MAT 108: PROBLEM SET 2

DUE TO FRIDAY OCT 11 2024

Abstract. This problem set corresponds to the second week of the course, covering material on proofs by induction and proofs by contradiction. It was posted online on Thursday Oct 3 and is due Friday Oct 11.

Purpose: The goal of this assignment is to practice proofs **by induction** and proofs **by contradiction**. It is important to realize that proofs by induction and by contradiction work in a variety of problems, from statements about divisibility, formulas, problems on combinatorics and theorems in analysis.

Proof by Induction: Let $P(k)$ be a statement that depends on a natural number $k \in \mathbb{N}$. (The most common example is a formula depending on k.) A proof by induction has following two steps:

1. First prove the statement $P(1)$, that is, the statement when you plug $k = 1$.
2. Second, you assume that the statement $P(k)$ is true, and show that then $P(k+1)$ must also be true.

If you successfully perform these two steps, you will have proven the statement $P(k)$ for all $k \in \mathbb{N}$. The fact that this is a *valid* method of proof was explained in class, and you can find the proof in Theorem 2.17 of the textbook.

Proof by Contradiction: The first step is to make the assumption that the conclusion you want to achieve is *false*. The second step is to arrive from there to a statement which you know to be *false*. Since you know this latter statement **is** false, the original assumption that our initial conclusion is false must be wrong. Thus the initial conclusion is true.

Proofs by contradiction were discussed in class and you can also look at Section 2.1 in the Textbook.

Task: Solve Problems 1 through 7 below. The first 2 problems and the last one will not be graded but I trust that you will work on them. **Problems 3 to 6 will be graded**, you only have to submit those. Either of these problems might appear in the exams.

Textbook: We will use "The Art of Proof: Basic Training for Deeper Mathematics" by M. Beck and R. Geoghegan. Please contact me *immediately* if you have not been able to get a copy.

Proofs by Contradiction

Problem 1. Show that there do not exist two integers $n, m \in \mathbb{Z}$ such that $n^4 - 4m = 2$.

Hint: Proof by contradiction, i.e. assume that there exist two integers $n, m \in \mathbb{Z}$ such that $n^4 - 4m = 2$ and reach a contradiction.

Problem 2. A natural number $n \in \mathbb{N}$ which is only divisible by 1 and n is said to be a *prime* number. Prove that there are infinitely many prime numbers.

Hint: Proof by contradiction, i.e. assume that there exist finitely many primes $\{p_1, p_2, \dots p_N\}$, and then try to reach a contradiction. (Clue: Consider the number $P = p_1 \cdot p_2 \cdot \dots \cdot p_N + 1$. Is this a prime ?)

Problem 3. (25 pts) Prove that there are infinitely many prime numbers that have residue 3 when divided by 4. Equivalently, prove that there are infinitely many prime numbers p of the form $p = 4k - 1$ for some natural number $k \in \mathbb{N}$.

For instance, $p = 2$ or $p = 5$ are prime numbers but they are not of the form $p = 4k - 1$ for any $k \in \mathbb{N}$. So not *every* prime number is of the form $p = 4k - 1$, this problem asks you to show that there are infinitely many of them.

Hint: Adapt your proof by contradiction in Problem 2 to this case.

Proofs by Induction

Problem 4. (25=5+10+10 pts) (Proposition 2.18 in Textbook) Prove the following three statements:

(i) For all $k \in \mathbb{N}$, $k^3 + 2k$ is divisible by 3.

(ii) For all $k \in \mathbb{N}$, $k^4 - 6k^3 + 11k^2 - 6k$ is divisible by 4.

(iii) For all $k \in \mathbb{N}$, $k^3 + 5k$ is divisible by 6.

Hint: In the induction step you might want to use the binomial formulas:

$$(n+1)^2 = (n+1)(n+1) = n^2 + 2n + 1,$$
$$(n+1)^3 = (n+1)(n+1)(n+1) = n^3 + 3n^2 + 3n + 1,$$
$$(n+1)^4 = (n+1)(n+1)(n+1)(n+1) = n^4 + 4n^3 + 6n^2 + 4n + 1.$$

Problem 5. (25=10+15 pts) Prove by induction the following two formulas:

(i) For all $k \in \mathbb{N}$, we have

$$1 + 2 + 3 + 4 + \dots + (k-1) + k = \frac{k(k+1)}{2}.$$

The left hand side is the sum of all the natural numbers less equal than k, i.e. from 1 to k, the latter included.

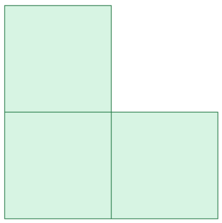
(ii) For all $k \in \mathbb{N}$, we have

$$1^2 + 2^2 + 3^2 + 4^2 + \dots + (k-1)^2 + k^2 = \frac{k(k+1)(2k+1)}{6}.$$

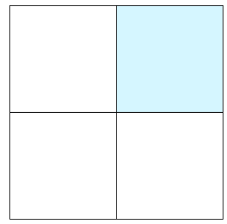
The left hand side is the sum of *the squares* of all the natural numbers less equal than k, i.e. from 1^2 to k^2, the latter included.

Problem 6. (25 pts) Let $k \in \mathbb{N}$ be a natural number. Consider a $2^k \times 2^k$ square board divided into equal square tiles of 1×1 size, like a chess board. (So the $2^k \times 2^k$ board is covered by 2^{2k} tiles.) Remove one tile from the $2^k \times 2^k$ board. Prove **by induction** that the remaining part of the board can be covered with *triomino* pieces, i.e. pieces made of three unit tiles with an L-shape.

I have depicted in Figure 1 the triomino pieces (Left) and an example of the case $k = 1$ (Right), where you can see a board of size $2^1 \times 2^1$ with one tile (the blue one) removed. It is clear in this case, that the board with one tile removed can be covered with triomino pieces, in this case, exactly one triomino piece (covering the three white tiles).



(A) A triomino piece.



(B) The $2^k \times 2^k$ board with one tile removed in the case $k = 1$, where the board is 2×2.

Figure 1. The art of tiling a board with a missing tile with triominos, as presented in Problem 6. The goal is to prove that you can always tile with triominos if a tile is missing in a $2^k \times 2^k$ board.

Problem 7. Let $k \in \mathbb{N}$ be a natural number. Consider k distinct straight lines in the plane. These are infinitely long straight lines, and we assume that no two such lines are parallel and no three such lines every intersect at a single point. Prove that k such lines divide the plane into $(k^2+k+2)/2$ regions.

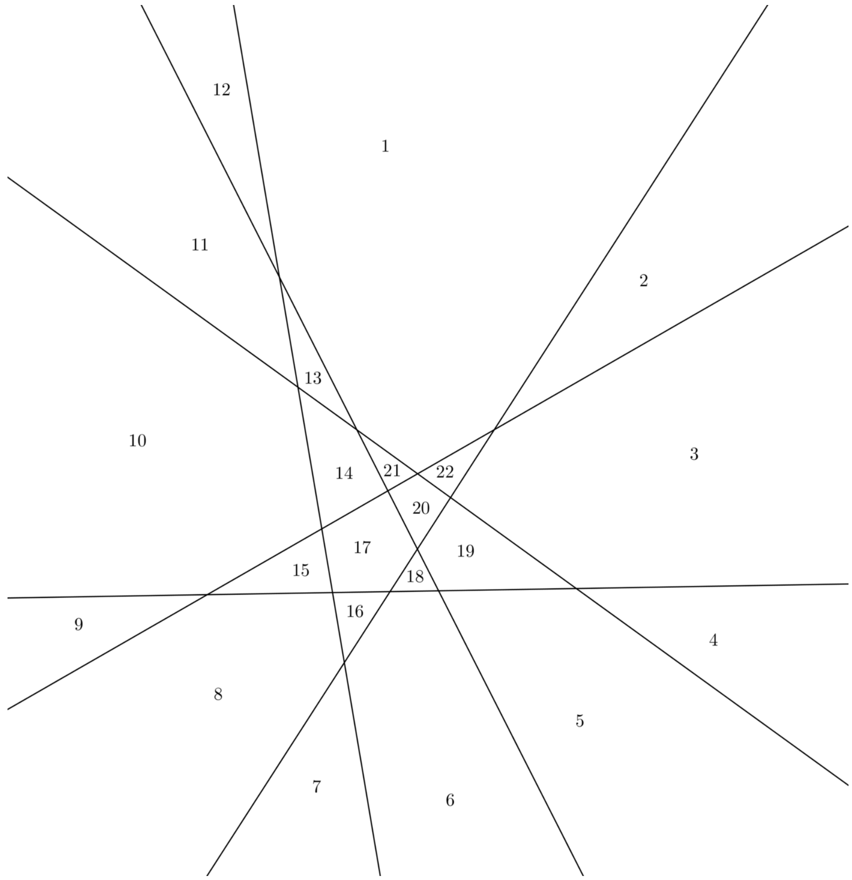


FIGURE 2. Six lines dividing the plane in 22 regions. This is the case $k = 6$ in Problem 7.

Hint: This can be proven by induction, but it is crucial in this problem that you play and experiment with this formula first. It will give you an intuition on how to prove the general case, by adding one line at a time and seeing how new regions appear.

For instance, for one line we have $k = 1$ and one line divides the plane into $(1^2 + 1 + 2)/2 = 2$ regions. By hand, try at least the formula for $k = 2, 3$ and $k = 4$. I have depicted the case $k = 6$ in Figure 2, where the plane is divided into $(k^2 + k + 2)/2 = (6^2 + 6 + 2)/2 = 22$ regions.