University of California Davis **Name (Print):** ____________________
Differential Equations MAT 108 **Student ID (Print):** ____________________

Second Midterm Exam **November 25 2024**
Time Limit: 50 Minutes

This examination document contains 8 pages, including this cover page, and 4 problems. You must verify whether there any pages missing, in which case you should let the instructor know. **Fill in** all the requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

The maximal grade is 100 points. Bonus points will add to your score up to such a grade of 100 points. You may *not* use your books, notes, the Internet, or any calculator.

You are required to show your work on each problem on this exam. The following rules apply:

(A) **If you use a lemma, proposition or theorem which we have seen in the class or in the book, you must indicate this** and explain why the theorem may be applied.

(B) **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive little credit.

(C) **(Justify your answer: mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive little credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

(D) If you need more space, use the back of the pages; clearly indicate when you did this.

Do not write in the table to the right.

Problem	Points	Score
1	25	
2	25	
3	25	
4	25	
Total:	100	

1. (25 points) Let $X = \{x \in \mathbb{R} : 3 \leq x^2 < 5, 0 < x\} = [\sqrt{3}, \sqrt{5})$.

   (a) (10 points) Show that $\inf(X) = \sqrt{3}$.

   Solution. By construction, $x \in X$ satisfies $\sqrt{3} \leq x$ and so $\sqrt{3}$ is a lower bound. Since $\sqrt{3} \in X$, it is the greatest lower bound and thus $\inf(X) = \sqrt{3}$ as required.

   (b) (10 points) Show that $\sup(X) = \sqrt{5}$.

   Solution. By construction, $x \in X$ satisfies $x < \sqrt{5}$ and so $\sqrt{5}$ is an upper bound. Let us show that $\sqrt{5}$ is the least upper bound. Indeed, suppose that $u = \sup(X)$ was a smaller upper bound, i.e. $u < \sqrt{5}$. (By (a) we also have $\sqrt{3} < u$.) Then the midpoint $\frac{u+\sqrt{5}}{2}$ would satisfy

   $$u < \frac{u + \sqrt{5}}{2} < \sqrt{5}.$$

   Therefore, the midpoint $\frac{u+\sqrt{5}}{2} \in X$ belong to X: this is a contradiction with the fact that u is an upper bound. Hence it must be that the assumption $u < \sqrt{5}$ is wrong and thus any upper bound u must satisfy $\sqrt{5} \leq u$. Thus $\sqrt{5}$ is the least upper boudn, as required.

(c) (5 points) Prove that $\sup(X) = \sqrt{5} \notin \mathbb{Q}$, i.e. it is not a rational number.

Solution. By contradiction, suppose that $\sqrt{5} = p/q$ for some $p, q \in \mathbb{Z}$ and $gcd(p, q) = 1$. The equality $\sqrt{5} = p/q$ implies that $5q^2 = p^2$ and thus 5 divides p^2. Since 5 is prime, 5 also divides p and therefore 5^2 divides p^2. Given that $p^2 = 5q^2$, this implies that 5^2 divides $5q^2$ and so 5 divides q^2. Again, this implies that 5 divides q and so $gcd(p, q) \neq 1$. Therefore we have reached a contradiction with $gcd(p, q) = 1$ and thus the initial equation $\sqrt{5} = p/q$ cannot hold, so $\sqrt{5}$ is irrational.

(d) (Bonus 5 points) Is the set X countable or uncountable? (Justify your answer.)

Solution. It is uncountable, as it contains the interval $(\sqrt{3}, \sqrt{5})$ which is in bijection with the interval $(0, 1)$ thus uncountable (as it bijects to $\mathbb{R}$, e.g. via $x \mapsto \ln(x)$).

2. (25 points) Solve the following two parts:

(a) (15 points) Prove that

$$x_n = \frac{5n^2 + 3n + 1}{n^2 + 2}$$

is a convergent sequence with limit $\lim_{n\to\infty} x_n = 5$.

Solution. Let $\epsilon > 0$ be given, we want to show that there exists an $N \in \mathbb{N}$ such that

$$\left| \frac{5n^2 + 3n + 1}{n^2 + 2} - 5 \right| < \varepsilon, \quad \forall n \geq N.$$

This inequality is equivalent to

$$\left| \frac{3n - 9}{n^2 + 2} \right| < \varepsilon, \quad \forall n \geq N.$$

Since $\frac{3n-9}{n^2+2} < \frac{3n-9}{n^2}$, this inequality will hold if we can argue that there exists an N' such that

$$\left| \frac{3}{n} - \frac{9}{n^2} \right| = \left| \frac{3n - 9}{n^2} \right| < \varepsilon, \quad \forall n \geq N'.$$

This is itself implied if we can argue that

$$\left| \frac{3}{n} \right| + \left| \frac{9}{n^2} \right| < \varepsilon, \quad \forall n \gg 1.$$

Since $\frac{1}{n^2} \leq \frac{1}{n}$, the above inequality is true for $n \gg 1$ by Proposition 10.4.

(b) (10 points) Show that the sequence $y_n = (-1)^n$ does *not* converge.

Solution. The subsequence $y_{2n} = 1$ is constant and converges to 1. The subsequence $y_{2n+1} = -1$ is constant and converges to -1. If y_n converged, by uniqueness of limits, all subsequences should converge to the same limit. Therefore y_n does not converge since the limit of y_{2n} is different than the limit of y_{2n+1}.

3. (25 points) Consider the recursive sequence (x_n), $n \in \mathbb{N}$, given by

$$x_{n+1} = \sqrt[3]{7x_n + 6}, \quad x_1 = 3.5.$$

(a) (10 points) Show that (x_n) is bounded below by 3, i.e. that $3 \leq x_n$ for all $n \in \mathbb{N}$.

Solution. By induction on n. The base case $n = 1$ holds because $x_1 = 3.5 \geq 3$. For the induction step, assume that $x_n \geq 3$. Then

$$x_{n+1} \geq 3 \iff \sqrt[3]{7x_n + 6} \geq 3 \iff 7x_n + 6 \geq 27 \iff x_n \geq 3.$$

The latter inequality $x_n \geq 3$ is true by induction hypothesis, and therefore so is the first inequality $x_{n+1} \geq 3$, as required.

(b) (5 points) Prove that (x_n) is decreasing, i.e. that $x_{n+1} \leq x_n$ for all $n \in \mathbb{N}$.

Solution. We must show that $x_{n+1} \leq x_n$, i.e. that $\sqrt[3]{7x_n + 6} \leq x_n$. This is equivalent to showing

$$0 < x_n^3 - 7x_n - 6.$$

For that, consider the function $g(x) = x^3 - 7x - 6 = (x + 1)(x + 2)(x - 3)$, which intersects the x-axis at $x = -1, -2, 3$. Therefore $g(x) \geq 0$ if $x \geq 3$ and thus $0 < x_n^3 - 7x_n - 6$, since (a) guarantees that $x_n \geq 3$.

(c) (5 points) Show that (x_n) converges.

Solution. The Monotone Convergence Theorem guarantees that a decreasing sequence bounded below converges. By (a) and (b) above, x_n is a decreasing sequence bounded below, and so it converges by the theorem.

(d) (5 points) Prove that the limit is $\lim_{n\to\infty} x_n = 3$.

Solution. By (c), the sequence converges. Since $x_{n+1} = \sqrt[3]{7x_n + 6}$, uniqueness of limits implies that the limit L must satisfy $L = \sqrt[3]{7L + 6}$, so L is a zero of $g(x) = x^3 - 7x - 6 = (x+1)(x+2)(x-3)$ and thus the possibilities are $L = -1, -2, 3$. Since $x_n \geq 3$ by (a), it must be that $L \geq 3$ and so the only correct choice is $L = 3$.

4. (25 points) Solve the following two problems:

   (a) (15 points) Consider the map

$$f : \mathbb{R} \longrightarrow \mathbb{R}, \quad f(x) = \frac{1}{x^2 + 1}.$$

   Show that f is not injective.

   Solution. It suffices to note that $f(1) = f(-1)$. Since $1 \neq -1$, f is not injective.

   (b) (10 points) Is f surjective? (Justify your answer.)

   Solution. No. Note that $f(x) \geq 0$ and so there are no x such that $f(x) = -1$, for instance. Therefore f is not surjective.

(c) (Bonus 5 points) Show that the set

$$X = \{x \in \mathbb{R} : f(x) \in \mathbb{Q}\}$$

is countable.

Solution. Since $\mathbb{Q}$ is countable and the equation

$$\frac{1}{1+x^2} = q, \quad q \in \mathbb{Q}$$

has at most two solutions, the set X is itself countable as well.