right| \leq \frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

(c) We first look at the sequence z_n. We can rewrite it as follows.

$$\frac{n!}{n^n} = \frac{n}{n} \cdot \frac{n-1}{n} \cdots \frac{1}{n}$$

We use Proposition 10.23(iv) so we consider the limit of each sequence $\left\{\frac{k}{n}\right\}$ where $1 \leq k \leq n$ as n approaches infinity. Notice that

$$\frac{n}{n} \cdot \frac{n-1}{n} \cdots \frac{1}{n} \leq 1 \cdot 1 \cdots \frac{1}{n}$$

so it suffices to look at the sequence $\left\{\frac{1}{n}\right\}$. Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that $\frac{1}{N} < \varepsilon$. For $n \geq N$, we have

$$\left|\frac{1}{n} - 0\right| \leq \frac{1}{N} < \varepsilon.$$

Hence, we get

$$0 \leq \lim_{n\to\infty} \frac{n!}{n^n} = \lim_{n\to\infty} \frac{n}{n} \cdot \lim_{n\to\infty} \frac{n-1}{n} \cdots \lim_{n\to\infty} \frac{1}{n} \leq 1 \cdot 1 \cdots \lim_{n\to\infty} \frac{1}{n} = 0$$

so

$$\lim_{n\to\infty} \frac{n!}{n^n} = 0.$$

(d) Let $\varepsilon > 0$. Let $N \in \mathbb{N}$ satisfy $N > \frac{3}{\varepsilon}$. For $n \geq N$, we have

$$\left|\frac{3n^2-1}{n^2+n} - 3\right| = \frac{3n+1}{n^2+n} < \frac{3}{n} \leq \frac{3}{N} < \varepsilon.$$

Problem 3. (10+10 points) Consider the following two sequences of real numbers

$$x_n = \frac{4n-3}{2^n}, \quad y_n = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{n^2}.$$

In this exercise we will show that they are convergent.

(a) Show that (x_n) is eventually decreasing and bounded below. By eventually decreasing it is meant that

$$x_{n+1} \leq x_n, \quad \text{for large enough } n \in \mathbb{N}.$$

(b) Show that (y_n) is increasing and bounded above.

Observation: By the Monotone Convergence Limit, you have proven that the limit of (y_n) actually exists. It is a real challenge to show that it is actually $\pi^2/6$.

Solution.

(a) We have

$$x_n = \frac{4n-3}{2^n}$$

so the next term is

$$x_{n+1} = \frac{4(n+1)-3}{2^{n+1}} = \frac{4n+1}{2^{n+1}}.$$

Notice that

$$x_n = \frac{2(4n-3)}{2^{n+1}}.$$

For n large enough, $2(4n-3) > 4n+1$. In fact, this happens when $n > 1$. Thus, $x_{n+1} > x_n$ so (x_n) is eventually decreasing. This sequence is bounded below because $x_n > 0$ for all $n \in \mathbb{N}$.

(b) The sequence (y_n) is increasing because

$$y_n = \frac{1}{1^2} + \cdots + \frac{1}{n^2} < \frac{1}{1^2} + \cdots + \frac{1}{n^2} + \frac{1}{(n+1)^2} = y_{n+1}.$$

We now show it is bounded above. Note $\frac{1}{n^2} < \frac{1}{n(n-1)} = \frac{1}{n-1} - \frac{1}{n}$. Then,

$$\frac{1}{2^2} + \cdots + \frac{1}{n^2} < \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) = 1 - \frac{1}{n}.$$

By adding $\frac{1}{1^2}$ to both sides, we obtain

$$\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2} < 2 - \frac{1}{n}.$$

Problem 4. (10+10 points) Consider the following sequence, defined recursively:

$$x_{n+1} = \frac{x_n}{2} + 1, \quad \forall n \in \mathbb{N}, \quad x_1 = 1.$$

(a) Show that (x_n) is increasing and bounded above.

(b) Prove that (x_n) converges and find its limit.

Solution.

(a) We use induction to prove (x_n) is increasing. If $n = 1$, then $x_2 = \frac{3}{2} > 1 = x_1$. Now assume $x_{k+1} > x_k$. We have

$$x_{k+2} = \frac{x_{k+1}}{2} + 1 > \frac{x_k}{2} + 1 = x_{k+1}.$$

We now show it is bounded above by 2 using induction. When $n = 1$, $x_2 = \frac{3}{2} < 2$. Assume $x_k < 2$. Then,

$$x_{k+1} = \frac{x_k}{2} + 1 < \frac{2}{2} + 1 = 2.$$

(b) We see that (x_n) is bounded below by 1. By Theorem 10.19, we conclude this sequence converges. We claim the limit is 2. Let $\varepsilon > 0$. Then,

$$|x_{n+1} - 2| = \left|\frac{x_n}{2} + 1 - 2\right| = \left|\frac{x_n - 2}{2}\right| = \left|\frac{x_{n-1} - 2}{2^2}\right| = \cdots = \left|\frac{x_1 - 2}{2^n}\right|.$$

The last few equalities follow from the definition of (x_n). For example,

$$\frac{x_n - 2}{2} = \frac{\left(\frac{x_{n-1}}{2} + 1\right) - 2}{2} = \frac{x_{n-1} - 2}{2^2}.$$

We are given $x_1 = 1$ so we consider the sequence $\left(\frac{1}{2^n}\right)$. Set $N \in \mathbb{N}$ such that $\frac{1}{N} < \varepsilon$. This sequence converges to 0 because

$$\left|\frac{1}{2^n} - 0\right| < \frac{1}{n} < \frac{1}{N} < \varepsilon$$

for $n \geq N$. Therefore,

$$|x_{n+1} - 2| < \varepsilon.$$

Problem 5. (5+10+5 points) Prove the following three statements:

(a) Any convergent sequence (x_n) is bounded.

(b) Let (x_n) and (y_n) be two convergent sequences, and suppose that their limits are $x_n \longrightarrow L$ and $y_n \longrightarrow M$. Show that the sequence $(x_n + y_n)$, obtained by summing them termwise, is a convergent sequence, and in fact

$$x_n + y_n \longrightarrow (L + M).$$

(c) There exist bounded sequences which are not convergent.

Solution.

(a) Let $(x_n)_{n\in\mathbb{N}}$ be a sequence convergent to $L \in \mathbb{R}$. Setting $\varepsilon = 1$ in the definition of limit (choosing 1 is completely arbitrary), we know that there exists a natural number $n_0 \in \mathbb{N}$ such that

$$|x_n - L| < 1$$

for all $n \geq n_0$. This simplifies to

$$L - 1 < x_n < L + 1$$

for all $n \geq n_0$. Consider the set of remaining terms

$$A := \{x_n : n < n_0\} = \{x_1, x_2, \ldots, x_{n_0-1}\}.$$

Let R^+ and R^- be the maximum and minimum, respectively, of the set

$$A \cup \{L - 1, L + 1\},$$

Note that R^+ and R^- exist because the set $A \cup \{L-1, L+1\}$ is finite, having at most $n_0 + 1$ elements. Now consider an arbitrary member x_n of our sequence. If $n < n_0$, then $n \in A$, so

$$R^- \leq \min(A) \leq x_n \leq \max(A) \leq R^+.$$

Finally, if $n \geq n_0$, then

$$R^- \leq L - 1 \leq x_n \leq L + 1 \leq R^+.$$

Therefore, $R^- < x_n < R^+$ for all $n \in \mathbb{N}$, so (x_n) is a bounded sequence.

(b) Let $\varepsilon > 0$. Then there exist natural numbers $n_1, n_2 \in \mathbb{N}$ such that

$$|x_n - L| < \frac{\varepsilon}{2} \qquad \text{for all } n \geq n_1, \text{ and}$$
$$|y_n - M| < \frac{\varepsilon}{2} \qquad \text{for all } n \geq n_2.$$

Set $n_0 := \max\{n_1, n_2\}$ (that is, n_0 is the greater of the two numbers n_1 and n_2). Then $n_0 \geq n_1$ and $n_0 \geq n_2$. Therefore, we have

$$|x_n - L| < \frac{\varepsilon}{2} \qquad \text{and} \qquad |y_n - M| < \frac{\varepsilon}{2}$$

for all $n \geq n_0$. Finally, we have

$$\begin{aligned} |(x_n + y_n) - (L + M)| &= |x_n - L + y_n - M| \\ &\leq |x_n - L| + |y_n - M| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon, \end{aligned}$$

where we used the triangle inequality in the second line. Therefore,

$$|(x_n + y_n) - (L + M)| < \varepsilon$$

for all $n \geq n_0$, so we conclude that the sequence $(x_n + y_n)$ converges to $L + M$.

(c) By Problem 6(a) below, the sequence $x_n = (-1)^n$ is not convergent. Furthermore, $\{x_n : n \in \mathbb{N}\} = \{1, -1\}$, which is bounded (if you like, because it is finite), so x_n is bounded.

Problem 6. (5+5+5+5 points) Prove or disprove each of the statements.[1]

(a) The sequence $x_n = (-1)^n$ is convergent.

(b) The sequence $x_n = \frac{(-1)^n}{n}$ is convergent.

(c) Let (x_n) be a sequence such that the sequence $|x_n|$ of absolute values converges. Then (x_n) converges.

(d) Let (x_n) and (y_n) be two unbounded sequences. Then the product sequence $(x_n \cdot y_n)$ is unbounded.

Solution.

(a) False. The sequence $x_n = (-1)^n$ is not convergent. Suppose for the sake of contradiction that x_n converges to L for some $L \in \mathbb{R}$. Then there exists an $n_0 \in \mathbb{N}$ such that

$$|x_n - L| < 1$$

for all integers $n \geq n_0$ (here we take the special case $\varepsilon = 1$ in the definition of limit). Notice that $2n_0 \geq n_0$ and $2n_0 + 1 \geq n_0$, so

$$1 > |x_{2n_0} - L| = |(-1)^{2n_0} - L| = |1 - L| = |L - 1|$$

and

$$1 > |x_{2n_0+1} - L| = |(-1)^{2n_0+1} - L| = |-1 - L| = |L + 1|.$$

These imply that

$$0 < L < 2 \quad \text{and} \quad -2 < L < 0,$$

respectively, which is a contradiction.

[1] If your claim is that a statement is false, then you must give a counter-example or an argument showing that the statement is indeed mathematically incorrect.

(b) True. We show that the sequence $x_n = \frac{(-1)^n}{n}$ converges to 0. Let $\varepsilon > 0$. There exists an integer $n_0 \in \mathbb{N}$ such that $n_0 > 1/\varepsilon$. Then, for any integer $n \geq n_0$, we have

$$|x_n - 0| = \left|\frac{(-1)^n}{n}\right| = \frac{1}{n} \leq \frac{1}{n_0} < \frac{1}{1/\varepsilon} = \varepsilon.$$

We conclude that $|x_n - 0| < \varepsilon$ for all $n \geq n_0$, so (x_n) converges to 0, as desired.

(c) False. By part (a), the sequence $x_n = (-1)^n$ is not convergent. But the sequence $|x_n| = |(-1)^n| = 1$ is constant, and therefore is convergent (by the Monotone Convergence Theorem, if you like).

(d) False. Define two sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ by

$$x_n = \begin{cases} n & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases} \quad \text{and} \quad y_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ n & \text{if } n \text{ is odd} \end{cases}.$$

Notice that $\{x_n : n \in \mathbb{N}\}$ is the set of all even natural numbers, and $\{y_n : n \in \mathbb{N}\}$ is the set of all odd natural numbers, so both sequences are unbounded. However, the product $(x_n y_n)_{n\in\mathbb{N}}$ has terms

$$x_n y_n = \begin{cases} n \cdot 0 & \text{if } n \text{ is even} \\ 0 \cdot n & \text{if } n \text{ is odd} \end{cases} = 0.$$

Therefore $\{x_n y_n : n \in \mathbb{N}\} = \{0\}$, so this sequence is bounded.

Remark: Whenever you see a false statement, it's always good to ask "why did this fail?" In other words, how could you strengthen the hypotheses of the statement so that the conclusion becomes true?

Notice that neither sequence in the counterexample above is monotone. If you wanted to modify the original hypotheses to make the conclusion true, would it be enough to assume that both sequences are monotone? What if you assume that just one is monotone?