SOLUTIONS TO PROBLEM SET 1

MAT 108

Abstract. These are the solutions to Problem Set 1 for MAT 108 in the Fall Quarter 2020. The problems were posted online on Tuesday Sep 29 and due Friday Oct 9 at 9:00am

Problem 1. Read the following statements and show that they are *false* by giving a counter-example.

(a) Let $n, m \in \mathbb{N}$ be two natural numbers, such that n is even and m is odd. Then $n + m$ is even.

(b) Let $n \in \mathbb{N}$ be an even natural number, then there exists an even natural number $m \in \mathbb{N}$ such that their sum $n + m$ is odd.

(c) Let $a, b, c \in \mathbb{N}$ be three non-zero natural numbers such that $a^2 + b^2 = c^2$. Then the three numbers must be $a = 3$, $b = 4$ and $c = 5$.

(d) Let $\alpha, \beta, \gamma \in [0, 2\pi)$ be the three angles. There exists a unique planar triangle whose interior angles are α, β and γ.

Solution.

(a) Let $n = 2$ and $m = 3$. We then get $n + m = 2 + 3 = 5$, which is not even.

(b) Let $n = 2$. If $m = 2k$ where $k \in \mathbb{N}$, then $n + m = 2 + 2k = 2(1 + k)$, which is not odd.

(c) Consider $a = 6$, $b = 8$, and $c = 10$. This satisfies the equation: $6^2 + 8^2 = 36 + 64 = 100 = 10^2$.

(d) Let $\alpha = 0$, $\beta = \frac{\pi}{3}$, and $\gamma = \frac{\pi}{4}$. These three angles won't form a planar triangle because they don't add up to π.

Problem 2. Read Sections 1.1 and 1.2 in the textbook, and carefully follow their proofs of Proposition 1.6 and Proposition 1.9. Prove, using the five Axioms in Section 1.1 (and Prop. 1.6 if need) the following two propositions:

Proposition (Proposition 1.7). If m is an integer, then $0 + m = m$ and $1 \cdot m = m$.

Proposition (Proposition 1.8). If m is an integer, then $(-m) + m = 0$.

Solution.

Proposition 1.7. Let m be an integer. By Axiom 1.2, there exists an integer 0 such that whenever $m \in \mathbb{Z}$, $m + 0 = m$. We rewrite the left hand side using Axiom 1.1(i): $m + 0 = 0 + m = m$. To prove the second part of the proposition, we use Axiom 1.3:

there exists an integer 1 such that $1 \neq 0$ and whenever $m \in \mathbb{Z}$, $m \cdot 1 = m$. We use Axiom 1.1(iv) to conclude $m \cdot 1 = 1 \cdot m = m$.

Proposition 1.8. Let m be an integer. By Axiom 1.4, for each $m \in \mathbb{Z}$, there exists an integer $-m$ such that $m + (-m) = 0$. We then use Axiom 1.1(i) to rewrite the left hand side: $m + (-m) = (-m) + m = 0$.

Problem 3. (10+10 pts) Let us take Axioms 1.1 through 1.5 in Section 1.1 as true, and assume Propositions 1.6 through 1.9 have been proven. Prove the following two propositions:

Proposition (Proposition 1.10). Let $m, x_1, x_2 \in \mathbb{Z}$. If m, x_1, x_2 satisfy the equations $m + x_1 = 0$ and $m + x_2 = 0$, then $x_1 = x_2$.

Proposition (Proposition 1.12). Let $x \in \mathbb{Z}$. If x has the property that for each integer m, $m + x = m$, then $x = 0$.

Solution.

Proposition 1.10. Let $m, x_1, x_2 \in \mathbb{Z}$. Assume m, x_1, x_2 satisfy the equations $m + x_1 = 0$ and $m + x_2 = 0$. In words, this proposition is saying every integer m has a unique additive inverse. Suppose x_1 and x_2 are additive inverses of m. Then,

$$x_1 \overset{\text{Axiom 1.2}}{=} x_1 + 0 = x_1 + (m + x_2) \overset{\text{Axiom 1.1}(ii)}{=} (x_1 + m) + x_2 = 0 + x_2 \overset{\text{Prop 1.7}}{=} x_2.$$

Proposition 1.12. Let $x \in \mathbb{Z}$ and assume x has the property that for each $m \in \mathbb{Z}$,

$$m + x = m. \tag{0.1}$$

By Axiom 1.4, for each $m \in \mathbb{Z}$, there exists an integer $-m$ such that

$$m + (-m) = 0.$$

By adding $-m$ to both sides of Equation 0.1, we get

$$(-m) + (m + x) = (-m) + m.$$

The right hand side of the equation is 0 as explained above. For the left hand side, we have

$$(-m) + (m + x) \overset{\text{Axiom 1.1}(ii)}{=} (-m + m) + x = 0 + x \overset{\text{Prop 1.7}}{=} x.$$

Hence, $x = 0$.

Problem 4. (20 pts) Discuss the difference between the following two statements and prove that at least **one** of them is false.

Statement (1). There exists a natural number $a \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, we have that $n + a = 7$.

Statement (2). For all $n \in \mathbb{N}$, there exists a natural number $a \in \mathbb{N}$ such that we have that $n + a = 7$.

Here 10 points are given for correctly pointing out the difference in the mathematical content between Statements 1 and 2, and 10 points are given for correctly proving that one of the statements is wrong.

Solution. Consider the equation $n + a = 7$. For the first statement, the number a is fixed and is not dependent of the number n. For the second statement, we are given any natural number n and are allowed to choose the appropriate number a. In other words, a depends on n. We provide a counterexample for the second statement. If $n = 100$, then there does not exist a natural number a such that $n + a = 100 + a = 7$.

Problem 5. (20 pts) Let us assume the following two *axioms*, as discussed in class:

A1. The area of a planar rectangle of sides $a, b \in \mathbb{R}$ is the product $a \cdot b$.

A2. The area of two planar figures which intersect at most along edges is the sum of the areas of each of the planar figures.

From the two Axioms A1 and A2 above, deduce that the area of a triangle with height $h \in \mathbb{R}$ and base $a \in \mathbb{R}$ equals the quantity $(a \cdot h)/2$.
Hint: try to cut the triangle in pieces and reassemble them to get a rectangle, then apply Axiom 2. Make sure to explain where are you using Axiom 2 in this argument.

Solution. We will proceed in two steps: first, we extend Axiom A1 to the case of parallelograms, and then we use that extension to proof the desired result.

Step 1: We claim the following extension of Axiom A1: *The area of a planar parallelogram with base $a \in \mathbb{R}$ and height $b \in \mathbb{R}$ is the product $a \cdot b$.* To prove this claim, we will construct a rectangle from any given parallelogram, and then apply Axiom A1.

Therefore, let P be a planar parallelogram with base a and height b. Draw a line segment in P of length b, perpendicular to the base, splitting P into a right triangle A and a trapezoid B, shown on the left side of Figure 1.

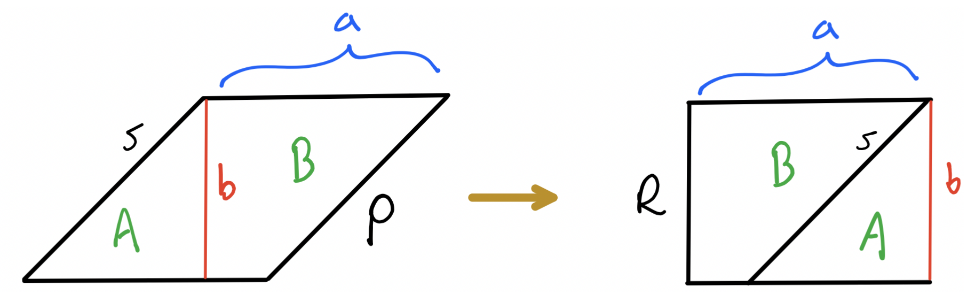

FIGURE 1

Notice that A and B intersect only along our drawn edge. Therefore, by Axiom A2, we have

$$\text{Area}(P) = \text{Area}(A) + \text{Area}(B).$$

Translate the right triangle A so that the hypotenuse s of A coincides with the edge of P originally opposite to s. Now A and B together form a rectangle R, and again they intersect only at one edge, s, shown on the right side of Figure 1. So Axiom A2 again gives us

$$\text{Area}(R) = \text{Area}(A) + \text{Area}(B),$$

which, along with our first equality, tells us $\text{Area}(P) = \text{Area}(R)$.

The sides of R are have lengths a and b by our construction, so by Axiom A1, we have $\text{Area}(R) = a \cdot b$. Combined with our previous result, we find $\text{Area}(P) = a \cdot b$, proving the claim.

Step 2 Let T be a triangle with base length a and height h. We want to prove that T has area $(a \cdot h)/2$. Take another copy of T, labeled T'. Rotate T' a full 180°, and then translate T' so that it shares a single non-base edge with T. Together, T and T' form a parallelogram Q, shown in Figure 2.

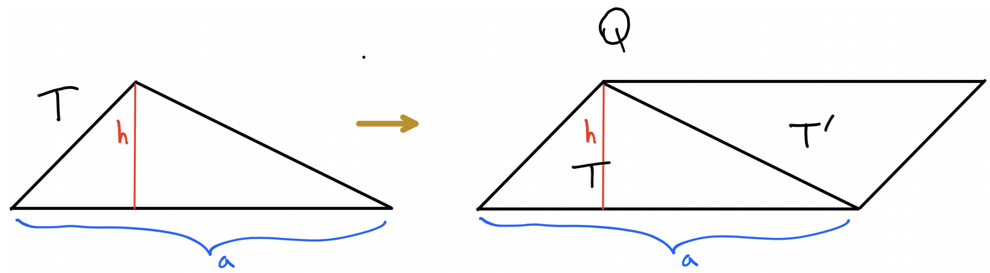

FIGURE 2

Axiom A2 then gives us

$$\text{Area}(Q) = \text{Area}(T) + \text{Area}(T').$$

Since T and T' are congruent, we have $\text{Area}(T') = \text{Area}(T)$, giving

$$\begin{aligned}\text{Area}(Q) &= \text{Area}(T) + \text{Area}(T')\\ &= \text{Area}(T) + \text{Area}(T)\\ &= 2 \cdot \text{Area}(T).\end{aligned}$$

By our construction, the parallelogram Q has base a and height h, so the proven claim in Step 1 tells us that Q has area $a \cdot h$. Putting everything together, we have

$$a \cdot h = \text{Area}(Q) = 2 \cdot \text{Area}(T),$$

from which we conclude that our original triangle T has area $(a \cdot h)/2$.

Problem 6. (20 pts) Following the Axioms in Problem 5, show that a trapezoid with height $h \in \mathbb{R}$ and two horizontal basis of length $b_1, b_2 \in \mathbb{R}$ has area $h \cdot (b_1 + b_2)/2$.

Solution. Let S be a trapezoid with height h and horizontal base lengths b_1 and b_2. If the remaining sides of S are not vertical, then all four vertices of S have distinct coordinates in the horizontal direction. Take the two vertices which are neither the leftmost nor the rightmost vertices of S (depending on the sign of the slopes of the non-horizontal sides of S, each of these vertices may be on the top or the bottom of S), and draw vertical line segments through S at these locations, as shown in Figure 3.

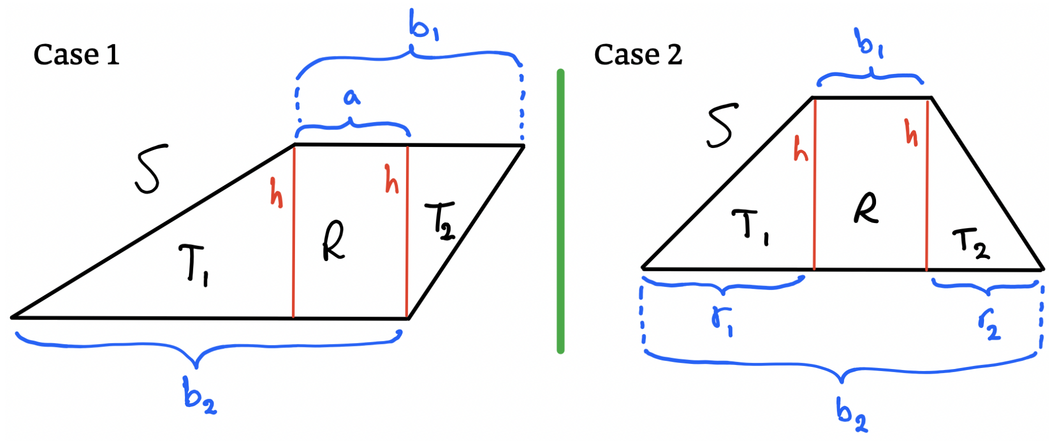

FIGURE 3

We have now split S into two triangles, T_1 and T_2, and a rectangle R, and by Axiom A2 we have

$$\text{Area}(S) = \text{Area}(T_1) + \text{Area}(T_2) + \text{Area}(R).$$

There are two main cases to consider:

Case 1: The non-horizontal sides of S have slopes of the same sign. This case is shown on the left of Figure 3. Referring to the labels in the diagram, we have

$$\begin{aligned}\text{Area}(S) &= \text{Area}(T_1) + \text{Area}(T_2) + \text{Area}(R) \\ &= \frac{(b_2 - a)\cdot h}{2} + \frac{(b_1 - a)\cdot h}{2} + a \cdot h \\ &= \frac{(b_1 + b_2)\cdot h}{2}.\end{aligned}$$

In the second line, we used the result of Problem 5 to calculate the areas of T_1 and T_2, along with Axiom A1 to calculate the area of R.

Case 2: The non-horizontal sides of S have slopes of opposite sign. This case is shown on the right of Figure 3. Referring to the labels in the diagram, we have

$$\begin{aligned}\text{Area}(S) &= \text{Area}(T_1) + \text{Area}(T_2) + \text{Area}(R) \\ &= \frac{r_1 \cdot h}{2} + \frac{r_2 \cdot h}{2} + b_1 \cdot h \\ &= \frac{(r_1 + r_2)\cdot h}{2} + b_1 \cdot h \\ &= \frac{(b_2 - b_1)\cdot h}{2} + b_1 \cdot h \\ &= \frac{(b_1 + b_2)\cdot h}{2}\end{aligned}$$

In the second line, we used the result of Problem 5 to calculate the areas of T_1 and T_2, along with Axiom A1 to calculate the area of R. In the fourth line, we used the fact that $b_2 = r_2 + b_1 + r_1$, which can be seen from the diagram.

In the remaining cases where S has one (respectively, two) vertical sides, then we'll have only one (respectively, zero) triangles. The calculations are then similar to those of Case 2, with $r_1 = 0$ and/or $r_2 = 0$.

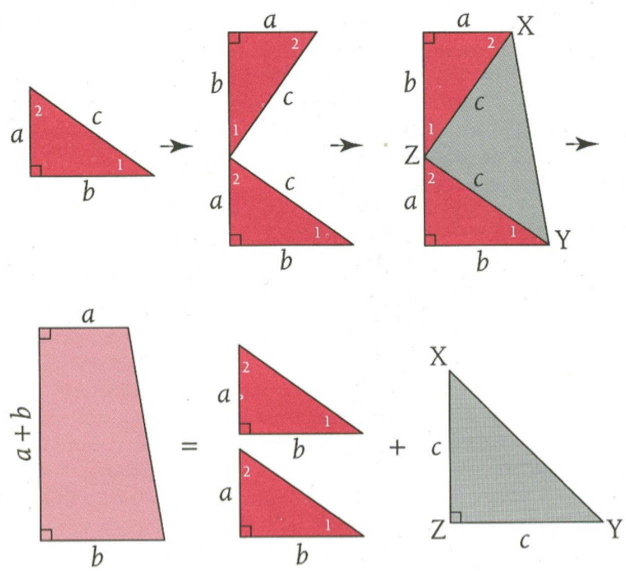

FIGURE 4. Setup for the proof of the Phytagorean Theorem as describe in Problem 7.

Problem 7. (20 pts) Let us prove the Phytagorean Theorem. Let T be a triangle with sides of length $a, b, c \in \mathbb{R}$ such that the interior angle between the a-side and the b-side is 90 degrees (a right angle). An example of such a triangle T is depicted in the upper-left corner of Figure 4. Your task is to show that

$$a^2 + b^2 = c^2.$$

This is called the Phytagorean Theorem. Here are the two steps that you might want to follow:

Step 1. (5pts) Construct the trapezoid of height $(a+b) \in \mathbb{R}$ and basis of length $a, b \in \mathbb{R}$, as in the lower left corner of Figure 4. Use the formula from Problem 6 to show that its area is $(a+b)(a+b)/2$.

Step 2. (10pts) Use the decomposition of this trapezoid as a union of triangles to show that the area of this trapezoid is also

$$2 \cdot (ab)/2 + c^2/2.$$

Step 3. (5pts) Using the Axioms A1 and A2, prove that the two areas that you have computed above must be equal and deduce that

$$a^2 + b^2 = c^2.$$

Fun fact: This proof of the Phytagorean Theorem was published in 1876 in the *New-England Journal of Education* by James A. Garfield, the 20th President of the United States. It is different from Phytagoras' proof, dating back to 500BC.

Solution. *Step 1*: Arrange T as in the upper left image in Figure 4. Make a copy of T, labeled T', and arrange T' as in the next image. That is, rotate T' $90°$ clockwise and translate it so that its leg b extends the leg a of T. Note that leg a of T' is parallel to leg b of T because the perimeter between them includes two right angles. Connecting the remaining vertices which have acute angles, we have the desired trapezoid R. By construction, R has height $a+b$ and base lengths a and b. It follows from Problem 6 that

$$\text{Area}(R) = \frac{(a+b)(a+b)}{2} = \frac{a^2+b^2}{2} + ab.$$

Step 2: By referencing the vertices indicated in the upper left corner of Figure 4, it is clear that R is the union of three triangles which intersect only on their edges, triangles T and T', and a gray triangle T''. Referring to the labels of the angles in the figure, note that angle 1 and angle 2 sum to $90°$ (because they are part of a right triangle), so T'' is a right triangle and hence has base c and height c. Therefore, by Axiom A2 and Problem 5,

$$\begin{aligned}\text{Area}(R) &= \text{Area}(T) + \text{Area}(T') + \text{Area}(T'') \\ &= \frac{a \cdot b}{2} + \frac{a \cdot b}{2} + \frac{c \cdot c}{2} \\ &= \frac{c^2}{2} + ab.\end{aligned}$$

Step 3: Comparing our two expressions for Area(R) found above, we conclude that $(a^2+b^2)/2 + ab = c^2/2 + ab$, which simplifies to $a^2+b^2=c^2$.

Problem 8. Consider the following sums:

$$S_0 = 0,$$
$$S_1 = \frac{1}{2},$$
$$S_2 = \frac{1}{2} + \frac{1}{4},$$
$$S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8},$$
$$S_4 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16},$$
$$S_5 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32},$$
$$S_6 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64},$$

where each sum S_n is obtained from the previous one by adding the fraction $1/2^n$.

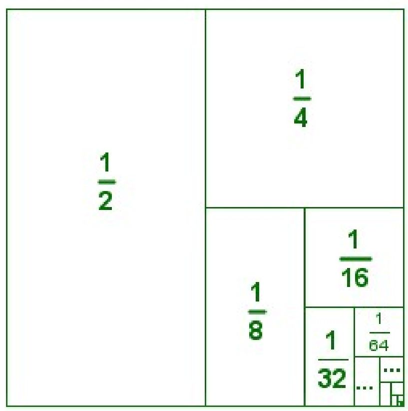

FIGURE 5. This is the Hint for Problem 8.

a. Evaluate the sums $S_1, S_2, \ldots, S_6, S_7, S_8$ and S_9.
   What do you observe when you compute these sums ?

b. Make a prediction of the approximate value of S_{100}. How about S_{10000} ?

Hint: Look at Figure 5.

Solution.

(a) For each n, you should find $S_n = (2^n - 1)/2^n = 1 - 2^{-n}$. Note that $S_n = S_{n-1} + 2^{-n}$.

(b) We predict that $S_{100} = 1 - 2^{-100} =$ and $S^{1000} = 1 - 2^{-1000}$.

Problem 9. (Optional) Explore the world from a scientific lens – on Campus, at home, wherever you are – and try to describe mathematically something you see and like. Whatever it is, keep it simple. Examples of things I like are rainbows, doors, how a baskteball spins, European stock prices or why you see a little cusp in the coffee mug when the light reflects on the surface of the coffee as in Figure 6. The following three steps might be helpful:

(1) Formulate a precise statement about the object of study,

(2) Phrase the statement in mathematical terms,

(3) Prove of disprove your statement.

FIGURE 6. This cusp appears due to the way rays bounce of the surface.