SOLUTIONS TO PROBLEM SET 4

MAT 108

Abstract. Solutions to Problem Set 4 for MAT 108 in the Fall Quarter 2024.

Problem 1. (Proposition 8.53) Prove that every non-empty subset of $\mathbb{R}$ that is bounded below has a greatest lower bound.

Solution. Let A be a nonempty subset of $\mathbb{R}$ that is bounded below. Construct a new set $\tilde{A} = \{-a | a \in A\}$. This set is bounded above because l being a lower bound of A implies $-l$ is an upper bound of $\tilde{A}$. In other words, $l \leq a$ for all $a \in A$ and negating this gives $-l \geq -a$ for all $a \in A$. By the Completeness Axiom, $s = \sup \tilde{A}$ exists. We claim $-s = \inf(A)$. By definition, s being a supremum of $\tilde{A}$ implies $-a \leq s$ for all $a \in A$. Multiply this inequality by -1 to get $a \geq -s$. Hence, $-s$ is a lower bound of A. Moreover, it has to be our greatest lower bound. If not, then suppose $-t$ is the infimum of A so $-t \leq a$ for all $a \in A$. This would imply $t \geq -a$, i.e. the supremum of $\tilde{A}$ is t, a contradiction.

Problem 2. (20 points, 5 each) Find the least upper bound $\sup(A)$, and the greatest lower bound $\inf(A)$ of the following subsets of the real numbers $\mathbb{R}$:

(a) $A = (-3.2, 7) \subseteq \mathbb{R}$, i.e. $A = \{x \in \mathbb{R} : -3.2 < x \text{ and } x < 7\} \subseteq \mathbb{R}$.

(b) $B = (-3.2, 7] \subseteq \mathbb{R}$, i.e. $A = \{x \in \mathbb{R} : -3.2 < x \text{ and } x \leq 7\} \subseteq \mathbb{R}$.

(c) $C = (0, \infty) \subseteq \mathbb{R}$, i.e. $A = \{x \in \mathbb{R} : 0 < x\} \subseteq \mathbb{R}$.

(d) $D = (-\infty, 4] \subseteq \mathbb{R}$, i.e. $A = \{x \in \mathbb{R} : x \leq 4\} \subseteq \mathbb{R}$.

Solution. We will make use of the fact that the average of two distinct real numbers lies strictly between those two numbers. That is, for real numbers $a < b$, we have

$$a = \frac{a}{2} + \frac{a}{2} < \frac{a}{2} + \frac{b}{2} < \frac{b}{2} + \frac{b}{2} = b,$$

so

$$a < \frac{a+b}{2} < b. \tag{0.1}$$

(a) We claim that $\inf(A) = -3.2$ and $\sup(A) = 7$. It is clear from the definition of A that these give a lower bound and upper bound, respectively. Let u be a lower bound for A, and suppose for the sake of contradiction that $u > -3.2$. Since u is a lower bound for A, we also have

$$u \leq 0 < 7.$$

Consider the average $r := \frac{-3.2+u}{2}$, which, by (0.1) satisfies
$$-3.2 < r < u < 7,$$
so $r \in A$. Since $r < u$, this contradicts the fact that u is a lower bound, so we conclude that $u \leq -3.2$ after all. Therefore, -3.2 is the greatest lower bound for A, as desired.

Similarly, Let v be an upper bound for A, and suppose for the sake of contradiction that $v < 7$. Since v is an upper bound for A, we also have
$$v \geq 0 > -3.2.$$
Consider the average $r := \frac{v+7}{2}$, which, by (0.1) satisfies
$$-3.2 < v < r < 7,$$
so $r \in A$. Since $r > v$, this contradicts the fact that v is a lower bound, so we conclude that $v \geq -3.2$ after all. Therefore, -3.2 is the least upper bound for A, as desired.

(b) The proof is nearly identical to Part (a). We claim that $\inf(B) = -3.2$ and $\sup(B) = 7$. It is clear from the definition of B that these give a lower bound and upper bound, respectively. Let u be a lower bound for B, and suppose for the sake of contradiction that $u > -3.2$. Since u is a lower bound for B, we also have
$$u \leq 7.$$
Consider the average $r := \frac{-3.2+u}{2}$, which, by (0.1) satisfies
$$-3.2 < r < u \leq 7,$$
so $r \in B$. Since $r < u$, this contradicts the fact that u is a lower bound, so we conclude that $u \leq -3.2$ after all. Therefore, -3.2 is the greatest lower bound for B, as desired.

Similarly, Let v be an upper bound for B, and suppose for the sake of contradiction that $v < 7$. Since v is an upper bound for B, we also have
$$v \geq 0 > -3.2.$$
Consider the average $r := \frac{v+7}{2}$, which, by (0.1) satisfies
$$-3.2 < v < r < 7,$$
so $r \in B$. Since $r > v$, this contradicts the fact that v is a lower bound, so we conclude that $v \geq -3.2$ after all. Therefore, -3.2 is the greatest lower bound for B, as desired.Alternatively, notice that $\max(B) = 7$, so a Proposition from Discussion 6 tells us that $\sup(B) = 7$.

(c) We claim that $\inf(C) = 0$ and that C has no supremum. The proof the the former is by now standard. It is clear from the definition of C that 0 is a lower bound. Let u be a lower bound for C, and suppose for the sake of contradiction that $u > 0$. Consider the average $r := \frac{0+u}{2}$, which, by (0.1) satisfies
$$0 < r < u,$$
so $r \in C$. Since $r < u$, this contradicts the fact that u is a lower bound, so we conclude that $u \leq 0$ after all. Therefore, 0 is the greatest lower bound for C, as desired.

To show that C has no supremum, we show that it has no upper bounds (this suffices because suprema are, in particular, upper bounds). Indeed, let $x \in \mathbb{R}$. If $x \leq 0$, then $x < 1$, but $1 \in C$, so x is not an upper bound for C. Otherwise, $x > 0$, and we have $x < x + 1$, but $x + 1 > x > 0$ is in C, so x is again not an upper bound. Having excluded all possible real numbers as upper bounds, we conclude that C has no upper bound.

(d) We claim that $\sup(D) = 4$ and that D has no infimum. The proof the the former is by now standard. It is clear from the definition of D that 4 is an upper bound. Let v be an upper bound for D, and suppose for the sake of contradiction that $v < 4$. Consider the average $r := \frac{v+4}{2}$, which, by (0.1) satisfies

$$v < r < 4,$$

so $r \in D$. Since $r > v$, this contradicts the fact that v is a lower bound, so we conclude that $v \geq 4$ after all. Therefore, 4 is the greatest lower bound for C, as desired.Alternatively, notice that $\max(D) = 4$, so a Proposition from Discussion 6 tells us that $\sup(D) = 4$.

To show that D has no infimum, we proceed as in Part (c) by showing that it has no lower bound (this suffices because infima are, in particular, lower bounds). Indeed, let $x \in \mathbb{R}$. If $x > 4$, then–because $4 \in D$–x is not a lower bound for D. Otherwise, $x \leq 4$, and we have $x > x - 1$, but $x - 1 < x \leq 4$ is in D, so x is again not a lower bound. Having excluded all possible real numbers as lower bounds, we conclude that D has no lower bound.

Problem 3. (10+10 points) Consider the set of real numbers

$$N = \left\{ 3 - \frac{1}{n} : n \in \mathbb{N} \right\}.$$

Find $\inf(N)$ and $\sup(N)$.

Solution. We claim $\sup(N) = 3$ and $\inf(N) = 2$. Since $3 > 3 - \frac{1}{n}$ for all $n \in \mathbb{N}$, we know 3 is un upper bound for N. We know for each $\varepsilon > 0$, there exists an $n \in \mathbb{N}$ such that $\frac{1}{n} < \varepsilon$. Then, $3 - \frac{1}{n} > 3 - \varepsilon$ so $3 - \varepsilon$ is not an upper bound for any $\varepsilon > 0$. Thus, 3 must be our least upper bound. Now we prove the infimum is 2. Note that 2 is a lower bound. Moreover, $3 - \frac{1}{n+1} > 3 - \frac{1}{n} \geq 2$ because $\frac{1}{n+1} < \frac{1}{n}$ for all $n \in \mathbb{N}$. Therefore, 2 is our greatest lower bound.

Problem 4. Consider the two following subsets of the real numbers

$$S = \left\{ \frac{n}{n+1} : n \in \mathbb{N} \right\} \subseteq \mathbb{R}, \quad T = \left\{ \frac{2n+1}{n+1} : n \in \mathbb{N} \right\} \subseteq \mathbb{R}.$$

Show that $\sup(S) = 1$, $\sup(T) = 2$ and $\inf(T) = 3/2$. Find $\inf(S)$.

Solution. Define

$$s_n = \frac{n}{n+1} \quad \text{and} \quad t_n = \frac{2n+1}{n+1}.$$

Then we have the sequences $(s_n)_{n\in\mathbb{N}}$ and $(t_n)_{n\in\mathbb{N}}$. Note that

$$t_n = \frac{2n+1}{n+1} = \frac{n+n+1}{n+1} = \frac{n}{n+1} + \frac{n+1}{n+1} = s_n + 1,$$

so our sets are $S = \{s_n : n \in \mathbb{N}$ and $T = \{s_n + 1 : n \in \mathbb{N}$. We show that the sequence $(s_n)_{n\in\mathbb{N}}$ is monotone. Indeed, for each $n \in \mathbb{N}$, we have

$$\begin{aligned} s_{n+1} - s_n &= \frac{n+1}{n+2} - \frac{n}{n+1} \\ &= \frac{(n+1)(n+1) - (n+2)n}{(n+2)(n+1)} \\ &= \frac{1}{(n+2)(n+1)} \\ &\geq 0, \end{aligned}$$

so $s_{n+1} \geq s_n$. In particular,

$$s_n - s_1 = \frac{n}{n+1} - \frac{1}{2} = \frac{2n - (n+1)}{2(n+1)} = \frac{n-1}{2(n+1)} \geq 0,$$

since $n \geq 1$, so $\frac{1}{2} = s_1 \leq s_n$. Therefore, $\frac{1}{2} \in S$ is a lower bound for S, and hence $\inf(S) = \frac{1}{2}$ by a Proposition from Discussion 6.

By the proof of the Monotone Convergence Theorem, the limit of $(s_n)_{n\in\mathbb{N}}$ exists and is equal to $\sup(S)$, so we now prove that $\lim_{n\to\infty} s_n = 1$. Let $\varepsilon > 0$, and let $n_0 \in \mathbb{N}$ be such that $\frac{1}{n_0} < \varepsilon$. Then, for all $n \geq n_0$ we have

$$|1 - s_n| = \left| 1 - \frac{n}{n+1} \right| = 1 - \frac{n}{n+1} = \frac{(n+1) - n}{n+1} = \frac{1}{n+1} \leq \frac{1}{n_0} < \varepsilon.$$

Note that in the second equality above we used the fact that $n < n+1$, which rearranges to $1 - \frac{n}{n+1} > 0$. This completes the proof that $\sup(S) = \lim_{n\to\infty} s_n = 1$.

The calculations for T follow from those for S. The sequence $(t_n)_{n\in\mathbb{N}}$ is monotone because

$$t_{n+1} - t_n = (s_{n+1} + 1) - (s_n + 1) = s_{n+1} - s_n \geq 0$$

for all $n \in \mathbb{N}$. In particular,

$$t_n - t_1 = (s_n + 1) - (s_1 + 1) = s_n - s_1 \geq 0$$

so $\frac{3}{2} = t_1 \in T$ is a lower bound for T, and hence $\inf(T) = \frac{3}{2}$ by a Proposition from Discussion 6.

By the proof of the Monotone Convergence Theorem, the limit of $(t_n)_{n\in\mathbb{N}}$ exists and is equal to $\sup(T)$, so we now prove that $\lim_{n\to\infty} t_n = 2$. Let $\varepsilon > 0$, and let $n_0 \in \mathbb{N}$ be such that $\frac{1}{n_0} < \varepsilon$. Then, for all $n \geq n_0$ we have

$$|2 - t_n| = |2 - (s_n + 1)| = |1 - s_n| < \varepsilon.$$

This completes the proof that $\sup(T) = \lim_{n\to\infty} t_n = 2$.

Problem 5. (10+5+5 points) Find an upper bound for each of the following three sets:

$$X = \left\{ \left(1 + \frac{1}{n}\right)^n : n \in \mathbb{N} \right\}, \quad Y = \left\{ \left(1 + \frac{1}{n^2}\right)^n : n \in \mathbb{N} \right\}, \quad Z = \left\{ \left(1 + \frac{1}{n}\right)^{n^2} : n \in \mathbb{N} \right\}.$$

Hint: Consider the following expansion

$$\left(1 + \frac{1}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k} = \sum_{k=0}^{n} \frac{1}{k!} \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\left(1 - \frac{3}{n}\right) \cdot \ldots \cdot \left(1 - \frac{k-1}{n}\right).$$

Solution.

(i) Let's look at the expansion:

$$\sum_{k=0}^{n} \frac{1}{k!} \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\left(1 - \frac{3}{n}\right) \cdot \ldots \cdot \left(1 - \frac{k-1}{n}\right).$$

In discussion, we proved that as n becomes larger, the value of $\frac{1}{n}$ becomes smaller and the infimum of the set $\left\{\frac{1}{n} | n \in \mathbb{N}\right\}$ is thus 0. Therefore, each term in the parenthesis is bounded above by 1 so it suffices to consider

$$\sum_{k=0}^{n} \frac{1}{k!}.$$

Therefore, we have the following.

$$\begin{aligned} \left(1 + \frac{1}{n}\right)^n &\leq \sum_{k=0}^{n} \frac{1}{k!} \\ &= \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} \\ &< 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n} + \cdots \\ &= 1 + \sum_{k=0}^{\infty} \frac{1}{2^k} \\ &= 3 \end{aligned}$$

The last equality follows since the sum of the infinite geometric series $\sum_{k=0}^{\infty} \frac{1}{2^k}$ is $\frac{1}{1-1/2} = 2$.

(ii) Notice that

$$\left(1+\frac{1}{n^2}\right)^n = \left(\left(1+\frac{1}{n^2}\right)^{n^2}\right)^{1/n}.$$

We know $\left(1+\frac{1}{n^2}\right)^{n^2}$ is bounded above by 3 from part (i). (If it's difficult to see, replace n^2 with a new variable z, for instance.) It is enough to then consider $3^{1/n}$. Using what we know about the behavior of $\frac{1}{n}$, we conclude it is bounded above by 3.

(iii) We claim that this set has no upper bound. Notice that

$$c_n := \left(1+\frac{1}{n}\right)^{n^2} = \left(\left(1+\frac{1}{n}\right)^{n}\right)^{n}.$$

This is similar to part (ii). By the Binomial Theorem, we have

$$\begin{aligned}\left(1+\frac{1}{n}\right)^n &= \sum_{k=0}^{n}\binom{n}{k}\frac{1}{n^k}\\ &= \binom{n}{0}\frac{1}{n^0}+\binom{n}{1}\frac{1}{n^1}+\sum_{k=2}^{n}\binom{n}{k}\frac{1}{n^k}\\ &= 1\cdot 1 + n\cdot\frac{1}{n}+\sum_{k=2}^{n}\binom{n}{k}\frac{1}{n^k}\\ &\geq 2,\end{aligned}$$

so $c_n \geq 2^n$. It now suffices to show that the sequence $(2^n)_{n\in\mathbb{N}}$ is unbounded, which we prove by showing that $2^n \geq n$ using induction. (This proves it is not bounded above since the natural numbers is not bounded above.) For the base case, we have $2^1 \geq 1$, which is true. Now assume $2^k \geq k$. We then have

$$2^{k+1} = 2^k \cdot 2 > k\cdot 2 \geq k+1.$$

The last inequality follows because $2k \geq k+1$ can be rewritten as $k \geq 1$, which is true.

Problem 6. Consider the subset $C_0 = [0,1] \subseteq \mathbb{R}$. Recursively, define the sets

$$C_{n+1} = \frac{C_n}{3} \cup \left(\frac{2}{3}+\frac{C_n}{3}\right),$$

for $n \geq 1$, where, if we let $A = [a,b]$, then the notation $A/3$ describes the interval $[a/3, b/3]$ and the notation $A + 2/3$ describe the interval $[a+2/3, b+2/3]$.

(a) Describe and draw the sets C_1, C_2, C_3 and C_4 as a union of explicit intervals.

(b) Show that the intersection $\cap_{n=1}^{\infty} C_n$ is non-empty.

Solution. Here is the extension of the notations $\frac{A}{3}$ and $A+\frac{2}{3}$ for arbitrary sets. Let $X \subseteq \mathbb{R}$ be an arbitrary subset, and let c be any real number. Then we define the new sets

$$c\cdot X := \{c\cdot x : x \in X\} \subseteq \mathbb{R} \quad \text{and} \quad X + c := \{x + c : x \in X\} \subseteq \mathbb{R}.$$

For $c \neq 0$, we also define $\frac{X}{c} := \frac{1}{c} \cdot X$.

(a) The set C_{n+1} is obtained from C_n by scaling all of C_n down to fit inside $[0, \frac{1}{3}]$, and then repeating this scaled copy in the translation to $[\frac{2}{3}, 1]$. It follows that C_{n+1} is given by deleting the open middle third of each interval in C_n. Explicitly,

$$C_0 = [0, 1]$$
$$C_1 = [0, \tfrac{1}{3}] \cup [\tfrac{2}{3}, 1]$$
$$C_2 = [0, \tfrac{1}{9}] \cup [\tfrac{2}{9}, \tfrac{1}{3}] \cup [\tfrac{2}{3}, \tfrac{7}{9}] \cup [\tfrac{8}{9}, 1]$$
$$C_3 = [0, \tfrac{1}{27}] \cup [\tfrac{2}{27}, \tfrac{1}{9}] \cup [\tfrac{2}{9}, \tfrac{7}{27}] \cup [\tfrac{8}{27}, \tfrac{1}{3}] \cup [\tfrac{2}{3}, \tfrac{19}{27}] \cup [\tfrac{20}{27}, \tfrac{7}{9}] \cup [\tfrac{8}{9}, \tfrac{25}{27}] \cup [\tfrac{26}{27}, 1]$$
$$C_4 = [0, \tfrac{1}{81}] \cup [\tfrac{2}{81}, \tfrac{1}{27}] \cup [\tfrac{2}{27}, \tfrac{7}{81}] \cup [\tfrac{8}{81}, \tfrac{1}{9}] \cup [\tfrac{2}{9}, \tfrac{19}{81}] \cup [\tfrac{20}{81}, \tfrac{7}{27}] \cup [\tfrac{8}{27}, \tfrac{25}{81}] \cup [\tfrac{26}{81}, \tfrac{1}{3}]$$
$$\cup [\tfrac{2}{3}, \tfrac{55}{81}] \cup [\tfrac{56}{81}, \tfrac{19}{27}] \cup [\tfrac{20}{27}, \tfrac{61}{81}] \cup [\tfrac{62}{81}, \tfrac{7}{9}] \cup [\tfrac{8}{9}, \tfrac{73}{81}] \cup [\tfrac{74}{81}, \tfrac{25}{27}] \cup [\tfrac{26}{27}, \tfrac{79}{81}] \cup [\tfrac{80}{81}, 1].$$

These are illustrated in Figure 1 below, taken from
`georgcantorbyelithompson.blogspot.com`

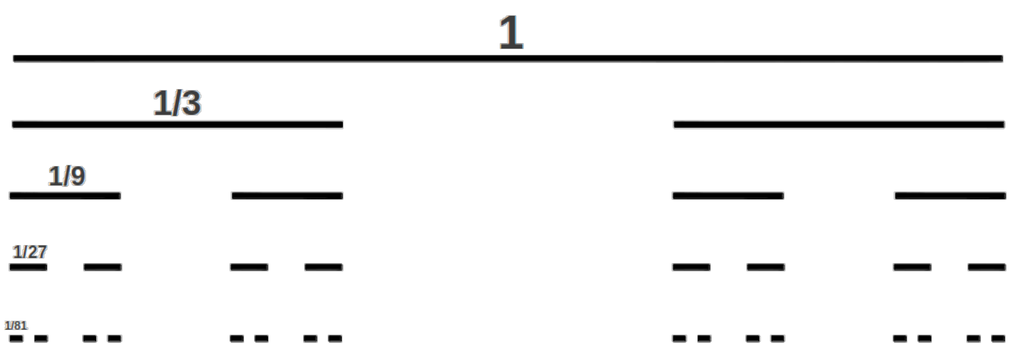

FIGURE 1. The sets C_0, C_1, C_2, C_3, and C_4.

(b) We will show that $0 \in C_n$ for all integers $n \geq 0$ by induction on n. For our base case $n = 0$, we have $0 \in [0, 1] = C_0$ (it's important that we're working with *closed* intervals). As our inductive hypothesis, suppose $0 \in C_n$ for some integer $n \geq 0$. Then

$$0 = \frac{0}{3} \in \frac{C_n}{3} \subseteq C_{n+1},$$

so $0 \in C_{n+1}$. We conclude that $0 \in C_n$ for all $n \geq 0$, so $0 \in \bigcap_{n=0}^{\infty} C_n$, and consequently $\bigcap_{n=0}^{\infty} C_n$ is not empty.

Note: The set $C_n \subseteq \mathbb{R}$ is a union of 2^n disjoint closed intervals. The above argument works similarly to show that any of the endpoints of these intervals persist in the further sets C_{n+1}, C_{n+2}, etc. (and of course, they're contained in C_{n-1}, C_{n-2}, etc. as well, since $C_0 \supset C_1 \supset C_2 \cdots$).

So each of these $2 \cdot 2^n$ points in the set C_n is in the intersection $\bigcap_{n=0}^{\infty} C_n$, and consequently the set $C := \bigcap_{n=0}^{\infty} C_n$ has infinitely many points! In fact, these persisting endpoints are the *only* elements of C. Notice the 2^{n+1} endpoints from C_n can all be written as rational numbers with common denominator 3^n.

The set $C := \bigcap_{n=0}^{\infty} C_n$ is called the Cantor set, and it exhibits a wide variety of strange phenomena that can occur in the real numbers $\mathbb{R}$.