MAT 108: PROBLEM SET 6

DUE TO FRIDAY NOV 22 2024

Abstract. This problem set corresponds to material on rational numbers, irrational numbers and cardinality (Chapters 11 and 13). It is due Friday Nov 22nd.

Purpose: The goal of this assignment is to practice problems on **rational numbers**, properties of **functions** between sets and cardinality. This particular problem set covers the material in Chapters 11 and 13.

Task: Solve Problems 1 through 7 below. The first problem will not be graded but I trust that you will work on it. Problems 2 to 5 will be graded, focus on these four problems. Problems 6 and 7 will not be graded, they are fun though and they are good examples of interesting problems regarding cardinality.

Textbook: We use "The Art of Proof: Basic Training for Deeper Mathematics" by M. Beck and R. Geoghegan. Please contact me *immediately* if you have not been able to get a copy.

Problem 1. (Theorem 11.8) Show that $\mathbb{Q} \subseteq \mathbb{R}$ is a dense subset, i.e. $\forall x, y \in \mathbb{R}$ there exists $q \in \mathbb{Q}$ such that $x < q < y$. Is the complement $\mathbb{R} \setminus \mathbb{Q}$ also dense ?

Problem 2. (16+10=26 points) Prove the following two statements:

(a) Show that $\sqrt[5]{3}$ is **not** a rational number.

(b) (Theorem 11.12) Let $p \in \mathbb{N}$ be a prime number, then $\sqrt{p}$ is irrational.

Problem 3. (25 points, 5 each) In the following instances of a function $f : X \longrightarrow Y$ between two sets X and Y, determine whether the function is an injection, a surjection and a bijection. You must provide a complete proof of each of your assertions.

(a) The function $f : \mathbb{N} \longrightarrow \mathbb{N}$, given by $f(n) = 4n + 6$,

(b) The function $f : \mathbb{Z} \longrightarrow \mathbb{Z}$, defined by $f(x) = -x^2 + 3x + 5$,

(c) The function $f : \mathbb{Z} \longrightarrow \mathbb{N} \cup \{0\}$, given by $f(x) = |x|$,

(d) The function $f : \mathbb{Q} \setminus \{0\} \longrightarrow \mathbb{Q} \setminus \{0\}$, given by $f(x) = 1/x^2$,

(e) The function $f : \mathbb{R} \longrightarrow \mathbb{R}$, given by $f(x) = 5x^3 - 9$.

Problem 4. (24 points, 6 each) Determine the cardinality of each of the following four sets, i.e. determine whether they are finite, countably infinite or uncountable. You must provide a complete proof of each of your assertions.

(a) The set $X_1 = \{2n : n \in \mathbb{N}\}$ of even natural numbers.

(b) The set $X_2 = \{(x, y) : x, y \in \mathbb{Q}\}$ of pairs of rational numbers.

(c) The set $X_3 = \{x \in \mathbb{R} : x > 3\}$ of positive real numbers greater than 3.

(d) The set $X_4 = \{x \in \mathbb{R}$ such that $x \neq \sqrt[n]{2}$ for any $n \in \mathbb{N}$, and $x \in \mathbb{R} \setminus \mathbb{Q}\}$ of irrational numbers which are not of the form $\sqrt[n]{2}$ for any $n \in \mathbb{N}$.

Problem 5. (15+10=25 points) Prove the following two statements:

(a) Let X, Y be two sets such that the set

$$X \times Y := \{(x, y) : \quad x \in X, y \in Y\},$$

is uncountable. Show that then either X or Y must be uncountable.

(b) Show that the set $[0, 1] \times [0, 1]$ is uncountable.

Problem 6. Consider the set of binary sequences

$$\mathcal{B} := \{f : \mathbb{N} \longrightarrow \{0, 1\}\}.$$

For instance, $s = 0001000100010001000010\ldots \in \mathcal{B}$ is an example of an element of this set. That is, elements of this set are infinite sequences of 0 and 1s.

Show that the set $\mathcal{B}$ is uncountable.

Hint: Use Theorem 13.31 in the textbook.

Problem 7. Consider the subset $C_0 = [0, 1] \subseteq \mathbb{R}$. Recursively, define the sets

$$C_{n+1} = \frac{C_n}{3} \cup \left(\frac{2}{3} + \frac{C_n}{3}\right),$$

for $n \geq 1$, where, if we let $A = [a, b]$, then the notation $A/3$ describes the interval $[a/3, b/3]$ and the notation $A + 2/3$ describe the interval $[a + 2/3, b + 2/3]$. This sets appeared in Problem 6 of Problem Set 4.

(a) Show that the intersection $\mathcal{C} := \cap_{n=1}^{\infty} C_n$ is infinite.

(b) Show that the intersection $\mathcal{C} := \cap_{n=1}^{\infty} C_n$ is uncountable.

Hint: For Part (a), show that the subset of **endpoints** of the intervals is countably infinite. For Part (b), construct a surjection to $[0,1]$.