MAT 108: PRACTICE PROBLEMS

DEPARTMENT OF MATHEMATICS - UC DAVIS

ABSTRACT. This document contains additional practice problems for the first two weeks of the MAT-108 course during Fall 2020, with a view towards the first midterm on Friday October 25th.

Purpose: The goal of this document is to provide practice problems on the different topics seen covered in the lectures until Friday Oct 16. I have posted this document in order to help you practice problems on these topics, with a view towards the first Midterm Exam on Friday Oct 11. This document includes material on the following topics:

(i) Proofs by Induction and by contradiction. This corresponds to the second week of the course, Problem Set 2 and Chapter 2 in the Textbook.

(ii) Problems on Recursion. This corresponds to the third week of the course, Problem Set 3 and Chapter 4 in the Textbook.

Note that the Midterm includes additional topics, to be covered in weeks three and four, including modular arithmetic. Thus in additional to the types of problems below, you should practice problems on modular arithmetic. There is an additional set of practice problems for that as well.

Textbook: We are using "The Art of Proof: Basic Training for Deeper Mathematics" by M. Beck and R. Geoghegan.

Suggestion: In the first four problems, I would recommend that you prove the first three cases (a), (b) and (c), and if you feel you need more practice then do the rest. It is more important that you know how to do the first three cases in the first four problems than all the cases in one of these four problems.

Problem 1. Prove the following formulas for sums.

(a) $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$,

(b) $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$,

(c) $\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$,

(d) $\sum_{k=1}^{n} k^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$,

Problem 2. Prove the following additional formulas for sums.

(a) $\sum_{k=0}^{n}(2k+1) = (n+1)^2$,

(b) $\sum_{k=1}^{n} 2k = n(n+1)$,

(c) $\sum_{k=1}^{n} k(k+1) = \frac{n(n+1)(n+2)}{3}$,

(d) $\sum_{k=0}^{n} 3^k = \frac{3^{n+1}-1}{2}$,

(e) $\sum_{k=1}^{n} k!k = (n+1)!-1$,

Problem 3. Prove the following inequalities. Be aware of the base case in each case.

(a) For all $n \in \mathbb{N}$, $n < 2^n$,

(b) For all $n \in \mathbb{N}$, $n^2 + 6n + 7 < 20n^2$,

(c) For $n \geq 4$, $n^2 \leq 2^n$,

(d) For $n \geq 4$, $2^n < n!$.

(e) For $n \geq 6$, $6(n+1) < 2^n$,

(f) For $n \geq 8$, $3n^2 + 3n + 1 < 2^n$,

(g) For $n \geq 12$, $5^n < n!$,

Problem 4 Show that the following divisibility statements are true.

- For all $n \in \mathbb{N}$, $4|(5^n - 1)$, i.e. 4 divides $5^n - 1$.
- For all $n \in \mathbb{N}$, $5|(11^n - 6)$.
- For all $n \in \mathbb{N}$, $6|(n^3 - n)$.
- For all $n \in \mathbb{N}$, $7|(2^{n+2} + 3^{2n+1})$.

Problem 5 Prove that there are infinitely primes of the form $6k + 5$ with $k \in \mathbb{N}$.

Problem 6 Show that there are no positive integer solutions $a, b \in \mathbb{N}$ to the equation $a^2 - b^2 = 1$.

Problem 7 Show that there are no positive integer solutions $a, b \in \mathbb{N}$ to the equation $a^2 - b^2 = 10$.

Problem 8 Let $a_n = 2^n + 1$, prove that a_n satisfies the recursion

$$a_{n+1} = 2a_n - 1, \quad a_1 = 3.$$

Problem 9. Let F_n be the nth Fibonacci number, defined by the recursion $F_{n+1} = F_n + F_{n-1}$ and $F_1 = F_2 = 1$. Prove that

$$\sum_{k=1}^{n} F_k^2 = F_n F_{n+1}.$$

Problem 10. Let A_n be be defined by the recursion $A_{n+1} = 2A_n + 1$ and $F_1 = \alpha$. Prove that

$$A_n = (\alpha + 1) \cdot 2^{n-1} - 1.$$

Problem 11. Let L_n be defined by the recursion $L_{n+1} = L_n + L_{n-1}$ and $L_0 = 2, L_1 = 1$. Prove that

$$L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n.$$

Problem 12. Let a_n be defined by the recursion $a_{n+1} = 7a_n - 10a_{n-1}$ and $a_0 = 2, a_1 = 3$. Find a closed formula for a_n.