University of California Davis **Name (Print):** ______________________

Differential Equations MAT 108 **Student ID (Print):** ______________________

Second Midterm Exam **November 25 2024**

Time Limit: 50 Minutes

This examination document contains 8 pages, including this cover page, and 4 problems. You must verify whether there any pages missing, in which case you should let the instructor know. **Fill in** all the requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

The maximal grade is 100 points. Bonus points will add to your score up to such a grade of 100 points. You may *not* use your books, notes, the Internet, or any calculator.

You are required to show your work on each problem on this exam. The following rules apply:

(A) **If you use a lemma, proposition or theorem which we have seen in the class or in the book, you must indicate this** and explain why the theorem may be applied.

(B) **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive little credit.

(C) **(Justify your answer: mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive little credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

(D) If you need more space, use the back of the pages; clearly indicate when you did this.

Problem	Points	Score
1	25	
2	25	
3	25	
4	25	
Total:	100	

Do not write in the table to the right.

1. (25 points) Let $X = \left\{x \in \mathbb{R} : 3 \leq x^2 < 5, 0 < x\right\} = [\sqrt{3}, \sqrt{5})$.

   (a) (10 points) Show that $\inf(X) = \sqrt{3}$.

   (b) (10 points) Show that $\sup(X) = \sqrt{5}$.

(c) (5 points) Prove that $\sup(X) = \sqrt{5} \notin \mathbb{Q}$, i.e. it is not a rational number.

(d) (Bonus 5 points) Is the set X countable or uncountable? (Justify your answer.)

2. (25 points) Solve the following two parts:

   (a) (15 points) Prove that

$$x_n = \frac{5n^2 + 3n + 1}{n^2 + 2}$$

   is a convergent sequence with limit $\lim_{n\to\infty} x_n = 5$.

   (b) (10 points) Show that the sequence $y_n = (-1)^n$ does *not* converge.

3. (25 points) Consider the recursive sequence (x_n), $n \in \mathbb{N}$, given by

$$x_{n+1} = \sqrt[3]{7x_n + 6}, \quad x_1 = 3.5.$$

(a) (10 points) Show that (x_n) is bounded below by 3, i.e. that $3 \leq x_n$ for all $n \in \mathbb{N}$.

(b) (5 points) Prove that (x_n) is decreasing, i.e. that $x_{n+1} \leq x_n$ for all $n \in \mathbb{N}$.

(c) (5 points) Show that (x_n) converges.

(d) (5 points) Prove that the limit is $\lim_{n\to\infty} x_n = 3$.

4. (25 points) Solve the following two problems:

   (a) (15 points) Consider the map

$$f : \mathbb{R} \longrightarrow \mathbb{R}, \quad f(x) = \frac{1}{x^2 + 1}.$$

   Show that f is not injective.

   (b) (10 points) Is f surjective? (Justify your answer.)

(c) (Bonus 5 points) Show that the set

$$X = \{x \in \mathbb{R} : f(x) \in \mathbb{Q}\}$$

is countable.