University of California Davis **Name (Print):** ______________________
Differential Equations MAT 108 **Student ID (Print):** ______________________

Practice Final Examination **December 13 2024**
Time Limit: 120 Minutes

This examination document contains 8 pages, including this cover page, and 5 problems. You must verify whether there any pages missing, in which case you should let the instructor know. **Fill in** all the requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, the Internet, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

(A) **If you use a lemma, proposition or theorem which we have seen in the class or in the book, you must indicate this** and explain why the theorem may be applied.

(B) **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive little credit.

(C) **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive little credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

(D) If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total:	100	

Do not write in the table to the right.

1. (20 points) Solve the following two parts:

   (a) (10 points) Prove that

$$\sqrt{n^4+1} \leq \frac{1}{2n^2} + n^2, \quad \forall n \in \mathbb{N}.$$

   (b) (10 points) Show that $\lim_{n\to\infty} \left(\sqrt{n^4+1} - n^2\right) = 0$.

2. (20 points) Consider the sequence $(x_n)_n \in \mathbb{N}$ given by the recursion

$$x_{n+1} = \sqrt{1 + x_n}, \quad x_1 = 1.$$

(a) (5 points) Write the first 5 terms of the sequence.

(b) (5 points) Show that (x_n) satisfies $x_n \leq 2$ for all $n \in \mathbb{N}$.

(c) (5 points) Prove that the sequence (x_n) is increasing.

(d) (5 points) Show that (x_n) is convergent and compute its limit $\lim_{n\to\infty} x_n$.

3. (20 points) Solve the following two parts:

   (a) (10 points) Show that 7 divides $n^7 - 14n^6 + 7n^3 - 211n$ for all $n \in \mathbb{N}$.

   (b) (10 points) Show that there are no solutions $x, y, z \in \mathbb{Z}$ to the equation

$$x^6 + 7y^4 + 2y^6 - 210x^4y + 15z^6 = 1056.$$

4. (20 points) Consider the function

$$f : \mathbb{Q} \longrightarrow \mathbb{Q}, \quad f(x) = 19x^5 + \frac{3}{2}.$$

(a) (4 points) Show that f is injective.

(b) (4 points) Show that there exists no $x \in \mathbb{Q}$ such that $f(x) = 39.5$.
(i.e. show that such x would be irrational.)

(c) (4 points) Is f surjective? (Justify your answer.)

(d) (4 points) Consider the set

$$X = \{(x, y) \in \mathbb{Q} \times \mathbb{Q} : \sqrt{f(x) - f(y)} \notin \mathbb{Q}\}.$$

Show X is countable.

(e) (4 points) Consider instead the function

$$g : \mathbb{R} \longrightarrow \mathbb{R}, \quad f(x) = 19x^5 + \frac{3}{2}$$

and the set

$$Z = \{(x, y) \in \mathbb{R} \times \mathbb{R} : \sqrt{g(x) - g(y)} \notin \mathbb{Q}\}.$$

Is Z countable or uncountable? (Justify your answer.)

5. (20 points) For each of the sentences below, circle **the unique correct answer**. (You do *not* need to justify your answer.)

(a) (4 points) The last two digits of $(98431207)^{1024}$ are

(a) 1. (b) 7. (c) 9. (d) 49. (e) 51.

(b) (4 points) The coefficient in front of x^4y^{10} in $(x+y)^{14}$ is

(a) 1. (b) 18. (c) 210. (d) 816. (e) 1330. (f) None of the above.

(c) (4 points) A closed formula for the recursion $x_n = 3x_{n-1} + 2$, $x_0 = 1$ is x_n equal

(a) $2^n - 1$ (b) $6^n - 1$. (c) $2 \cdot 3^n - 1$. (d) $3^n - 1$. (e) $3 \cdot 2^n - 1$.

(d) (4 points) The set $\{x \in \mathbb{R} : x^2 + ax + b = 0 \text{ for some } a, b \in \mathbb{Q}\}$ is

(a) countably finite (b) countably infinite (c) uncountable (d) None of above.

(e) (4 points) Let X, Y be finite sets such that the cardinality of X is strictly less than the cardinality of Y. Given a function $f : X \longrightarrow Y$, then f

(a) cannot be injective (b) cannot be surjective (c) might be bijective
(d) None of above.