MAT 108: PROBLEM SET 4

DUE TO FRIDAY NOV 8 2024

Abstract. This problem set covers material on the real numbers (Chapter 8). It was posted online on Monday Oct 28 and is due Friday Nov 8.

Purpose: The goal of this assignment is to practice problems on the set of **real numbers**. This particular problem set covers the material in Chapter 8, with a view towards Chapter 10.

Task: Solve Problems 1 through 6 below. The first and last problems will not be graded but I trust that you will work on it. Problems 2 to 5 will be graded.

Textbook: We use "The Art of Proof: Basic Training for Deeper Mathematics" by M. Beck and R. Geoghegan. Please contact me *immediately* if you have not been able to get a copy.

Problem 1. (Proposition 8.53) Prove that every non-empty subset of $\mathbb{R}$ that is bounded below has a greatest lower bound.

Problem 2. (25 points, 5 each) Find the least upper bound $\sup(A)$, and the greatest lower bound $\inf(A)$ of the following subsets of the real numbers $\mathbb{R}$:

(a) $A = (-3.2, 7) \subseteq \mathbb{R}$, i.e. $A = \{x \in \mathbb{R} : -3.2 < x \text{ and } x < 7\} \subseteq \mathbb{R}$.

(b) $B = (-3.2, 7] \subseteq \mathbb{R}$, i.e. $B = \{x \in \mathbb{R} : -3.2 < x \text{ and } x \leq 7\} \subseteq \mathbb{R}$.

(c) $C = (0, \infty) \subseteq \mathbb{R}$, i.e. $C = \{x \in \mathbb{R} : 0 < x\} \subseteq \mathbb{R}$.

(d) $D = (-\infty, 4] \subseteq \mathbb{R}$, i.e. $D = \{x \in \mathbb{R} : x \leq 4\} \subseteq \mathbb{R}$.

(e) $E = \mathbb{R}$.

Problem 3. (25 points) Consider the set of real numbers

$$N = \left\{3 - \frac{1}{n} : n \in \mathbb{N}\right\}.$$

Find $\inf(A)$ and $\sup(A)$.

Problem 4. (25 points) Consider the two following subsets of the real numbers

$$S = \left\{ \frac{n}{n+1} : n \in \mathbb{N} \right\} \subseteq \mathbb{R}, \quad T = \left\{ \frac{2n+1}{n+1} : n \in \mathbb{N} \right\} \subseteq \mathbb{R}.$$

Show that $\sup(S) = 1$, $\sup(T) = 2$ and $\inf(T) = 3/2$. Find $\inf(S)$.

Problem 5. (10+5+5 points) Find an upper bound for each of the following three sets:

$$X = \left\{ \left(1 + \frac{1}{n}\right)^n : n \in \mathbb{N} \right\}, \quad Y = \left\{ \left(1 + \frac{1}{n^2}\right)^n : n \in \mathbb{N} \right\}, \quad Z = \left\{ \left(1 + \frac{1}{n}\right)^{n^2} : n \in \mathbb{N} \right\}.$$

Hint: Consider the following expansion

$$\left(1 + \frac{1}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k} = \sum_{k=0}^{n} \frac{1}{k!} \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\left(1 - \frac{3}{n}\right) \cdot \ldots \cdot \left(1 - \frac{k-1}{n}\right).$$

Problem 6. (Optional) Consider the subset $C_0 = [0, 1] \subseteq \mathbb{R}$. Recursively, define the sets

$$C_{n+1} = \frac{C_n}{3} \cup \left(\frac{2}{3} + \frac{C_n}{3}\right),$$

for $n \geq 1$, where, if we let $A = [a, b]$, then the notation $A/3$ describes the interval $[a/3, b/3]$ and the notation $A + 2/3$ describe the interval $[a + 2/3, b + 2/3]$.

(a) Describe and draw the sets C_1, C_2, C_3 and C_4 as a union of explicit intervals.

(b) Show that the intersection $\cap_{n=1}^{\infty} C_n$ is non-empty.