SOLUTIONS TO PROBLEM SET 6

MAT 108

ABSTRACT. Solutions to Problem Set 6 for MAT 108 in Fall Quarter 2024.

Problem 1. (Theorem 11.8) Show that $\mathbb{Q} \subseteq \mathbb{R}$ is a dense subset, i.e. $\forall x, y \in \mathbb{R}$ there exists $q \in \mathbb{Q}$ such that $x < q < y$. Is the complement $\mathbb{R} \setminus \mathbb{Q}$ also dense ?

Solution. For the proof that $\mathbb{Q}$ is dense in $\mathbb{R}$, refer to Theorem 11.8 on pp. 108-109 of our textbook.

The set of irrational numbers $\mathbb{R} \setminus \mathbb{Q}$ is also dense in $\mathbb{R}$: Let $x < y$ be real numbers. By the density of $\mathbb{Q}$ in $\mathbb{R}$, there exists a rational number $q \in \mathbb{Q}$ such that $x < q < y$. Then $y - q > 0$, so by Proposition 10.4 there exists a natural number $n \in \mathbb{N}$ such that $y - q > \frac{1}{n}$. We then have

$$x < q + \frac{1}{n\sqrt{2}} < q + \frac{1}{n} < y.$$

Finally, $q + \frac{1}{n\sqrt{2}}$ is irrational by our results on unary and binary operations from Discussion 8 (make sure you can prove this directly, though).

Problem 2. Prove the following two statements:

(a) Show that $\sqrt[5]{3}$ is **not** a rational number.

(b) (Theorem 11.12) Let $p \in \mathbb{N}$ be a prime number, then $\sqrt{p}$ is irrational.

Solution.

(a) Suppose $\sqrt[5]{3} = \frac{a}{b}$ where $a, b \in \mathbb{Z}$ and $b \neq 0$. Also, assume $(a, b) = 1$. We then get $3b^5 = a^5$. It follows that b divides a^5 but since $(a, b) = 1$, we conclude $(a^5, b) = 1$. In other words, $b = 1$ so $3 = a^5$, a contradiction.

(b) Suppose $\sqrt{p} = \frac{a}{b}$ where a, b are defined the same as before. We then get $pb^2 = a^2$. Since p divides a^2, we know p divides a. Hence, $a = pA$ for some integer A. Substituting this expression into our original equation gives $pb^2 = (pA)^2 = p^2A^2$. In other words, $b^2 = pA^2$. We similarly conclude p divides b. However, this is a contradiction because the greatest common divisor of a and b is 1. Hence, $\sqrt{p}$ is irrational.

Problem 3. In the following instances of a function $f : X \longrightarrow Y$ between two sets X and Y, determine whether the function is an injection, a surjection and a bijection. You must provide a complete proof of each of your assertions.

(a) The function $f : \mathbb{N} \longrightarrow \mathbb{N}$, given by $f(n) = 4n + 6$,

(b) The function $f : \mathbb{Z} \longrightarrow \mathbb{Z}$, defined by $f(x) = -x^2 + 3x + 5$,

(c) The function $f : \mathbb{Z} \longrightarrow \mathbb{N} \cup \{0\}$, given by $f(x) = |x|$,

(d) The function $f : \mathbb{Q} \setminus \{0\} \longrightarrow \mathbb{Q} \setminus \{0\}$, given by $f(x) = 1/x^2$,

(e) The function $f : \mathbb{R} \longrightarrow \mathbb{R}$, given by $f(x) = 5x^3 - 9$.

Solution.

(a) This function is injective but not surjective. It is not surjective because there is no solution to $f(n) = 1$, for instance. This function is injective because $f(x) = f(y)$ $(4x + 6 = 4y + 6)$ implies $x = y$.

(b) This function is neither injective nor surjective. It is not surjective because there is no solution to $f(x) = 10$, for instance. It is not injective because $f(1) = f(2)$.

(c) This function is surjective but not injective. It is not injective because $f(-1) = f(1)$. We show it is surjective. Suppose $f(x) = y$ so $|x| = y$. Then, we can let $x = y$ or $x = -y$ if $y \neq 0$. If $y = 0$, then let $x = 0$.

(d) This function is neither surjective nor injective. It is not injective because $f(1) = f(-1)$. It is not surjective because there is no solution to $f(x) = \frac{1}{2}$.

(e) This function is bijective. Suppose $f(x) = f(y)$ so $5x^3 - 9 = 5y^3 - 9$. We can simplify this to $x^3 = y^3$, which only holds if $x = y$. Thus, the function is injective. Now suppose $f(x) = b$ so $b = 5x^3 - 9$. Then, we can let $x = \sqrt[3]{\frac{b+9}{5}}$, which has a solution since we're working in $\mathbb{R}$.

Problem 4. Determine the cardinality of each of the following four sets, i.e. determine whether they are finite, countably infinite or uncountable. You must provide a complete proof of each of your assertions.

(a) The set $X_1 = \{2n : n \in \mathbb{N}\}$ of even natural numbers.

(b) The set $X_2 = \{(x, y) : x, y \in \mathbb{Q}\}$ of pairs of rational numbers.

(c) The set $X_3 = \{x \in \mathbb{R} : x > 3\}$ of positive real numbers greater than 3.

(d) The set $X_4 = \{x \in \mathbb{R}$ such that $x \neq \sqrt[n]{2}$ for any $n \in \mathbb{N}$, and $x \in \mathbb{R} \setminus \mathbb{Q}\}$ of irrational numbers which are not of the form $\sqrt[n]{2}$ for any $n \in \mathbb{N}$.

Solution.

(a) This set is countably infinite. Consider the map $f : \mathbb{N} \to X_1$ given by $n \mapsto 2n$. This function is a bijection. It is injective because $2x = 2y$ implies $x = y$. It is surjective by construction. If we are given a number $2n \in X_1$, then we can let our value in the domain be n.

(b) This set is countably infinite. Since $\mathbb{Q}$ is countably infinite by Corollary 13.18, we conclude the cartesian product is also countably infinite using Problem 5(a).

(c) This set is uncountable. Note $X_3 = (3, \infty)$, an open interval. We prove every open interval (a, b) has the same cardinality as $\mathbb{R}$. First, consider the case $(0, 1)$. Define $f : (0, 1) \to \mathbb{R}$ by $x \mapsto \tan\left(\pi x - \frac{\pi}{2}\right)$. This map is a bijection, implying $(0, 1)$ is uncountable. We now construct a bijection from $(0, 1)$ to (a, b) and this finishes our proof. Define $g : (0, 1) \to (a, b)$ by $x \mapsto (b - a)x + a$.

(d) This set is uncountable. It suffices to prove the following: if A is uncountable and B is countable, then $A \setminus B$ is uncountable. For our problem, A is the set of irrational numbers and B is the set of irrational numbers of the form $\sqrt[n]{2}$ where $n \in \mathbb{N}$. We explain why the latter set is countable. Note the set of irrational numbers of the form $\sqrt[n]{2}$ is equivalent to $\mathbb{N}$. We prove every number $\sqrt[n]{2}$ is irrational. The proof is similar to Problem 2(a). Suppose $\sqrt[n]{2} = \frac{a}{b}$ where $(a, b) = 1$. We then get $2b^n = a^n$. We have b divides a^n so $(b, a^n) = 1$. In other words, $b = 1$ so $2 = a^n$, a contradiction. Now we prove the first statement. Suppose $A \setminus B$ is countable. Then, since B is countable, $(A \setminus B) \cup B$ is countable by Proposition 13.19. Observe that $A \subset ((A \setminus B) \cup B)$. In other words, an uncountable set is a subset of a countable set, a contradiction. Thus, $A \setminus B$ is uncountable.

Problem 5. Prove the following two statements:

(a) Let X, Y be two sets such that the set

$$X \times Y := \{(x, y) : \quad x \in X, y \in Y\},$$

is uncountable. Show that then either X or Y must be uncountable.

(b) Show that the set $[0, 1] \times [0, 1]$ is uncountable.

Solution:

(a) Suppose X and Y are both countable. Then there exist injections

$$f : X \longrightarrow \mathbb{N} \qquad \text{and} \qquad g : Y \longrightarrow \mathbb{N}$$

Define the function

$$h : X \times Y \longrightarrow \mathbb{N} \times \mathbb{N}$$

by $h((x, y)) = (f(x), g(y))$ for $(x, y) \in X \times Y$. Suppose $h((x, y)) = h((x', y'))$ for $(x, y), (x', y') \in X \times Y$. Then

$$(f(x), g(y)) = (f(x'), g(y')),$$

so $f(x) = f(x')$ and $g(y) = g(y')$. Since f and g are injective, we conclude that $x = x'$ and $y = y'$, so $(x, y) = (x', y')$. Therefore, h is injective, and

$$\text{card}(X \times Y) \leq \text{card}(\mathbb{N} \times \mathbb{N}).$$

By Corollary 13.16, $\mathbb{N} \times \mathbb{N}$ is countable, so $X \times Y$ is countable.

(b) The set $[0, 1]$ is uncountable by Theorem 13.25, so it suffices to construct a surjection from $[0, 1] \times [0, 1]$ to $[0, 1]$. That is, take the function

$$f : [0, 1] \times [0, 1] \longrightarrow [0, 1]$$

defined by $f((x, y)) = x$ for $(x, y) \in [0, 1] \times [0, 1]$. If $a \in [0, 1]$, then $f((a, 0)) = a$, so f is surjective (you could replace 0 with any element of $[0, 1]$).

Note: Visualize f geometrically as the restriction of a linear map $\mathbb{R}^2 \longrightarrow \mathbb{R}$ to a square. Interpret the parenthetical comment above geometrically.

Problem 6. (20 points) Consider the set of binary sequences

$$\mathcal{B} := \{f : \mathbb{N} \longrightarrow \{0, 1\}\}.$$

For instance, $s = 000100010001000100010\ldots \in \mathcal{B}$ is an example of an element of this set. That is, elements of this set are infinite sequences of 0 and 1s.

Show that the set $\mathcal{B}$ is uncountable.
Hint: Use Theorem 13.31 in the textbook.

Solution: By Theorem 13.31, it suffices to show that $\text{card}(\mathcal{B}) = \text{card}(P(\mathbb{N}))$. That is, we show that $\mathcal{B}$ is in bijection with the power set of $\mathbb{N}$, so $\mathcal{B}$ must have larger cardinality than $\mathbb{N}$ (which is the definition of uncountable).

In fact, a more general fact is true: for any set X, the power set $P(X)$ is in bijection with the set of functions $2^X := \{f : X \longrightarrow \{0,1\}\}$ (this is why the notation 2^X is often used for $P(X)$ itself). The proof of this general fact is similar to the way we treated the finite case in Problem 4(d) of Problem Set 3. Specifically, we define a function

$$T : 2^X \longrightarrow P(X).$$

To the element $f \in 2^X$, we associate a subset

$$T(f) := \{x \in X : f(x) = 1\} \subseteq X$$

Then $T(f) \in P(X)$ as desired. We show that T is a bijection by showing that it is injective and surjective. Suppose $f \neq g$ are two elements of 2^X. Since f and g are not equal, there exists some $x \in X$ such that $f(x) \neq g(x)$. If $f(x) = 1$, then $g(x) = 0$, so $x \in T(f)$ and $x \neq T(g)$, and we conclude that $T(f) \neq T(g)$. Similarly, if $f(x) = 0$, then $g(x) = 1$, so $x \neq T(f)$ and $x = T(g)$, and we again conclude that $T(f) \neq T(g)$. Therefore, T is injective.

Let $A \in P(X)$ be an arbitrary subset of X. Then define $f \in 2^X$ by

$$f_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}.$$

Note that $T(f_A) = A$, which proves that T is surjective, hence bijective.

In fact, the function $R : P(X) \longrightarrow 2^X$ defined above as $R(A) = f_A$ is the inverse to T. We just proved that $T(R(A)) = A$ for all $A \in P(X)$, and one can similarly show that $R(T(f)) = f$ for all $f \in 2^X$. This is another proof that T is a bijection.

Another way to see that $\mathcal{B}$ is uncountable is to associate each sequence $f \in \mathcal{B}$ with the binary expansion $0.f(1)f(2)f(3)f(4)\cdots_2$ (the subscript 2 indicates that we are expanding in base 2), which is a real number in $[0,1]$. Such an association is a surjection of $\mathcal{B}$ to $[0,1]$ because every number in $[0,1]$ can be given a binary expansion. Therefore, $\text{card}(\mathcal{B}) \geq \text{card}([0,1])$. Since $[0,1]$ is uncountable by Theorem 13.25, so is $\mathcal{B}$.

Note: The association $\mathcal{B} \longrightarrow [0,1]$ described above, while surjective, is not injective. This is because some numbers in $[0,1]$ can be represented by more than one binary expansion, i.e. one number in $[0,1]$ would be the image of multiple distinct sequences in $\mathcal{B}$. For example, the number $\frac{1}{10} \in [0,1]$ can be represented in binary as $\frac{1}{10} = 0.1000\cdots_2$ and $\frac{1}{10} = 0.01111\cdots_2$.

Problem 7. Consider the subset $C_0 = [0,1] \subseteq \mathbb{R}$. Recursively, define the sets

$$C_{n+1} = \frac{C_n}{3} \cup \left(\frac{2}{3} + \frac{C_n}{3}\right),$$

for $n \geq 1$, where, if we let $A = [a,b]$, then the notation $A/3$ describes the interval $[a/3, b/3]$ and the notation $A + 2/3$ describe the interval $[a + 2/3, b + 2/3]$. This sets appeared in Problem 6 of Problem Set 4.

(a) Show that the intersection $\mathcal{C} := \cap_{n=1}^{\infty} C_n$ is infinite.

(b) Show that the intersection $\mathcal{C} := \cap_{n=1}^{\infty} C_n$ is uncountable.

Hint: For Part (a), show that the subset of **endpoints** of the intervals is countably infinite. For Part (b), construct a surjection to $[0, 1]$.

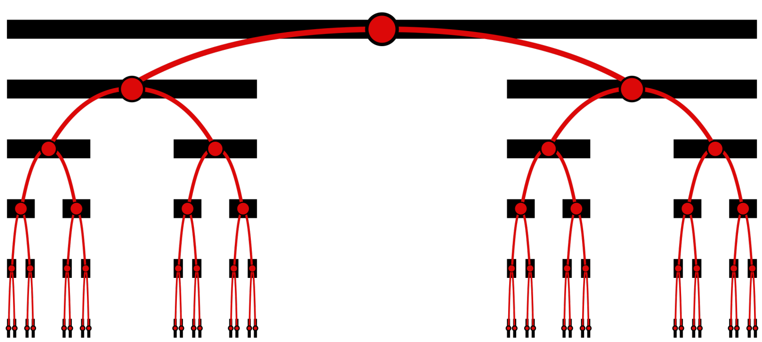
Solution

(a) *Sketch*: As remarked in the Solution to Problem 6 of Problem Set 4, the set C_n has 2^{n+1} endpoints, all of which are in $\mathcal{C}$. Let A_n denote the set of these 2^{n+1} endpoints. Note that for each $n \in \mathbb{N}$, $A_{n-1} \subseteq A_n$, and $A_n \setminus A_{n-1}$ contains $2^{n+1} - 2^n = 2^n$ elements, so each set C_n introduces exactly 2^n new endpoints. Therefore, the total set of endpoints is

$$\bigcup_{n=0}^{\infty} A_n = \bigsqcup_{n=1}^{\infty} A_n \setminus A_{n-1},$$

where the $\bigsqcup$ notation indicates a union of *disjoint* sets. The latter set has cardinality $2^0 + 2^1 + 2^2 + 2^3 + \cdots = \infty$. Note that this set is a countable union of countable (in fact, finite) sets, so it is countably infinite by Proposition 13.19. Since $\mathcal{C}$ contains this infinite set, $\mathcal{C}$ must be infinite.

(b) *Sketch*: Any element is $\mathcal{C}$ is uniquely specified by a path downward on the tree shown below (taken from the Wikipedia article on the Cantor set).



That is, any (infinite) sequence of LEFT or RIGHT moves determines a unique element of $\mathcal{C}$, and each element of $\mathcal{C}$ can be given by such a sequence. Therefore, $\mathcal{C}$ bijects to the set $\{f : \mathbb{N} \longrightarrow \{\text{LEFT}, \text{RIGHT}\}\}$ of binary sequences, which is clearly in bijection with the set $\mathcal{B}$ from Problem 6.

Seen another way, the elements of $\mathcal{C}$ are those real numbers in $[0, 1]$ which have an expansion in base 3 consisting of only the digits 0 and 2 (no 1's). Such expansions are unique, so $\mathcal{C}$ bijects to the set $\mathcal{B}$ of binary sequences from Problem 6.

With either viewpoint, we see that $\mathcal{C}$ is uncountable because $\mathcal{B}$ is uncountable by Problem 6. See the Wikipedia article for an explicit surjection to $[0, 1]$ which employs the ideas discussed above.