SOLUTIONS TO PROBLEM SET 3

MAT 108

Abstract. These are the solutions to Problem Set 3 for MAT 108 in the Fall Quarter 2024. The problems were due Friday Oct 18.

Problem 1. Consider a play-off tournament with $2n$ players. In the first round of this tournament teams are paired-up and each team plays a unique game. Let a_n be the number of possible pairings for this first round. Describe the first 10 terms of the sequence (a_n), $n \in \mathbb{N}$, and find the recursion that a_n satisfies.

Solution. Given $2n$ players, there are $2n-1$ ways to pair someone with person 1. We are then left with $2n-2$ people so the next person has $2n-3$ possible choices. If we keep repeating this, then we see that there are a total of $(2n-1)\cdot(2n-3)\cdots 3\cdot 1$ pairings. The recursive formula is then

$$a_{2n} = (2n-1)a_{2n-2}.$$

where a_{2n} represents the number of ways to pair $2n$ players. Note if $n = 1$, then there is one way to pair $2(1) = 2$ players. Using this fact, we can compute the first ten terms of the sequence using our recursive formula above.

Problem 2. (10+10 points) Consider a unit square divided into four squares, of equal area and sides. From this division, remove the square in the upper right part, leaving the initial unit square with a 1/4 of its area removed. See the second piece in Figure 1.

Now, repeat this process with the remaining three squares which were part of the initial subdivision and have not been removed. You will obtain the third piece in Figure 1. Keep iterating this process, and let a_n be the number of filled squares in the nth step.

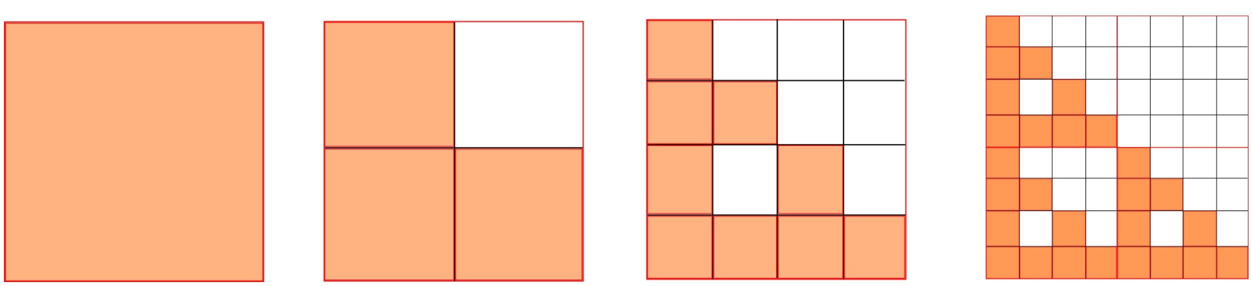

Figure 1. The sequences of squares in Problem 2 for $n = 1, 2, 3$ and 4.

The first step is to consider the square, so $a_1 = 1$. Figure 1 depicts the cases a_1, a_2, a_3 and a_4, and we see that the sequence thus starts with

$$(a_1, a_2, a_3, a_4, \ldots) = (1, 3, 9, 27, \ldots).$$

(a) Find the recursion satisfied by a_n. Given a closed formula for a_n.

(b) Let A_n the area covered by the orange squares, i.e. the squares which have not been removed, in the nth step. So $A_1 = 1$, $A_2 = 3/4$ and so on. Find the recursion satisfied by A_n and give a closed formula for A_n.

Solution. For a fixed number $c \in \mathbb{R}$, the recursion $r_{n+1} = c \cdot r_n$ with initial value $r_1 = a$ has solution r_n given by

$$r_n = a \cdot c^{n-1}.$$

To prove it, notice that this formula gives us $r_1 = a \cdot c^0 = a$ and satisfies

$$r_{n+1} = a \cdot c^{(n+1)-1} = a \cdot c^n = a \cdot c \cdot c^{n-1} = c \cdot (a \cdot c^{n-1}) = c \cdot r_n,$$

as desired

(a) Given the n^{th} diagram in this sequence, each orange square will be replaced in the next iteration by exactly 3 orange squares in the $(n+1)^{\text{th}}$ diagram. Therefore, our recursion is

$$a_{n+1} = 3 \cdot a_n,$$

with initial value $a_1 = 3$. By the general discussion above, a closed formula for a_n is

$$a_n = 3^{n+1}.$$

(b) Let b_n be the number of squares in the n^{th} diagram. Then $A_n = a_n/b_n$. Each step separates a given square into a 2×2 sub-grid, so the sequence b_n satisfies

$$b_{n+1} = 4 \cdot b_{n+1}$$

with initial value $b_1 = 1$. Therefore, the area covered by the orange squares satisfies the recursion

$$A_{n+1} = \frac{a_{n+1}}{b_{n+1}} = \frac{3 \cdot a_n}{4 \cdot b_n} = \frac{3}{4} \cdot A_n$$

with initial value $A_1 = \frac{a_1}{b_1} = \frac{1}{1} = 1$. By the general discussion above, a closed formula for A_n is

$$A_n = \left(\frac{3}{4}\right)^{n-1}.$$

Problem 3. (10+10 points) Let $n, k \in \mathbb{N}$ be natural numbers with $k \leq n$.

(a) Show that the following formula holds

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}.$$

This is Corollary 4.20 in the Textbook, you can either give a proof by induction, direct computation or by combinatorial interpretation.

(b) Show that the following formula holds

$$\binom{n}{k} = \binom{n}{n-k},$$

either by induction, direct computation or by combinatorial interpretation.

Solution.

(a) We give a combinatorial proof. Let the left hand side count the number of ways to choose k people from a class having n students and 1 professor. There are $\binom{n}{k}$ choices that do not include the professor and $\binom{n}{k-1}$ choices that do include the professor. Note we are only selecting $k-1$ students for the latter option because we already chose the professor as our first person. The result then follows.

(b) We prove this using direct computation.

$$\binom{n}{k} = \frac{n!}{(n-k)!k!} = \frac{n!}{(n-k)!(n-(n-k))!} = \binom{n}{n-k}$$

Problem 4. (5+5+5+5 points) Let $a, b \in \mathbb{N}$ be natural numbers.

(a) What is the coefficient of a^5b^6 in the expansion of $(a+b)^{11}$?

(b) Show that the following formula holds

$$\sum_{k=0}^{k=n} (-1)^k \binom{n}{k} = 0.$$

(c) Show that the following formula holds

$$\sum_{k=0}^{k=n} \binom{n}{k} = 2^n.$$

(d) Show that a set with n elements has exactly 2^n distinct subsets.

Hint: For Part (d), interpret the equality in (c) combinatorially.

Solution.

(a) The binomial theorem tells us that

$$(a+b)^{11} = \sum_{k=0}^{11} \binom{11}{k} a^k b^{11-k}.$$

The a^5b^6 term occurs for $k=5$, so the coefficient in front of a^5b^6 is

$$\binom{11}{5} = 462.$$

(b) Consider the binomial expansion

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}.$$

Evaluating at $a=-1$ and $b=1$, we have

$$((-1)+1)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k \cdot 1^{n-k}.$$

Using $((-1)+1)^n = 0^n = 0$ and $1^{n-k} = 1$ for any k, we get the desired equation.

(c) Now take $a = b = 1$ in the binomial expansion. We arrive at

$$(1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^k \cdot 1^{n-k},$$

which simplifies to the desired equation.

(d) Let X be a set with n elements. Specifying a subset A of X (which may be empty, all of X, or anything in between) is the same thing as *choosing* some elements of X to be in our set A. Specifically, if A has exactly $0 \leq k \leq n$ elements, then specifying A is the same thing as choosing k elements from our n element set X. There are $\binom{n}{k}$ possible ways to make this selection, so there are $\binom{n}{k}$ subsets of X which have exactly k elements.

The total number of subsets of X is the number of subsets of X with 0 elements, plus the number of subsets of X with 1 element, plus the number of subsets of X with 2 elements, and on and on, up to the number of subsets of X with all n elements. By the above discussion, these numbers are $\binom{n}{0}$, $\binom{n}{1}$, $\binom{n}{2}$,..., $\binom{n}{n}$, so the total number of subsets of X is

$$\sum_{k=0}^{k=n} \binom{n}{k}.$$

By Part (c), this is 2^n.

Note: There is a way to prove this result without using Part (c). Specifying a subset $A \subseteq X$ is the same thing as looking at each element $x \in X$ individually, and deciding whether x *is* or *is not* in A. Therefore, specifying a subset A is the same as making n independent *binary* choices, and there are 2^n ways to do this.

Since this alternative proof did not use Part (c), it can be combined with the first proof to give an alternative proof of Part (c).

Problem 5. (20 points) Let us consider parenthesis, as used in a sentence for the written English language. The only rule for a single parenthesis is that it must open, we write "(" for that, and it must close, in which case we write ")".

If we use a parenthesis inside another parenthesis, we must make sure that we close the parenthesis inside first, before closing the external parenthesis. Otherwise it is not correctly written. For instance, ()() is correct, but)(() is not correct. Let P_n be the number of ways in which n parenthesis can be written correctly. The start of the sequence is

$$(P_0, P_1, P_2, P_3, \ldots) = (1, 1, 2, 5, \ldots),$$

corresponding to no parenthesis, which gives $P_0 = 1$, the unique parenthesis (), which gives $P_1 = 1$, $P_2 = 2$ because we can write ()() and (()) and finally there are five ways, $P_3 = 5$, of correctly writing three parenthesis:

$$()()(), ()(()), (())(), (()()), ((())).$$

Show that P_n satisfy the following recursion

$$P_{n+1} = \sum_{k=0}^{n} P_k P_{n-k}.$$

Solution. Suppose we want to correctly write $n+1$ parentheses. To prove that P_{n+1} satisfies the recursion above, it suffices to construct a bijection from P_{n+1} to $\bigcup_{k=0}^{n} P_k \times P_{n-k}$. Let s be our desired string. Let's start with our first parenthesis "(." We know this must be closed off so write $s = (a)b$ where a and b are unique strings with n_a and n_b correctly written parentheses, respectively. Since there are $n+1$ parentheses and we have one represented by the red parenthesis, it follows that $n_a + n_b$ is the number of ways to correctly write $(n+1) - 1 = n$ parentheses. Hence, our desired map sends our string s to the pair $[a, b]$. Suppose $n_a = k$ and $n_b = n - k$. We then see that there are $P_k P_{n-k}$ ways to correctly write $s = (a)b$. Therefore, we have $P_{n+1} = \sum_{k=0}^{n} P_k P_{n-k}$.

As an example, if $n = 3$, then we have

$$P_{2+1} = P_0 P_2 + P_1 P_1 + P_2 P_0,$$

which corresponds to

$a = 0,\ b = 2$	$a = 1,\ b = 1$	$a = 2,\ b = 0.$
()()(), ()(())	(())()	(()()), ((()))

Problem 6. (20 points) A triangulation of a polygon in the plane is a subdivision of the polygon into triangular pieces by adding edges that go between vertices. Let C_n be the number of triangulations of a polygon in the plane with $n+2$ sides. Find a recursion for C_{n+1} in terms of the previous elements in the sequence.

Example: C_1 are triangulations of the triangle, and there is only one, so $C_1 = 1$. This correspond to the first row in Figure 2. The next term C_2 counts the number of triangulations of a square, we can add either of the two diagonals to divide into triangles, so there are two ways and $C_2 = 2$. This corresponds to the second row in Figure 2. Figure 2 depicts the fact that $C_3 = 5$ and $C_4 = 14$. The sequence thus starts as

$$(C_1, C_2, C_3, C_4, \ldots) = (1, 2, 5, 14, \ldots)$$

Hint: Establish a bijection between the set of triangulations and the ways of ordering parenthesis. This will allow you to conclude that the C_n satisfy the same recursion as in numbers in Problem 5.

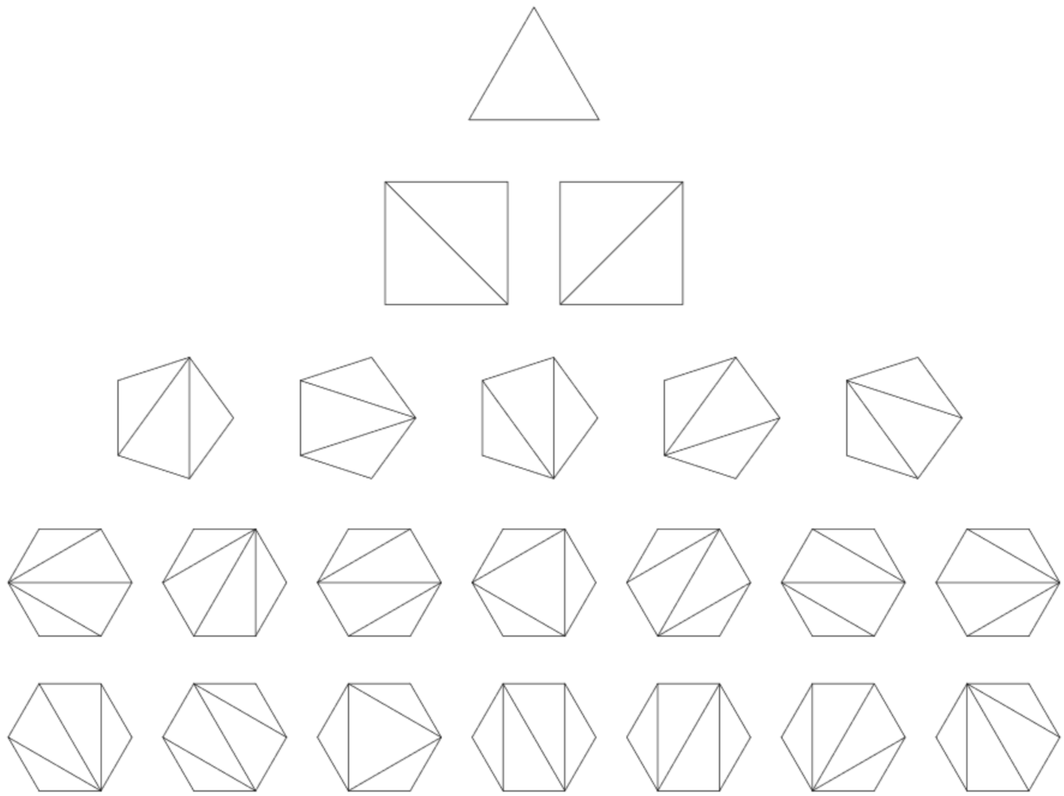

FIGURE 2. Triangulations of polygons of $n+2$ sides for $n=1,2,3$ and 4, appearing in Problem 6. The first row is $n=1$, triangulations of a triangle, the second row is $n=2$, which are triangulations of a square. The third row is the case $n=3$, which are triangulations of a pentagon and the fourth and fifth rows depict the $n=4$ case, triangulating an hexagon.

Solution.To standardize the enumeration of triangulations of a polygon, we will consider any $(n+2)$-gon to have a horizontal base edge L with vertex x on the left and vertex y on the right. Label the remaining vertices $0,1,\dots,n-1$, starting from the vertex clockwise from x and increasing clockwise (so vertex $n-1$ is adjacent to y).

By convention, define $C_0 = 1$, which we will think of as counting the single "empty" triangulation on a single edge (which you can think of as a "2-gon" whose sides have been collapsed onto each other). We will prove the identity

$$C_n = \sum_{k=0}^{n} C_k C_{n-k} \tag{0.1}$$

using a bijection similar to that in the solution to Problem 5. Let B_n be the set of all triangulations of an $(n+2)$-gon, and set $B_0 = \{\emptyset\}$ to be the set with a single element, the empty triangulation on one edge. Notice that $|B_n| = C_n$ (where $|B_n|$ refers to the *cardinality* of B_n, i.e. the number of elements of the set B_n), so to prove 0.1 it suffices to provide a bijection

$$\varphi : B_n \longrightarrow \bigcup_{k=0}^{n} B_k \times B_{n-k}$$

To define φ on a triangulated $(n+3)$-gon $W \in B_n$, we must associate to W an ordered pair $(X, Y) \in B_k \times B_{n-k}$. The base edge L of W is the edge of a unique triangle T with vertices x and y. Let k be the label of the remaining vertex of T, and note that $k \in \{0, 1, \ldots, n\}$. Deleting the edge L, we are left with two triangulated polygons which intersect at the single vertex k. One of these polygons–call it X–contains the vertices $x, 0, 1, \ldots, k$, and the other–call it Y–contains the vertices $k, k+1, \ldots, n, y$.

Therefore, X is a $(k+2)$-gon and Y is a $(n-k+2)$-gon. Separating the triangulated polygons X and Y, orient them both so that the edges belonging to the deleted triangle T are horizontal. Now (X, Y) is a uniquely-defined element of $B_k \times B_{n-k}$, as desired. We then set $\varphi(W) = (X, Y)$. Note that if $k = 0$, then $X = \emptyset$ is the empty triangulation on a single edge, the unique element of B_0. Similarly, if $k = n$, then $Y = \emptyset$ instead.

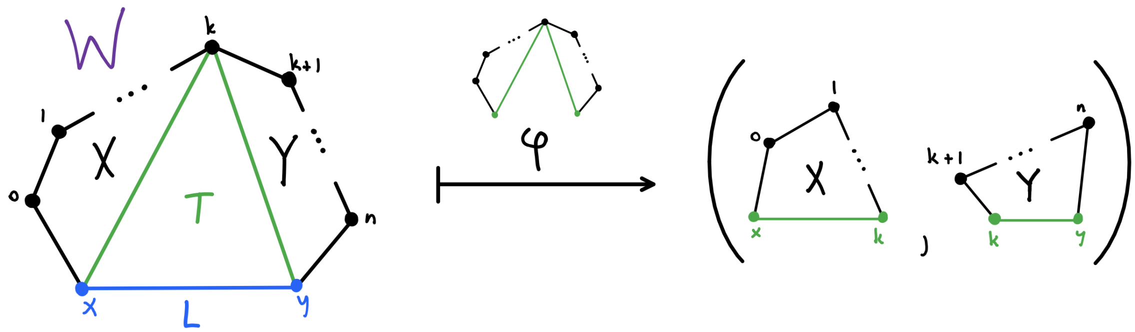

FIGURE 3

To prove that φ is a bijection, we will construct an inverse function

$$\psi : \bigcup_{k=0}^{n} B_k \times B_{n-k} \longrightarrow B_n.$$

Let (X', Y') be an ordered pair in $B_k \times B_{n-k}$ for some $k \in \{0, 1, \ldots, n\}$. Label the vertices of both in the standardized manner. Attach X' to Y' by gluing the y vertex of X' to the x vertex of Y'. Now draw a new edge L which connects the x vertex of $'X$ to the y vertex of Y'. Recall that X' had k triangles, Y had $n-k$ triangles, and L created a single new triangle with the original base edges of X' and Y'.

Since we haven't destroyed any triangles in the process, we now have $n+1$ triangles triangulating the newly-constructed $(n+3)$-gon which we call W'. Orienting W' so that the edge L is horizontal, we have a uniquely-defined element of B_n, so we set $\psi(X', Y') = W'$. Note that if $k = 0$, we are gluing to Y' the single edge X', and then completing the triangle. Similarly, if $k = n$, we are gluing to X' the single edge Y'.

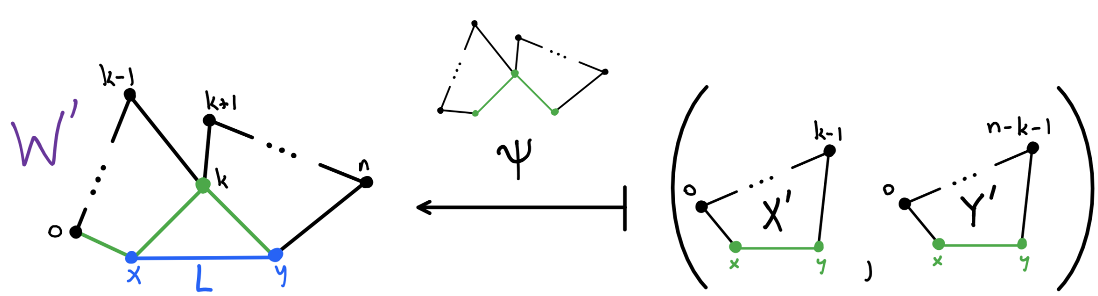

FIGURE 4

From the diagrams above, it is clear that the "building" operation ψ is exactly the inverse of the "decomposing" operation φ. That is, for any $W \in B_n$, we have $\psi(\varphi(W)) = W$, and for any $(X', Y') \in \bigcup_{k=0}^{n} B_k \times B_{n-k}$, we have $\varphi(\psi(X', Y'))$. Therefore, φ (and ψ) is a bijection, as desired.

Problem 7. The United States Postal Service in Davis delivers mail everyday to the ten adjacent Campus Buildings located along Shields Avenue. Suppose that two adjacent building never receive mail on the same day, but no more than two houses in a row get no mail. How many different possibilities of mail delivery are there ?

Example: Imagine that there were only two buildings, instead of ten. Then there would be *three* options for the mail being delivered:

(i) Neither building gets mail.

(ii) Only the first building receives mail.

(iii) Only the second building receives mail.

This tells us that there are *three* mail delivery possibilities for three buildings. □

Hint: Consider the possibilities in terms of the first two buildings, and build a recursion that recursively gives you the answer for *any given* number of building. Then evaluate your recursion for the case of ten buildings.

Solution. Define $f(n)$ to be the number of ways to distribute mail to n buildings. We break this problem up into two cases.

(i) Assume the first building receives mail. Then, since no two adjacent buildings receive mail on the same day, the second building cannot receive any mail. If the third building receives mail, then we know the fourth building cannot receive mail. On the other hand, if the fourth building receives mail, the third building cannot receive mail. Thus, there are $f(n-3)$ ways to deliver mail if the first building receives mail.

(ii) Assume the first building does not receive mail. If the second building does not receive mail, then the third building must receive mail and vice versa. We use a similar reasoning to conclude there are $f(n-2)$ ways to deliver mail if the first building does not receive mail.

Therefore, our recursive equation is

$$f(n) = f(n-2) + f(n-3).$$

We then have the following.

$$\begin{aligned}
f(1) &= 2 \qquad & f(4) &= f(2) + f(1) = 5 \\
f(2) &= 3 & f(5) &= f(3) + f(2) = 7 \\
f(3) &= 4 & f(6) &= f(4) + f(3) = 9 \\
& & f(7) &= f(5) + f(4) = 12 \\
& & f(8) &= f(6) + f(5) = 16 \\
& & f(9) &= f(7) + f(6) = 21 \\
& & f(10) &= f(8) + f(7) = 28
\end{aligned}$$

Problem 8. Consider a regular hexagon in the plane, as depicted in the upper left of Figure 5. It has six vertices, which are depicted in red thick dots.

Let A and B be two adjacent side of the hexagon, and insert a smaller copy of the same hexagon such that the A and B side of the copy are strictly included in the A and B sides of the initial hexagon. This is depicted in the upper right of Figure 5. Now the A and B sides of the initial copy have each been divided by a red dot (from the smaller hexagon), so introduced a red dot in the middle of all its remaining edges.

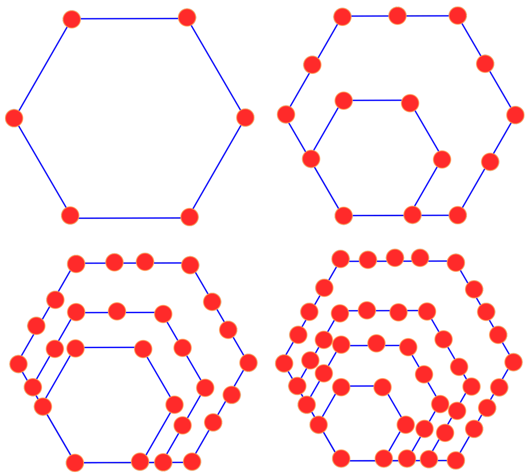

FIGURE 5. The hexagons and points in Problem 8 for $n = 1, 2, 3$ and 4.

Iterate the above process, introducing an smaller copy of the hexagon inside the previously inserted copies, and adding red dots in the edges so that each hexagon has exactly the same number of red dots per side. Let a_n be the number of vertices of the resulting polygonal figure in the plane. The first four cases are depicted in Figure 5, so the sequence begins with

$$(a_1, a_2, a_3, a_4, \ldots) = (6, 15, 28, 45, \ldots).$$

(a) Find a recursion for a_n.

(b) Find a closed formula for a_n.

Solution.

(a) Finding a recursion for a_n amounts to answering the question: *given the diagram with n hexagons, how many vertices are added to arrive at the next diagram with $n+1$ hexagons?*

To arrive at the next diagram, we insert a small hexagon inside all previously drawn hexagons from the n^{th} diagram. In particular, we add 3 vertices on the newly-drawn edges (the vertices connecting the four new edges that we drew). We also add a single vertex to each of the edges A and B. Together, these add 5 new vertices.

Finally, for each of the n hexagons from the n^{th} diagram (i.e. all but the newest hexagon), we add a vertex to each of the four sides not shared with any other hexagon. Together, these add $4n$ vertices. Therefore,

$$a_{n+1} = a_n + 4n + 5 \tag{0.2}$$

is our recursion for a_{n+1} in terms of a_n.

Note: We can view the $(n+1)^{\text{th}}$ diagram in a different, but equivalent way. Rather than taking the n^{th} diagram, adding an extra hexagon on the inside, and then adding vertices to each edge, we can take the n^{th} diagram and add one hexagon *on the outside*. This new hexagon has 3 new corners (which get vertices) and n vertices are added on each of the 4 new sides drawn. Finally, one vertex must be added to each of the sides A and B. This gives the same recursion as in Equation (0.2), but the process has the benefit that we only have to modify two of our previously-drawn hexagons.

(b) We make the claim

$$a_n = (2n+1)(n+1).$$

To prove that this is correct, we verify that it obeys the recursion found in Part (a). First, $a_1 = (2 \cdot 1 + 1)(1+1) = 6$, as desired. Next

$$\begin{aligned} a_{n+1} - a_n &= (2(n+1)+1)((n+1)+1) - (2n+1)(n+1) \\ &= (2n^2 + 7n + 6) - (2n^2 + 3n + 1) \\ &= 4n + 5, \end{aligned}$$

which implies Equation (0.2), as desired. Therefore, our guess $(2n+1)(n+1)$ is a closed formula for a_n, as it satisfies both the initial value a_1 and the recursion relation (0.2).

Note: This is not part of the proof, but here is an explanation for where the guess $a_n = (2n+1)(n+1)$ might come from. Iterating our recursion in Equation (0.2), we are led to the following calculation:

$$\begin{aligned} a_n &= 4(n-1) + 5 + a_{n-1} \\ &= 4(n-1) + 5 + (4(n-2) + 5 + a_{n-2}) \\ &= 4((n-1) + (n-2)) + 5 \cdot 2 + (4(n-3) + 5 + a_{n-3}) \\ &= 4((n-1) + (n-2) + (n-3)) + 5 \cdot 3 + (4(n-4) + 5 + a_{n-4}) \\ &\ \ \vdots \\ &= 4((n-1) + (n-2) + \cdots + 3 + 2) + 5(n-2) + (4 \cdot 1 + 5 + a_1) \\ &= 4((n-1) + (n-2) + \cdots + 3 + 2 + 1) + 5(n-1) + 6 \\ &= 2n(n-1) + 5(n-1) + 6 \\ &= (2n+1)(n+1). \end{aligned}$$

Notice that we used the identity $\sum_{j=1}^{k} j = \frac{k(k+1)}{2}$ in the fourth line from the bottom, setting $k = n - 1$.