MAT 108: PROBLEM SET 3

DUE TO FRIDAY OCT 18 2024

Abstract. This problem set corresponds to the third week of the course, covering material on arguments and proofs with sequences and *recursion*. It was posted online on Thursday Oct 10 and is due Friday Oct 18.

Purpose: The goal of this assignment is to practice problems on **recursion**. This includes the material on the following topics:

(i) Recursive definitions of sequences: this includes arithmetic and geometric sequences, and especially The Sum of Geometric Series (Proposition 4.13),

(ii) Factorial numbers, the binomial coefficient (Theorem 4.19), combinatorial interpretation of factorial numbers and binomial coefficients, and The Binomial Theorem (Theorem 4.21),

(iii) Closed formulas for recursive sequences.

Recursion has been discussed in class, starting Monday Oct 8, and the material is complemented and detailed in Chapter 4 in the Textbook.

Task: Solve Problems 1 through 8 below. The first problem will not be graded but I trust that you will work on it. **Only Problems 2 to 5 will be graded**. Problems 1, 6, 7 and 8 will not be graded and you do not have to submit them, but I encourage you to try them. Either of the these problems might appear in the exams.

Textbook: We will use "The Art of Proof: Basic Training for Deeper Mathematics" by M. Beck and R. Geoghegan. Please contact me *immediately* if you have not been able to get a copy.

Problem 1. Consider a play-off tournament with $2n$ players. In the first round of this tournament teams are paired-up and each team plays a unique game. Let a_n be the number of possible pairings for this first round. Describe the first 10 terms of the sequence (a_n), $n \in \mathbb{N}$, and find the recursion that a_n satisfies.

Problem 2. (10+15 points) Consider a unit square divided into four squares, of equal area and sides. From this division, remove the square in the upper right part, leaving the initial unit square with a 1/4 of its area removed. See the second piece in Figure 1.

Now, repeat this process with the remaining three squares which were part of the initial subdivision and have not been removed. You will obtain the third piece in Figure 1. Keep iterating this process, and let a_n be the number of filled squares in the nth step.

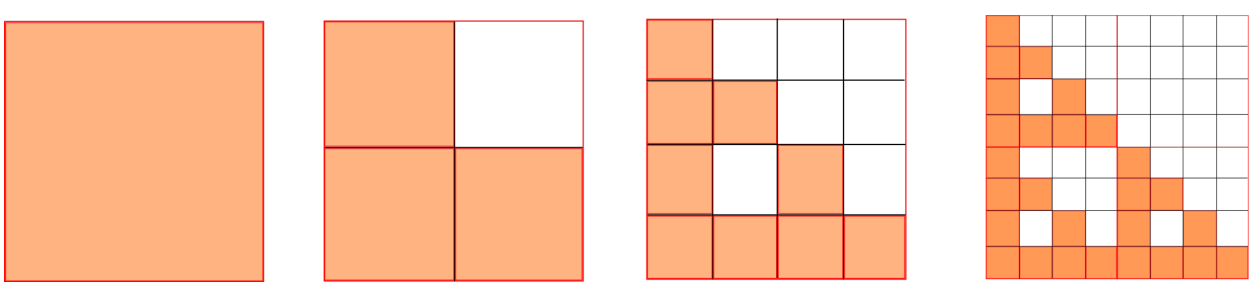

Figure 1. The sequences of squares in Problem 2 for $n = 1, 2, 3$ and 4.

The first step is to consider the square, so $a_1 = 1$. Figure 1 depicts the cases a_1, a_2, a_3 and a_4, and we see that the sequence thus starts with

$$(a_1, a_2, a_3, a_4, \ldots) = (1, 3, 9, 27, \ldots).$$

(a) Find the recursion satisfied by a_n. Given a closed formula for a_n.

(b) Let A_n the area covered by the orange squares, i.e. the squares which have not been removed, in the nth step. So $A_1 = 1$, $A_2 = 3/4$ and so on. Find the recursion satisfied by A_n and give a closed formula for A_n.

Problem 3. (15+10 points) Let $n, k \in \mathbb{N}$ be natural numbers with $k \leq n$.

(a) Show that the following formula holds

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}.$$

This is Corollary 4.20 in the Textbook, you can either give a proof by induction, direct computation or by combinatorial interpretation.

(b) Show that the following formula holds

$$\binom{n}{k} = \binom{n}{n-k},$$

either by induction, direct computation or by combinatorial interpretation.

Problem 4. (5+5+5+10 points) Let $a, b \in \mathbb{N}$ be natural numbers.

(a) What is the coefficient of a^5b^6 in the expansion of $(a+b)^{11}$?

(b) Show that the following formula holds

$$\sum_{k=0}^{k=n} (-1)^k \binom{n}{k} = 0.$$

(c) Show that the following formula holds

$$\sum_{k=0}^{k=n} \binom{n}{k} = 2^n.$$

(d) Show that a set with n elements has exactly 2^n distinct subsets.

Hint: For Part (d), interpret the equality in (c) combinatorially.

Problem 5. (25 points) Let us consider parenthesis, as used in a sentence for the written English language. The only rule for a single parenthesis is that it must open, we write "(" for that, and it must close, in which case we write ")".

If we use a parenthesis inside another parenthesis, we must make sure that we close the parenthesis inside first, before closing the external parenthesis. Otherwise it is not correctly written. For instance, ()() is correct, but)(() is not correct. Let P_n be the number of ways in which n parenthesis can be written correctly. The start of the sequence is

$$(P_0, P_1, P_2, P_3, \ldots) = (1, 1, 2, 5, \ldots),$$

corresponding to no parenthesis, which gives $P_0 = 1$, the unique parenthesis (), which gives $P_1 = 1$, $P_2 = 2$ because we can write ()() and (()) and finally there are five ways, $P_3 = 5$, of correctly writing three parenthesis:

$$()()(), ()(()), (())(), (()()), ((())).$$

Show that P_n satisfy the following recursion

$$P_{n+1} = \sum_{k=0}^{n} P_k P_{n-k}.$$

Problem 6. A triangulation of a polygon in the plane is a subdivision of the polygon into triangular pieces by adding edges that go between vertices. Let C_n be the number of triangulations of a polygon in the plane with $n+2$ sides. Find a recursion for C_{n+1} in terms of the previous elements in the sequence.

Example: C_1 are triangulations of the triangle, and there is only one, so $C_1 = 1$. This correspond to the first row in Figure 2. The next term C_2 counts the number of triangulations of a square, we can add either of the two diagonals to divide into triangles, so there are two ways and $C_2 = 2$. This corresponds to the second row in Figure 2. Figure 2 depicts the fact that $C_3 = 5$ and $C_4 = 14$. The sequence thus starts as

$$(C_1, C_2, C_3, C_4, \ldots) = (1, 2, 5, 14, \ldots)$$

Hint: Establish a bijection between the set of triangulations and the ways of ordering parenthesis. This will allow you to conclude that the C_n satisfy the same recursion as in numbers in Problem 5.

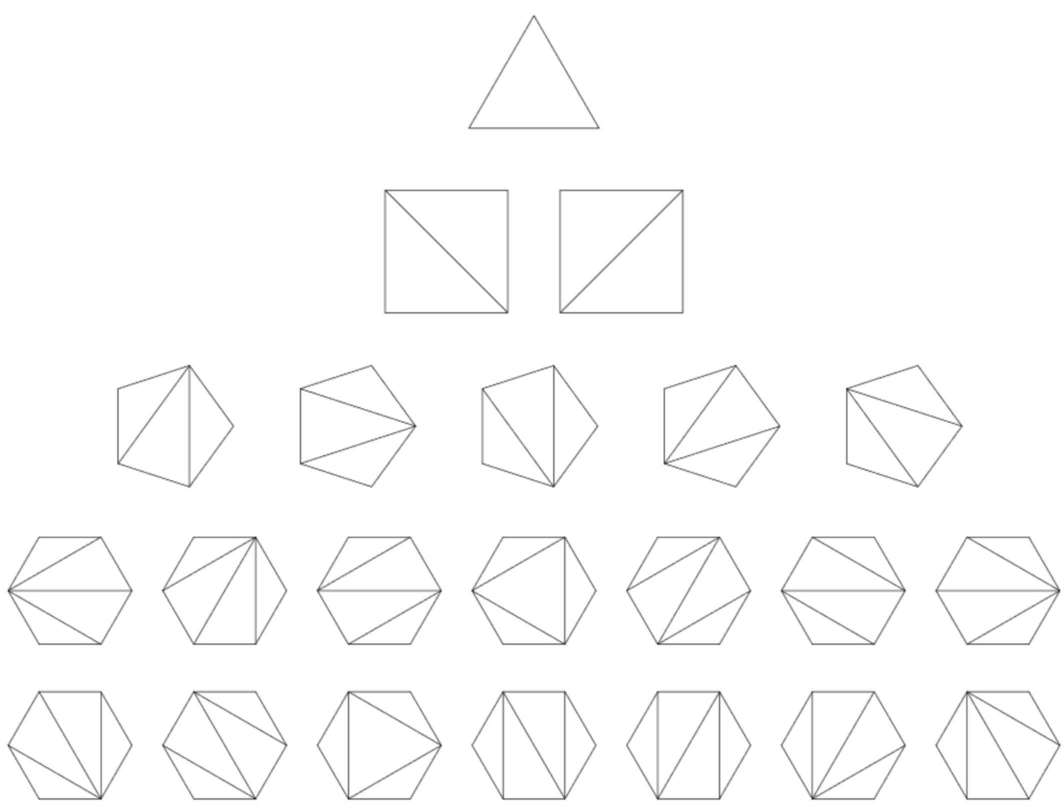

FIGURE 2. Triangulations of polygons of $n+2$ sides for $n = 1, 2, 3$ and 4, appearing in Problem 6. The first row is $n = 1$, triangulations of a triangle, the second row is $n = 2$, which are triangulations of a square. The third row is the case $n = 3$, which are triangulations of a pentagon and the fourth and fifth rows depict the $n = 4$ case, triangulating an hexagon.

Problem 7. The United States Postal Service in Davis delivers mail everyday to the ten adjacent Campus Buildings located along Shields Avenue. Suppose that two adjacent building never receive mail on the same day, but no more than two houses in a row get no mail. How many different possibilities of mail delivery are there ?

Example: Imagine that there were only two buildings, instead of ten. Then there would be *three* options for the mail being delivered:

(i) Neither building gets mail.
(ii) Only the first building receives mail.
(iii) Only the second building receives mail.

This tells us that there are *three* mail delivery possibilities for three buildings. □

Hint: Consider the possibilities in terms of the first two buildings, and build a recursion that recursively gives you the answer for *any given* number of building. Then evaluate your recursion for the case of ten buildings.

Problem 8. Consider a regular hexagon in the plane, as depicted in the upper left of Figure 3. It has six vertices, which are depicted in red thick dots.

Let A and B be two adjacent side of the hexagon, and insert a smaller copy of the same hexagon such that the A and B side of the copy are strictly included in the A and B sides of the initial hexagon. This is depicted in the upper right of Figure 3. Now the A and B sides of the initial copy have each been divided by a red dot (from the smaller hexagon), so introduced a red dot in the middle of all its remaining edges.

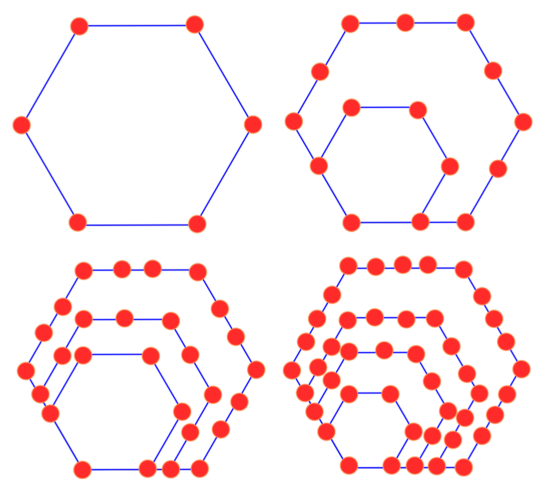

FIGURE 3. The hexagons and points in Problem 8 for $n = 1, 2, 3$ and 4.

Iterate the above process, introducing an smaller copy of the hexagon inside the previously inserted copies, and adding red dots in the edges so that each hexagon has exactly the same number of red dots per side. Let a_n be the number of vertices of the resulting polygonal figure in the plane. The first four cases are depicted in Figure 3, so the sequence begins with

$$(a_1, a_2, a_3, a_4, \ldots) = (6, 15, 28, 45, \ldots).$$

(a) Find a recursion for a_n.

(b) Find a closed formula for a_n.