University of California Davis **Name (Print):** KEY
Differential Equations MAT 108 **Student ID (Print):** ____________________

Practice Midterm Examination II **October 25 2024**
Time Limit: 50 Minutes

This examination document contains 6 pages, including this cover page, and 4 problems. You must verify whether there any pages missing, in which case you should let the instructor know. **Fill in** all the requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, the Internet, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

(A) **If you use a lemma, proposition or theorem which we have seen in the class or in the book, you must indicate this** and explain why the theorem may be applied.

(B) **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive little credit.

(C) **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive little credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

(D) If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	25	
2	25	
3	25	
4	25	
Total:	100	

1. (25 points) Show that the following formulas hold:

   (a) (15 points) Prove that

$$\sum_{k=1}^{n} 2k = n(n+1), \quad \forall n \in \mathbb{N}.$$

Proof. Let $P(n)$ denote $\sum_{k=1}^{n} 2k = n(n+1)$.

Base case: Observe $\sum_{k=1}^{1} 2k = 2(1) = 2 = 1(1+1)$. Thus $P(1)$ holds.

Assume $P(n)$ holds for some $n \in \mathbb{N}$. Thus $\sum_{k=1}^{n} 2k = n(n+1)$. We want to show $\sum_{k=1}^{n+1} 2k = (n+1)(n+2)$. Observe

$$\begin{aligned}
\sum_{k=1}^{n+1} 2k &= \sum_{k=1}^{n} 2k + 2(n+1) \\
&= n(n+1) + 2(n+1) && \text{by the inductive hypothesis} \\
&= (n+1)(n+2) && \text{by factoring.}
\end{aligned}$$

Thus $P(n+1)$ holds.

Therefore, by induction, $\sum_{k=1}^{n} 2k = n(n+1), \quad \forall n \in \mathbb{N}.$ □

   (b) (10 points) Show that

$$\sum_{k=1}^{n} (2k)^2 = \frac{2n(n+1)(2n+1)}{3}, \quad \forall n \in \mathbb{N}.$$

Proof. Let $P(n)$ denote $\sum_{k=1}^{n} (2k)^2 = \frac{2n(n+1)(2n+1)}{3}$.

Base case: Observe that $\sum_{k=1}^{1} (2k)^2 = 2^2 = 4$ and $\frac{1}{3}2(1)(1+1)(2+1) = \frac{1}{3}4 \cdot 3 = 4$. So $\sum_{k=1}^{1} (2k)^2 = \frac{1}{3}2(1)(1+1)(2+1)$. Thus $P(1)$ holds.

Assume $P(n)$ holds for some $n \in \mathbb{N}$. That is, $\sum_{k=1}^{n} (2k)^2 = \frac{2n(n+1)(2n+1)}{3}$. We want to show

$$\sum_{k=1}^{n+1} (2k)^2 = \frac{2(n+1)(n+2)(2(n+1)+1)}{3}.$$

Observe

$$
\begin{aligned}
\sum_{k=1}^{n+1}(2k)^2 &= \sum_{k=1}^{n}(2k)^2 + (2(n+1))^2 \\
&= \frac{2n(n+1)(2n+1)}{3} + (2(n+1))^2 \qquad \text{by the inductive hypothesis} \\
&= \frac{2n(n+1)(2n+1) + 12(n+1)^2}{3} \\
&= \frac{2(n+1)[n(2n+1) + 6(n+1)]}{3} \\
&= \frac{2(n+1)[2n^2 + n + 6n + 6]}{3} \\
&= \frac{2(n+1)[2n^2 + 7n + 6]}{3} \\
&= \frac{2(n+1)[(2n+3)(n+2)]}{3} \\
&= \frac{2(n+1)(n+2)(2(n+1)+1)}{3}.
\end{aligned}
$$

Therefore, $\sum_{k=1}^{n+1}(2k)^2 = \frac{2}{3}(n+1)(n+2)(2(n+1)+1)$ as desired. That is, $P(n+1)$ holds.

Therefore,

$$
\sum_{k=1}^{n}(2k)^2 = \frac{2n(n+1)(2n+1)}{3}, \quad \forall n \in \mathbb{N}.
$$

□

2. (25 points) Solve the following two parts:

(a) (10 points) Consider the sequence $(x_n)_n \in \mathbb{N}$ given by the recursion

$$x_{n+1} = 2x_n - 1, \quad x_1 = 3.$$

Find the first 5 terms of the sequence.

$x_1 = 3$
$x_2 = 2(x_1) - 1 = 2(3) - 1 = 6 - 1 = 5$
$x_3 = 2(x_2) - 1 = 2(5) - 1 = 9$
$x_4 = 2x_3 - 1 = 2(9) - 1 = 17$
$x_5 = 2x_4 - 1 = 2(17) - 1 = 33$

(b) (15 points) For the sequence in (a), find a closed formula for the nth term x_n.

$x_1 = 3$
$x_2 = 2(3) - 1$
$x_3 = 2(2(3) - 1) - 1 = 2^2 \cdot 3 - 2 - 1$
$x_4 = 2(2^2 \cdot 3 - 2 - 1) - 1 = 2^3 \cdot -2^2 - 2 - 1$
$x_5 = 2(2^3 \cdot -2^2 - 2 - 1) - 1 = 2^4 \cdot 3 - 2^3 - 2^2 - 2 - 1$

Observing the pattern, we see

$$\begin{aligned} x_n &= 2^{n-1} \cdot 3 - \sum_{k=0}^{n-2} 2^k \\ &= 2^{n-1} \cdot 3 - (2^{n-1} - 1) \\ &= 2^{n-1}(3 - 1) + 1 \\ &= 2^n + 1. \end{aligned}$$

Note we find $S = \sum_{k=0}^{n-2} 2^k = 2^{n-1} - 1$ by the following:

$$2S - S = \sum_{k=1}^{n-1} 2^k - \sum_{k=0}^{n-2} 2^k = 2^{n-1} - 2^0 = 2^{n-1} - 1.$$

Answer: Oor closed form is $x_n = 2^{n-1} - 1$.

Double checking our closed we see, this does match: $x_1 = 2^1 + 1 = 3$
$x_2 = 2^2 + 1 = 5$
$x_3 = 2^3 + 1 = 9$
$x_4 = 2^4 + 1 = 17$
$x_5 = 2^5 + 1 = 33$

3. (25 points) Solve the following two parts:

   (a) (10 points) Let $x \in \mathbb{Z}$, prove that we must have one of the following three options: either $x^3 \equiv 0 \pmod 9$, $x^3 \equiv -1 \pmod 9$ or $x^3 \equiv 1 \pmod 9$.

   Proof. Let $x \in \mathbb{Z}$. We know $x = 3q+r$ for some $q \in \mathbb{Z}$ and some integer $r \in \{0, 1, 2\}$ by the division algorithm. By the binomial theorem,

$$\begin{aligned} x^3 &= (3q+r)^3 \\ &= \binom{3}{0}(3m)^3 + \binom{3}{1}(3m)^2 r + \binom{3}{2}(3m)r^2 + \binom{3}{3}r^3 \\ &= 9(3m^3) + 9(3m^2 r) + 9(mr^2) + r^3. \end{aligned}$$

   By the above, we see

$$x^3 \mod 9 \equiv r^3.$$

   Since $r \in \{0, 1, 2\}$, $r^3 \in \{0, 1, 8\}$. Thus r^3 is equivalent to $0 \mod 3$, $1 \mod 3$, or $8 \equiv -1 \mod 3$. Therefore, either $x^3 \equiv 0 \pmod 9$, $x^3 \equiv -1 \pmod 9$ or $x^3 \equiv 1 \pmod 9$. □

   (b) (15 points) Show that there are no solutions $x, y, z \in \mathbb{Z}$ to the equation

$$x^3 + y^3 + z^3 = 2029.$$

   Proof. Assume, for the sake of contradiction, that are are integer solutions to the equation $x^3 + y^3 + z^3 = 2029$. Thus there exists integers n, m, k such that $n^3 + m^3 + k^3 = 2029$. We know by the above that n^3, m^3, and k^3 are all respectively equivalent to 0, 1, or -1 modulo 9. So $n^3 + m^3 + k^3$ is 0, 1, 2, 3, -1, -2, -3 modulo 9. Further observe that $2029 \equiv 4 \mod 9$ (or likewise equivalent to $-5 \mod 9$) since $2029 - 4 = 2025 = 225 * 9$. Thus the equation does not hold modulo 9. This is a contradiction. Therefore, there are no no solutions $x, y, z \in \mathbb{Z}$ to the equation $x^3 + y^3 + z^3 = 2029$. □

4. (25 points) For each of the sentences below, circle **the unique correct answer**. (You do *not* need to justify your answer.)

   (a) (5 points) The residue of 70071^6 divided by 7 is

   (a) 0. (b) 1. (c) 2. (d) 6.

   Answer: b

$$\begin{aligned} 70071^6 &\equiv (71)^6 \mod 7 \\ &\equiv 1^6 \mod 7 \end{aligned}$$

   (b) (5 points) The equation $x^2 + y^2 = 1$, $x, y \in \mathbb{Z}$ has

   (a) No solutions. (b) Exactly two solutions.
   (c) Infinitely many solutions. (d) None of the above.
   Answer : d
   has four solutions $(x, y) \in \{(0, 1)(1, 0), (0, -1), (-1, 0)\}$

   (c) (5 points) Let $x = 1298$, then $x \equiv \pmod 5$ is

   (a) 0 (b) 1

   (c) 2 (d) 3

   Answer: d
   since $1298 - 3 = 1295$ which is a multiple of 5

   (d) (5 points) Let $x_n = 3x_{n-1}$ with $x_1 = 1$, then x_3 is

   (a) 1 (b) 3 (c) 9
   (d) 27 (e) None of the above.
   Answer: c since $x_3 = 3(x_2) = 3^2 x_1 = 9(1) = 9$

   (e) (5 points) If $x \in \mathbb{Z}$, then x^{12} divided by 13 has resiude

   (a) Always 0. (b) Always 1. (c) Either 0 or 1. (d) 0,1 or 2. (e) 2.
   Answer: c
   because either 13 divides x or the corollary of Fermat's Little Theorem applies and $x^{13-1} \equiv 1 \mod 13$ as 13 is prime