University of California Davis
Differential Equations MAT 108

Name (Print): ______________________
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Practice Midterm Examination **October 25 2024**
Time Limit: 50 Minutes

This examination document contains 8 pages, including this cover page, and 4 problems. You must verify whether there any pages missing, in which case you should let the instructor know. **Fill in** all the requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, the Internet, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

(A) **If you use a lemma, proposition or theorem which we have seen in the class or in the book, you must indicate this** and explain why the theorem may be applied.

(B) **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive little credit.

(C) **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive little credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

(D) If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	25	
2	25	
3	25	
4	25	
Total:	100	

1. (25 points) Show that the following inequalities hold:

   (a) (15 points) Prove that

$$\sum_{k=1}^{n} \frac{1}{\sqrt{k}} < 2\sqrt{n}, \quad \forall n \in \mathbb{N}.$$

Proceed by induction on $n \in \mathbb{N}$.

Base case $(n = 1)$: $\sum_{k=1}^{1} \frac{1}{\sqrt{k}} = \frac{1}{\sqrt{1}} = 1 < 2 = 2\sqrt{1}$

Inductive step: Suppose there is some $n \in \mathbb{N}$ st. $\sum_{k=1}^{n} \frac{1}{\sqrt{k}} < 2\sqrt{n}$. Then,

$$\begin{aligned}\sum_{k=1}^{n+1} \frac{1}{\sqrt{k}} &= \sum_{k=1}^{n} \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{n+1}} \\ &< 2\sqrt{n} + \frac{1}{\sqrt{n+1}}.\end{aligned}$$

by the inductive hypothesis. Then, observe the following equality:

$$\begin{aligned}2\sqrt{n+1} - 2\sqrt{n} &= \frac{2(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{(\sqrt{n+1} + \sqrt{n})} \\ &= \frac{2((n+1) - n)}{\sqrt{n+1} + \sqrt{n}} \\ &= \frac{2}{\sqrt{n+1} + \sqrt{n}}.\end{aligned}$$

Thus, $2\sqrt{n+1} - 2\sqrt{n} = \frac{2}{\sqrt{n+1}+\sqrt{n}} \Rightarrow 2\sqrt{n} = 2\sqrt{n+1} - \frac{2}{\sqrt{n+1}+\sqrt{n}}$. Plugging in this equality to the above inequality yields:

$$\begin{aligned}\sum_{k=1}^{n+1} \frac{1}{\sqrt{k}} &< 2\sqrt{n} + \frac{1}{\sqrt{n+1}} \\ &= 2\sqrt{n+1} - \frac{2}{\sqrt{n+1} + \sqrt{n}} + \frac{1}{\sqrt{n+1}} \\ &= 2\sqrt{n+1} + \frac{-2\sqrt{n+1} + \sqrt{n+1} + \sqrt{n}}{\sqrt{n+1}(\sqrt{n+1} + \sqrt{n})} \\ &= 2\sqrt{n+1} + \frac{-\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1}(\sqrt{n+1} + \sqrt{n})}.\end{aligned}$$

Since $n \geq 1$, $\sqrt{n} < \sqrt{n+1} \Rightarrow -\sqrt{n+1} + \sqrt{n} < 0$ and $\sqrt{n+1}(\sqrt{n+1} + \sqrt{n}) > 0$.

Thus, $\frac{-\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}(\sqrt{n+1}+\sqrt{n})} < 0$. So,

$$\begin{aligned}\sum_{k=1}^{n+1} \frac{1}{\sqrt{k}} &< 2\sqrt{n+1} + \frac{-\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}(\sqrt{n+1}+\sqrt{n})} \\ &< 2\sqrt{n+1} + 0 \\ &= 2\sqrt{n+1}.\end{aligned}$$

Thus by induction, $\sum_{k=1}^{n} \frac{1}{\sqrt{k}} < 2\sqrt{n}$ for all $n \in \mathbb{N}$.

(b) (10 points) For $n \geq 6$ and $n \in \mathbb{N}$, show that

$$5n + 5 \leq n^2.$$

Proceed by induction on $n \in \mathbb{N}$ st. $n \geq 6$.

Base case (n = 6): $5 \cdot 6 + 5 = 35 \leq 36 = 6^2$.

Inductive step: Suppose there is some $n \in \mathbb{N}$ st. $n \geq 6$ and $5n + 5 \leq n^2$. Then,

$$\begin{aligned}5(n+1) + 5 &= 5n + 5 + 5 \\ &\leq n^2 + 5.\end{aligned}$$

by the inductive hypothesis. Since $n \geq 6$, $2n + 1 \geq 2 \cdot 6 + 1 = 13 \geq 5$. Thus, plugging in for the 5 in the above inequality yields:

$$\begin{aligned}5(n+1) + 5 &\leq n^2 + 5 \\ &\leq n^2 + (2n+1) \\ &= (n+1)^2.\end{aligned}$$

Thus by induction, $5n + 5 \leq n^2$ for all $n \in \mathbb{N}$ st. $n \geq 6$.

2. (25 points) Solve the following two parts:

   (a) (10 points) Consider the sequence $(x_n)_n \in \mathbb{N}$ given by the recursion

$$x_{n+1} = x_n + (n-1), \quad x_1 = 19.$$

Find x_{2020}.

Since 2020 is a large index, we should find a closed formula for the sequence to find x_{2020}. Observe the following pattern:

$$\begin{aligned}
&x_1 = 19 \\
&x_2 = x_1 + (2-1) = 19 + (2-1) = 19 + 1 \\
&x_3 = x_2 + (3-1) = 19 + 1 + (3-1) = 19 + 1 + 2 \\
&x_4 = x_3 + (4-1) = 19 + 1 + 2 + (4-1) = 19 + 1 + 2 + 3 \\
&\ \vdots \\
&x_n = 19 + \sum_{i=1}^{n-1} i.
\end{aligned}$$

From problem set 2, we know that for any $k \in \mathbb{N}$, $\sum\limits_{i=1}^{k} i = \frac{k(k+1)}{2}$. Thus,

$$\sum_{i=1}^{n-1} i = \frac{(n-1)(n-1+1)}{2} = \frac{(n-1)n}{2}.$$

Thus,

$$\begin{aligned}
x_n &= 19 + \frac{(n-1)n}{2} \\
\Rightarrow x_{2020} &= 19 + \frac{2019 \cdot 2020}{2} = 2039190.
\end{aligned}$$

   (b) (15 points) Consider the sequence $(x_n)_n$, $n \in \mathbb{N} \cup \{0\}$ defined recursively as

$$x_n = 7x_{n-1} - 10x_{n-2}, \quad x_0 = 2, x_1 = 3.$$

Find a closed formula for x_n.

Recall that for a recurrence relation of the form $x_n = bx_{n-1} + cx_{n-2}$, the closed formula is of the form $x_n = c_+ r_+^n + c_- r_-^n$, where $r_+, r_- \in \mathbb{R}$ are the roots of the polynomial equation $x^2 - bx - c = 0$ and $c_+, c_- \in \mathbb{R}$ are constant coefficients determined using the initial values of the recurrence relation.

The roots of $x^2 - 7x + 10 = 0$ are $x = \frac{7 \pm \sqrt{49-40}}{2} = \frac{7 \pm \sqrt{9}}{2} = \frac{7 \pm 3}{2}$. So, let

$$r_+ := 5 \text{ and } r_- := 2.$$

Then, our closed formula is of the form $x_n = c_+ 5^n + c_- 2^n$. Plugging in the initial values, we get:

$$2 = x_0 = c_+ 5^0 + c_- 2^0 = c_+ + c_-$$
$$3 = x_1 = c_+ 5^1 + c_- 2^1 = 5c_+ + 2c_-.$$

To solve this system of equations, note that $2 = c_+ + c_- \Rightarrow c_- = 2 - c_+$. Thus,

$$\begin{aligned} 3 &= 5c_+ + 2c_- \\ &= 5c_+ + 2(2 - c_+) \\ &= 3c_+ + 4 \\ \Rightarrow -1 &= 3c_+ \\ \Rightarrow -\frac{1}{3} &= c+ . \end{aligned}$$

Also, $c_- = 2 - c_+ \Rightarrow c_- = 2 + \frac{1}{3} = \frac{7}{3}$. So, the closed formula of the sequence is:

$$x_n = -\frac{1}{3} \cdot 5^n + \frac{7}{3} \cdot 2^n.$$

3. (25 points) Solve the following two parts:

(a) (10 points) Show that the coefficient in front of x^4y^{19} in $(x+y)^{23}$ is 8855.

The binomial theorem states that: if $a, b \in \mathbb{Z}$ and $k \in \mathbb{Z}_{\geq 0}$, then

$$(a+b)^k = \sum_{m=0}^{k} \binom{k}{m} a^k b^{m-k}.$$

Thus, the coefficient x^4y^{19} is $\binom{23}{4}$, since

$$(x+y)^{23} = \binom{23}{0}x^0y^{23} + \cdots + \binom{23}{4}x^4y^{19} + \cdots + \binom{23}{23}x^{23}y^0.$$

Then,

$$\begin{aligned}\binom{23}{4} &= \frac{23!}{4!(23-4)!} \\ &= \frac{23 \cdot 22 \cdot 21 \cdot 20 \cdot 19!}{4!\ 19!} \\ &= \frac{23 \cdot 22 \cdot 21 \cdot 20}{4 \cdot 3 \cdot 2} \\ &= 23 \cdot 11 \cdot 7 \cdot 5 \\ &= 8855\end{aligned}$$

is the coefficient of x^4y^{19} in $(x+y)^{23}$.

(b) (15 points) Consider the expression $(x+y)^n$, show that the coefficient in front of x^ky^{n-k} is the same as the coefficient in front of $x^{n-k}y^k$.

From the binomial theorem, in $(x+y)^n$, the coefficient in front of $x^ky^{n-k}, x^{n-k}y^k$ is $\binom{n}{k}, \binom{n}{n-k}$ respectively. Then,

$$\begin{aligned}\binom{n}{k} &= \frac{n!}{k!(n-k)!} \\ &= \frac{n!}{(n-k)!k!} \\ &= \frac{n!}{(n-k)!(n-(n-k))!} \\ &= \binom{n}{n-k}\end{aligned}$$

4. (25 points) Solve the following two problems:

   (a) (15 points) Show that there exists no integers $x, y \in \mathbb{Z}$ such that

$$4x^3 - 7y^3 = 2003.$$

Suppose for contradiction that there exists integers $x, y \in \mathbb{Z}$ st. $4x^3 - 7y^3 = 2003$. Consider the equation in $\mathbb{Z}_7$:

$$4x^3 - 7y^3 \equiv 4x^3 \pmod 7$$

Note that for any equivalence class $a \in \mathbb{Z}_7$,

$$a^3 \equiv \begin{cases} 0 & \text{if } x \equiv 0 \pmod 7 \\ 1 & \text{if } x \equiv 1, 2, 4 \pmod 7 \\ 6 & \text{if } x \equiv 3, 5, 6 \pmod 7 \end{cases}$$

$$\Rightarrow 4a^3 \equiv \begin{cases} 0 & \text{if } x \equiv 0 \pmod 7 \\ 4 & \text{if } x \equiv 1, 2, 4 \pmod 7 \\ 3 & \text{if } x \equiv 3, 5, 6 \pmod 7 \end{cases}$$

Thus, $4x^3$ is equivalent to 0, 3, or 4 modulo 7. But, 2003 is equivalent to 1 modulo 7. Since every integer is assigned exactly one equivalence class in $\mathbb{Z}_7$, it cannot be the case that $4x^3 - 7y^3 = 2003$. So, by contradiction, no such $x, y \in \mathbb{Z}$ exists.

   (b) (10 points) Show that the last two digits of 62^{48} are 96.

To find the last n digits of any integer, find the equivalence class modulo 10^n. So in this case, take 62^{48} mod 100.

Start by noting that $62^{48} = (60 + 2)^{48}$. Apply the binomial theorem,

$$62^{48} = (60 + 2)^{48} = \sum_{k=0}^{48} \binom{48}{k} 60^k 2^{48-k}.$$

Observe that $60^2 = 3600 \equiv 0 \pmod{100}$. Then, for any $k \geq 2$,

$$60^k = 60^2 60^{k-2} \equiv 0 \pmod{100}.$$

Then,

$$\begin{aligned}
62^{48} = (60+2)^{48} &= \sum_{k=0}^{48} \binom{48}{k} 60^k 2^{48-k} \\
&= \binom{48}{0} 60^0 \; 2^{48} + \binom{48}{1} 60^1 \; 2^{47} + \binom{48}{2} 60^2 \; 2^{46} + \ldots \\
&\equiv \binom{48}{0} 60^0 \; 2^{48} + \binom{48}{1} 60^1 \; 2^{47} + 0 \pmod{100} \\
&\equiv 2^{48} + 48 \cdot 60 \cdot 2^{47} \pmod{100} \\
&\equiv 2^{47}(2 + 48 \cdot 60) \pmod{100} \\
&\equiv 2^{47}(2 + 80) \pmod{100} \\
&\equiv 2^{47}(82) \pmod{100} \\
&\equiv (2^9)^5 \; 2^2 \cdot 82 \pmod{100} \\
&\equiv (112)^5 \; 4 \cdot 82 \pmod{100} \\
&\equiv (12)^5 \; 4 \cdot 82 \pmod{100} \\
&\equiv (12^2)^2 \; 12 \cdot 4 \cdot 82 \pmod{100} \\
&\equiv (144)^2 \; 12 \cdot 4 \cdot 82 \pmod{100} \\
&\equiv (44)^2 \; 12 \cdot 4 \cdot 82 \pmod{100} \\
&\equiv 1936 \cdot 12 \cdot 4 \cdot 82 \pmod{100} \\
&\equiv 36 \cdot 12 \cdot 4 \cdot 82 \pmod{100} \\
&\equiv 32 \cdot 4 \cdot 82 \pmod{100} \\
&\equiv 128 \cdot 82 \pmod{100} \\
&\equiv 28 \cdot 82 \pmod{100} \\
&\equiv 28 \cdot (-18) \pmod{100} \\
&\equiv -504 \pmod{100} \\
&\equiv 96 \pmod{100}.
\end{aligned}$$

So, the last two digits of 62^{48} is 96.