MAT 108: PRACTICE PROBLEMS

DEPARTMENT OF MATHEMATICS - UC DAVIS

ABSTRACT. This document contains additional practice problems for the first two weeks of the MAT-108 course during Fall 2024, with a view towards the first midterm on Friday October 25th.

Purpose: The goal of this document is to provide practice problems on the different topics seen covered in the lectures. I have posted this document in order to help you practice problems on these topics, with a view towards the first Midterm Exam on Friday Oct 11. This document includes material on the following topics:

(i) Proofs by Induction and by contradiction. This corresponds to the second week of the course, Problem Set 2 and Chapter 2 in the Textbook.

(ii) Problems on Recursion. This corresponds to the third week of the course, Problem Set 3 and Chapter 4 in the Textbook.

Note that the Midterm includes additional topics, to be covered in weeks three and four, including modular arithmetic. Thus in additional to the types of problems below, you should practice problems on modular arithmetic. There is an additional set of practice problems for that as well.

Textbook: We are using "The Art of Proof: Basic Training for Deeper Mathematics" by M. Beck and R. Geoghegan.

Suggestion: In the first four problems, I would recommend that you prove the first three cases (a), (b) and (c), and if you feel you need more practice then do the rest. It is more important that you know how to do the first three cases in the first four problems than all the cases in one of these four problems.

Problem 1. Prove the following formulas for sums.

(a) $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$,

(b) $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$,

(c) $\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$,

(d) $\sum_{k=1}^{n} k^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$,

Solution.

For each part, we may give a proof by induction.

(a) For the base case, we check if the statement holds for $n = 1$. Indeed, we have $1 = \frac{1(1+1)}{2}$. Now suppose for some $n \in \mathbb{N}$,
$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}.$$
For $n+1$, we then have
$$\begin{aligned}\sum_{k=1}^{n+1} k &= \sum_{k=1}^{n} k + (n+1)\\ &= \frac{n(n+1)}{2} + (n+1)\\ &= \frac{n(n+1)+2(n+1)}{2}\\ &= \frac{(n+1)((n+1)+1)}{2}\end{aligned}$$
where the second equality is given by the inductive hypothesis. The result follows by induction.

(b) For the base case, we can check that $1^2 = \frac{1(1+1)(2\cdot 1+1)}{6}$. Now suppose that for some $n \in \mathbb{N}$, we have
$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$
For $n+1$, we have
$$\begin{aligned}\sum_{k=1}^{n+1} k^2 &= \sum_{k=1}^{n} k^2 + (n+1)^2\\ &= \frac{n(n+1)(2n+1)}{6} + (n+1)^2\\ &= \frac{(n+1)(n(2n+1)+6(n+1)}{6}\\ &= \frac{(n+1)(2n^2+7n+6)}{6}\\ &= \frac{(n+1)(n+2)(2n+3)}{6}\\ &= \frac{(n+1)((n+1)+1)(2(n+1)+1)}{6}\end{aligned}$$
where the second equality is given by the inductive hypothesis. The result follows by induction.

(c) For the base case, we check $1^3 = \frac{1^2(1+1)^2}{4}$. Now suppose that for some $n \in \mathbb{N}$, we have
$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}.$$

For $n+1$, we get

$$\begin{aligned}\sum_{k=1}^{n+1} k^3 &= \sum_{k=1}^{n} k^3 + (n+1)^3\\ &= \frac{n^2(n+1)^2}{4} + (n+1)^3\\ &= \frac{n^2(n+1)^2 + 4(n+1)^3}{4}\\ &= \frac{(n+1)^2(n^2+4(n+1))}{4}\\ &= \frac{(n+1)^2((n+1)+1)^2}{4}\end{aligned}$$

where the second equality is given by the inductive hypothesis. The result follows by induction.

(d) For the base case, we can check that

$$\frac{1(1+1)(2\cdot 1+1)(3\cdot 1^2+3\cdot 1-1)}{30} = \frac{2\cdot 3\cdot 5}{30} = 1^4.$$

We could suppose for the inductive hypothesis that the formula holds for n, and then show it holds for $n+1$ as well. However, note that it is equivalent to suppose that the formula holds for $n-1$, where $n>1$, and then show it holds for n as well. The latter approach will lead to nicer formulas. So, suppose that for some $n>1$, we have

$$\begin{aligned}\sum_{k=1}^{n-1} k^4 &= \frac{(n-1)((n-1)+1)(2(n-1)+1)(3(n-1)^2+3(n-1)-1)}{30}\\ &= \frac{n(n-1)(2n-1)(3n^2-3n-1)}{30}.\end{aligned}$$

For n, we have

$$\begin{aligned}\sum_{k=1}^{n} k^4 &= \sum_{k=1}^{n-1} k^4 + n^4\\ &= \frac{n(n-1)(2n-1)(3n^2-3n-1)}{30} + n^4\\ &= \frac{n((n-1)(2n-1)(3n^2-3n-1)+30n^3)}{30}\\ &= \frac{n(6n^4+15n^3+10n^2-1)}{30}\\ &= \frac{n(n+1)(6n^3+9n^2+n-1)}{30}\\ &= \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}\end{aligned}$$

where the second equality is given by the inductive hypothesis. The result follows by induction.

Problem 2. Prove the following additional formulas for sums.

(a) $\sum_{k=0}^{n}(2k+1) = (n+1)^2$,

(b) $\sum_{k=1}^{n} 2k = n(n+1)$,

(c) $\sum_{k=1}^{n} k(k+1) = \frac{n(n+1)(n+2)}{3}$,

(d) $\sum_{k=0}^{n} 3^k = \frac{3^{n+1}-1}{2}$,

(e) $\sum_{k=1}^{n} k!k = (n+1)! - 1$,

Solution.

For each part, we may give a proof by induction.

(a) For the base case, we check for $n = 0$. This gives $(2 \cdot 0 + 1) = (0 + 1)^2$ which is true. Now suppose that for some $n \in \mathbb{N}$, we have

$$\sum_{k=0}^{n}(2k+1) = (n+1)^2.$$

Checking for $n + 1$, we have

$$\begin{aligned}
\sum_{k=0}^{n+1}(2k+1) &= \sum_{k=0}^{n}(2k+1) + (2(n+1)+1) \\
&= (n+1)^2 + 2n + 3 \\
&= n^2 + 4n + 4 \\
&= (n+2)^2 \\
&= ((n+1)+1)^2
\end{aligned}$$

where the second equality is given by the inductive hypothesis. The result follows by induction.

(b) For the base case, we check that $2 \cdot 1 = 1(1+1)$. Now suppose for some $n \in \mathbb{N}$ that

$$\sum_{k=1}^{n} 2k = n(n+1).$$

Checking for $n + 1$, we have

$$\begin{aligned}
\sum_{k=1}^{n+1} 2k &= \sum_{k=1}^{n} 2k + 2(n+1) \\
&= n(n+1) + 2(n+1) \\
&= (n+1)(n+2) \\
&= (n+1)((n+1)+1)
\end{aligned}$$

where the second equality is given by the inductive hypothesis. The result follows by induction.

(c) For the base case, we check that $1(1+1) = \frac{1(1+1)(1+2)}{3}$, which is true. Now suppose that for some $n \in \mathbb{N}$ we have

$$\sum_{k=1}^{n} k(k+1) = \frac{n(n+1)(n+2)}{3}.$$

Checking for $n+1$:

$$\begin{aligned}\sum_{k=1}^{n+1} k(k+1) &= \sum_{k=1}^{n} k(k+1) + (n+1)(n+2)\\ &= \frac{n(n+1)(n+2)}{3} + (n+1)(n+2)\\ &= \frac{(n+3)(n+1)(n+2)}{3}\\ &= \frac{(n+1)((n+1)+1)((n+1)+2)}{3}\end{aligned}$$

where the second equality is given by the inductive hypothesis. The result follows by induction.

(d) For the base case, we can check that $3^0 = \frac{3^{0+1}-1}{2}$, which is true. Now suppose that for some $n \in \mathbb{N}$, we have

$$\sum_{k=0}^{n} 3^k = \frac{3^{k+1}-1}{2}.$$

For $n+1$, we have

$$\begin{aligned}\sum_{k=0}^{n+1} 3^k &= \sum_{k=0}^{n} 3^k + 3^{n+1}\\ &= \frac{3^{n+1}-1}{2} + 3^{n+1}\\ &= \frac{3 \cdot 3^{n+1}-1}{2}\\ &= \frac{3^{(n+1)+1}-1}{2}.\end{aligned}$$

The result follows by induction.

(e) For the base case, we check that $1! \cdot 1 = (1+1)! - 1$, which is true. Now suppose that for some $n \in \mathbb{N}$, we have

$$\sum_{k=1}^{n} k!k = (n+1)! - 1.$$

For $n+1$, we have

$$\begin{aligned}\sum_{k=1}^{n+1} k!k = \sum_{k=1}^{n} k!k + (n+1)!(n+1)\\ &= (n+1)! - 1 + (n+1)!(n+1)\\ &= (n+1)!(n+2) - 1\\ &= ((n+1)+1)! - 1.\end{aligned}$$

The result follows by induction.

Problem 3. Prove the following inequalities. Be aware of the base case in each case.

(a) For all $n \in \mathbb{N}$, $n < 2^n$,

(b) For all $n \in \mathbb{N}$, $n^2 + 6n + 7 < 20n^2$,

(c) For $n \geq 4$, $n^2 \leq 2^n$,

(d) For $n \geq 4$, $2^n < n!$.

(e) For $n \geq 6$, $6(n+1) < 2^n$,

(f) For $n \geq 8$, $3n^2 + 3n + 1 < 2^n$,

(g) For $n \geq 12$, $5^n < n!$,

Solution.

(a) For the base case, we have $n = 1$, and we can verify that $1 < 2^1 = 2$. Note that since 2^n is increasing, we also know that $1 < 2^n$ for all $n \in \mathbb{N}$. Now suppose that for some $n \in \mathbb{N}$, $n < 2^n$. Considering $n + 1$, we have

$$\begin{aligned} n + 1 &< 2^{n+1} \\ &< 2^n + 2^n \\ &= 2^{n+1}. \end{aligned}$$

The first inequality follows by the inductive hypothesis. The second follows because $1 < 2^n$ for all $n \in \mathbb{N}$, and the statement thus follows by induction.

(b) For the base case we consider $n = 1$. We then have $1^2 + 6 \cdot 1 + 7 = 14$, and $20 \cdot 1^2 = 20$. Since $14 < 20$, the base case follows. Now suppose that for some $n \in \mathbb{N}$, $n^2 + 6n + 7 < 20n^2$. Checking $n + 1$, we have

$$\begin{aligned} 20(n + 1)^2 &= 20n^2 + 40n + 20 \\ &> (n^2 + 6n + 7) + (40n + 20) \end{aligned}$$

We can also compute

$$(n + 1)^2 + 6 \cdot (n + 1) + 7 = (n^2 + 6n + 7) + (2n + 7)$$

The claim follows if we can show that $40n + 20 > 2n + 7$ whenever $n \geq 1$. Indeed, we have

$$\begin{aligned} 40n + 20 - (2n + 7) &= 38n + 13 \\ &> 38 + 13 \\ &> 0, \end{aligned}$$

so the claim follows by induction.

(c) For the base case, we may check that $4^2 \leq 2^4$, which is true because $4^2 = 2^4 = 16$. Now suppose that for some $n \in \mathbb{N}$ with $n \geq 4$, we have $n^2 \leq 2^n$. Note that n and 2^n are positive numbers, so the inequality $n^2 \leq 2^n$ is equivalent to the inequality

$$\frac{n^2}{2^n} \leq 1.$$

Considering $n + 1$, we have

$$\begin{aligned} \frac{(n + 1)^2}{2^{n+1}} &= \frac{(n + 1)^2 \cdot n^2}{2^{n+1} \cdot n^2} \\ &= \frac{(n + 1)^2}{2n^2} \cdot \frac{n^2}{2^n} \\ &\leq \frac{n^2 + 2n + 1}{2n^2} \\ &= \frac{1}{2} + \frac{1}{n} + \frac{1}{2n^2}. \end{aligned}$$

Note that we used the inductive hypothesis for the inequality. Now, considering that $n \geq 4$, the sum $\frac{1}{n} + \frac{1}{2n^2}$ will be less than or equal to $\frac{1}{4} + \frac{1}{4}$. The last expression in the above calculation is therefore less than or equal to 1. This is precisely what we desired, so the statement follows by induction.

(d) The base case is given by $n = 4$, and we may verify that $2^4 = 16$, and $4! = 24$. Since $16 < 24$, the base case follows. Now suppose for some $n \in \mathbb{N}$ with $n \geq 4$ that $2^n < n!$. Considering

$n+1$, we have

$$\begin{aligned} 2^{n+1} &= 2 \cdot 2^n \\ &< 2 \cdot n! \\ &< (n+1) \cdot n! \\ &= (n+1)! \end{aligned}$$

where the first inequality follows from the inductive hypothesis, and the second from the fact that $n > 2$. By induction, the statement follows.

(e) The base case is given by $n = 6$, and we need to check that $6(6+1) < 2^6$. Since $6(6+1) = 42$ and $2^6 = 64$, the base case is satisfied. Now suppose for some natural number $n \geq 6$ that $6(n+1) < 2^n$. Considering $n+1$, we have

$$\begin{aligned} 6((n+1)+1) &= 6(n+1) + 6 \\ &< 2^n + 6 \end{aligned}$$

Since 2^n is increasing, and since $2^6 > 6$, we may further conclude:

$$\begin{aligned} 2^n + 6 &< 2^n + 2^n \\ &= 2^{n+1}. \end{aligned}$$

We see then that $6((n+1)+1) < 2^{n+1}$, so the statement follows by induction.

(f) The base case is given by $n = 8$, and we wish to show that $3 \cdot 8^2 + 3 \cdot 8 + 1 < 2^8$. Rather than calculate these numbers by hand, we can factor out 8 from each of them. We see that

$$3 \cdot 8^2 + 3 \cdot 8 + 1 < 8(3 \cdot 8 + 3 + 1) = 8 \cdot 28$$

while

$$2^8 = 8 \cdot 2^5 = 8 \cdot 32.$$

The base case follows because $28 < 32$. Now suppose that for some $n \geq 8$ we have $3n^2+3n+1 < 2^n$. For $n+1$, we have

$$\begin{aligned} 2^{n+1} &= 2 \cdot 2^n \\ &> 2 \cdot (3n^2 + 3n + 1) \\ &= 6n^2 + 6n + 2. \end{aligned}$$

We also compute

$$3(n+1)^2 + 3(n+1) + 1 = 3n^2 + 9n + 7.$$

The desired result has now been reduced to showing that the quantity

$$6n^2 + 6n + 2 - (3n^2 + 9n + 7) = 3n^2 - 3n - 5$$

is positive whenever $n \geq 8$. For this we can briefly use induction again. For $n = 8$, we have

$$3 \cdot 8^2 - 3 \cdot 8 - 5 > 8(3 \cdot 8 - 3 - 1) = 8 \cdot 20 > 0.$$

Then, supposing that $3n^2 - 3n - 5$ is positive for some $n \geq 8$, we have

$$\begin{aligned} 3(n+1)^2 - 3(n+1) - 5 &= (3n^2 - 3n - 5) + 3n + 3 \\ &> 0 + 0. \end{aligned}$$

The result follows by induction.

(g) The base case is given by $n = 12$, so we wish to show that $5^{12} < 12!$. This will require some work in order to easily compute it by hand. We may compute the ratio $12!/5^{12}$. This is now an exercise in grouping terms until we have reduced to something much simpler:

$$\begin{aligned} \frac{12!}{5^{12}} &= \frac{12}{5} \cdot \frac{11}{5} \cdot \frac{10}{5} \cdot \frac{7 \cdot 4}{5 \cdot 5} \cdot \frac{9 \cdot 3}{5 \cdot 5} \cdot \frac{5 \cdot 6}{5 \cdot 5} \cdot \frac{8 \cdot 2}{5^3} \\ &> \frac{2^3 \cdot 8 \cdot 2}{5^3} \\ &= \frac{128}{125} \\ &> 1. \end{aligned}$$

Indeed, $12!/5^{12} > 1$, so $12! > 5^{12}$. Now suppose for some natural number n with $n \geq 12$ that $5^n < n!$. For $n+1$, we have

$$\begin{aligned} 5^{n+1} &= 5 \cdot 5^n \\ &< 5 \cdot n! \\ &< (n+1) \cdot n! \\ &= (n+1)! \end{aligned}$$

where the inequalities follow respectively from the hypotheses that $5^n < n!$ and that $n \geq 12$. By induction, the result follows.

Problem 4 Show that the following divisibility statements are true.

- For all $n \in \mathbb{N}$, $4|(5^n - 1)$, i.e. 4 divides $5^n - 1$.
- For all $n \in \mathbb{N}$, $5|(11^n - 6)$.
- For all $n \in \mathbb{N}$, $6|(n^3 - n)$.
- For all $n \in \mathbb{N}$, $7|(2^{n+2} + 3^{2n+1})$.

Solution.

- For $n = 1$, we can check that $4|(5^1 - 1)$, because $5^1 - 1 = 4$. Now suppose for some $n \in \mathbb{N}$ that $4|(5^n - 1)$, so that there is some integer k satisfying $4k = 5^n - 1$. Considering $n+1$, we have
$$\begin{aligned} 5^{n+1} - 1 &= 5 \cdot 5^n - 1 \\ &= 5 \cdot (5^n - 1 + 1) - 1 \\ &= 5 \cdot (5^n - 1) + 5 - 1 \\ &= 5 \cdot 4k + 4 \\ &= 4(5k + 4). \end{aligned}$$
Since $5k + 4$ is an integer, we see that $5^{n+1} - 1$ is again a multiple of 4, and the result follows by induction.
- For the base case, we can check that $5|(11^1 - 6)$. Indeed, $11 - 6 = 5$, which is certainly a multiple of 5. Now suppose for some $n \in \mathbb{N}$ that $5|(11^n - 6)$, so there is some integer k with $11^n - 6 = 5k$. Considering $n = 1$, we have
$$\begin{aligned} 11^{n+1} - 6 &= 11 \cdot 11^n - 6 \\ &= 11 \cdot (11^n - 6 + 6) - 6 \\ &= 11 \cdot (11^n - 6) + 66 - 6 \\ &= 11 \cdot 5k + 60 \\ &= 5(11k + 12). \end{aligned}$$
Since $11k + 12$ is an integer, we see that $11^{n+1} - 6$ is again a multiple of 5.
- For the base case, we may verify that $6|(1^3 - 1)$. Indeed, $1^3 - 1 = 0$, and $6 \cdot 0 = 0$. Now suppose for some $n \in \mathbb{N}$ that $6|(n^3 - n)$, so there is an integer k for which $n^3 - n = 6k$. Considering $n+1$, we have
$$\begin{aligned} (n+1)^3 - (n+1) &= n^3 + 3n^2 + 3n + 1 - n - 1 \\ &= (n^3 - n) + 3n^2 + 3n \\ &= 6k + 3n(n+1). \end{aligned}$$
Given that n is a natural number, either n or $n+1$ is even. Therefore, the term $3n(n+1)$ is always an even multiple of 3, or equivalently a multiple of 6. Since the sum of two multiples of 6 is again a multiple of 6, it follows that 6 divides $(n+1)^3 - (n+1)$. By induction, the result holds for all n.

- For the base case, we wish to show that $7|(2^{1+2}+3^{2\cdot 1+1})$. Indeed, $2^3+3^3=35$, and $35=5\cdot 7$. Suppose now that for some $n\in\mathbb{N}$, there is an integer k for which $2^{n+2}+3^{2n+1}=7k$. Considering $n+1$, we have

$$\begin{aligned} &2^{n+1+2}+3^{2(n+1)+1}\\ &\qquad = 2\cdot 2^{n+2}+9\cdot 3^{2n+1}\\ &\qquad = 9\cdot(2^{n+2}+3^{2n+1})-7\cdot 2^{n+2}\\ &\qquad = 9\cdot 7k-7\cdot 2^{n+2}. \end{aligned}$$

  This value, being the sum of two multiples of 7, is again a multiple of 7, so the statement follows by induction.

Problem 5 Prove that there are infinitely primes of the form $6k+5$ with $k\in\mathbb{N}$.

Solution. Suppose to the contrary that there are only finitely many, say N distinct primes of the form $6k+5$ for $k\in\mathbb{N}$. We may then enumerate them in a set $\{p_1,p_2,\ldots,p_N\}$. Define the integer $Q=p_1p_2\cdots p_N$, and take $P=6Q-1$. We may equialently express P in the form

$$Q=6(Q-1+1)-1=6(Q-1)+5$$

so P is an integer of the form $6k+5$. Furthermore, since $Q>0$, P is a positive integer. We may express P as the product of odd primes, because P itself is odd. Furthermore, all prime factors of P must be of the form $6k+1$ or $6k+5$ for $k\in\mathbb{N}$, because any integer of the form $6k+3$ is a multiple of 3. We now wish to show that P has some prime factor of the form $6k+5$.

For integers $k_1,k_2,\ldots,k_n$, the integers $6k_i+1$ all have residue 1 modulo 6, and so by the laws of modular arithmetic the product

$$(6k_1+1)(6k_2+1)\cdots(6k_n+1)$$

must also have residue 1 modulo 6. Indeed, since P has residue 5 modulo 6, we see that its factorization into odd primes must contain at least one factor of the form $6k+5$. Such a number must be one of the primes p_i on our list, so we have $p_i|Q$, and thus $p_i|4Q$ as well. Since $p_i|4Q$ and $p_i|P$, p_i must divide the difference $4Q-P=1$. This is a contradiction, because p_i is a positive integer greater than 1, so it cannot be a divisor of 1.

We reach a contradiction, so we may conclude that there are indeed infinitely many primes of the form $6k+5$.

Problem 6 Show that there are no positive integer solutions $a,b\in\mathbb{N}$ to the equation $a^2-b^2=1$.

Solution. Suppose, to the contrary, that there are natural numbers a and b which satisfy $a^2-b^2=1$. We may factor $a^2-b^2=(a+b)(a-b)$, so that $(a+b)(a-b)=1$. We know that $a+b$ is strictly greater than 1, because a and b are each integers which are at least 1. But now we see that $a+b$ is a divisor of 1. This is impossible, because $a+b$ is an integer greater than 1, so we have a contradiction.

Problem 7 Show that there are no positive integer solutions $a,b\in\mathbb{N}$ to the equation $a^2-b^2=10$.

Solution. Consider the equation $a^2-b^2=10$ modulo 4. This becomes the equation $a^2-b^2\equiv 2 \mod 4$. The possible residues modulo 4 are 0, 1, 2, and 3. Their squares are 0, 1, 0, and 1 respectively. In particular, we see that the pairwise difference between any two squares modulo 4 must be 0, 1, or -1. We see then that there are no integers a and b for which $a^2-b^2\equiv 2 \mod 4$. In particular, the equation $a^2-b^2=10$ has no integer solutions in a and b.

Problem 8 Let $a_n=2^n+1$, prove that a_n satisfies the recursion

$$a_{n+1}=2a_n-1,\quad a_1=3.$$

Solution. We can check the base case by direct computation. By definition, $a_1 = 2^1 + 1 = 3$, so indeed $a_1 = 3$. Now consider the term a_{n+1}. We have

$$\begin{aligned} a_{n+1} &= 2^{n+1} + 1 \\ &= 2 \cdot 2^n + 1 \\ &= 2 \cdot (2^n + 1 - 1) \\ &= 2 \cdot (2^n + 1) - 2 + 1 \\ &= 2 \cdot a_n - 1 \end{aligned}$$

as desired.

Problem 9. Let F_n be the nth Fibonacci number, defined by the recursion $F_{n+1} = F_n + F_{n-1}$ and $F_1 = F_2 = 1$. Prove that

$$\sum_{k=1}^{n} F_k^2 = F_n F_{n+1}.$$

Solution. The base case is given by $n = 1$. We have $F_1^2 = 1$, and $F_1 \cdot F_2 = 1 \cdot 1 = 1$, so the base case follows. Suppose now that for some $n \in \mathbb{N}$, we have

$$\sum_{k=1}^{n} F_k^2 = F_n F_{n+1}.$$

Considering $n + 1$, we have

$$\begin{aligned} \sum_{k=1}^{n+1} F_k^2 &= F_{n+1}^2 + \sum_{k=1}^{n} F_k^2 \\ &= F_{n+1}^2 + F_n F_{n+1} \\ &= F_{n+1}(F_{n+1} + F_n) \\ &= F_{n+1} F_{n+2} \end{aligned}$$

As desired. The result follows by induction.

Problem 10. Let A_n be be defined by the recursion $A_{n+1} = 2A_n + 1$ and $A_1 = \alpha$. Prove that

$$A_n = (\alpha + 1) \cdot 2^{n-1} - 1.$$

Solution. Define a sequence whose nth term is $B_n := (\alpha+1) \cdot 2^{n-1} - 1$. We wish to show that $B_1 = \alpha$ and $B_{n+1} = 2A_n + 1$. We can check that $B_1 = (\alpha+1) \cdot 2^{1-1} - 1 = \alpha$. Furthermore, considering $n+1$, we have

$$\begin{aligned} B_{n+1} &= (\alpha + 1) \cdot 2^{n+1-1} - 1 \\ &= 2(\alpha + 1) \cdot 2^{n-1} - 1 \\ &= 2\left((\alpha + 1) \cdot 2^{n-1} - 1\right) + 2 - 1 \\ &= 2B_n + 1 \end{aligned}$$

As desired. It follows by induction that $B_n = A_n$, for all $n \in \mathbb{N}$, for they both satisfy the same initial value and recursive conditions that uniquely define A_n.

Problem 11. Let L_n be defined by the recursion $L_{n+1} = L_n + L_{n-1}$ and $L_0 = 2, L_1 = 1$. Prove that

$$L_n = \left(\frac{1 + \sqrt{5}}{2}\right)^n + \left(\frac{1 - \sqrt{5}}{2}\right)^n.$$

Solution. First, define the real numbers

$$r_1 := \frac{1 + \sqrt{5}}{2},\ r_2 := \frac{1 - \sqrt{5}}{2}.$$

We may check that each r_i satisfies the equation $r^2 = r + 1$. Indeed, we calculate:

$$\begin{aligned} r_1^2 &= \left(\frac{1+\sqrt{5}}{2}\right)^2 \\ &= \frac{(1+\sqrt{5})^2}{4} \\ &= \frac{1+2\sqrt{5}+5}{4} \\ &= \frac{1+\sqrt{5}}{2} + 1 \\ &= r_1 + 1 \end{aligned}$$

and similarly

$$\begin{aligned} r_2^2 &= \left(\frac{1-\sqrt{5}}{2}\right)^2 \\ &= \frac{(1-\sqrt{5})^2}{4} \\ &= \frac{1-2\sqrt{5}+5}{4} \\ &= \frac{1-\sqrt{5}}{2} + 1 \\ &= r_2 + 1. \end{aligned}$$

Now we show by induction that $L_n = r_1^n + r_2^n$. For the base cases, we have $n = 0$ and $n = 1$, and we may verify:

$$\begin{aligned} r_1^0 + r_2^0 &= 2 \\ r_1^1 + r_2^1 &= \frac{1+\sqrt{5}+1-\sqrt{5}}{2} \\ &= 1 \end{aligned}$$

and this verifies the base cases. Now suppose that for some $n \in \mathbb{N}$, we have $L_n = r_1^n + r_2^n$ and $L_{n-1} = r_1^{n-1} + r_2^{n-1}$. Checking for $n + 1$, we have

$$\begin{aligned} L_{n+1} &= L_n + L_{n-1} \\ &= r_1^n + r_2^n + r_1^{n-1} + r_2^{n-1} \\ &= r_1^{n-1}(r_1 + 1) + r_2^{n-1}(r_2 + 1) \\ &= r_1^{n-1} r_1^2 + r_2^{n-1} r_2^2 \\ &= r_1^{n+1} + r_2^{n+1} \end{aligned}$$

which is what we desired. By induction, we see that $L_n = r_1^n + r_2^n$ for all n.

Problem 12. Let a_n be defined by the recursion $a_{n+1} = 7a_n - 10a_{n-1}$ and $a_0 = 2, a_1 = 3$. Find a closed formula for a_n.

Solution. In general, suppose b_n and b'_n are two sequences such that $b_{n+1} = 7b_n - 10b_{n-1}$, and such that b'_n satisfies the same recurrence relation. Given any two real numbers c and d, we may verify that the sequence $s_n = cb_n + db'_n$ satisfies the same recursive rule. Indeed,

$$\begin{aligned} s_{n+1} &= cb_{n+1} + db'_{n+1} \\ &= c(7b_n - 10b_{n-1}) + d(7b'_n - 10b'_{n-1}) \\ &= 7(cb_n + db'_n) - 10(cb_{n-1} + db'_{n-1}) \\ &= 7s_n - 10s_{n-1}. \end{aligned}$$

We therefore see that solutions to this recurrence relation are closed under real linear combinations.

Now we seek some number r such that the sequence $b_n = r^n$ satisfies the relation $b_{n+1} = 7b_n - 10b_{n-1}$. In particular, when $n = 1$, this gives

$$r^2 - 7r + 10 = 0$$
$$(r-5)(r-2) = 0.$$

So $r = 5$ and $r = 2$ both give solutions to this recursion. The sequence a_n that we seek will be some linear combination $a_n = c \cdot 2^n + d \cdot 5^n$. The initial terms $a_0 = 2$ and $a_1 = 3$ can be used to solve for c and d:

$$c \cdot 2^0 + d \cdot 5^0 = 2$$
$$c \cdot 2^1 + d \cdot 5^1 = 3.$$

Subtracting two of the first equation from the second equation gives $3d = -1$, so $d = -1/3$. Substituting this into the first equation, we get $c = 2 + 1/3 = 7/3$. Finally, the sequence a_n has a closed form given by

$$a_n = \frac{7}{3} \cdot 2^n - \frac{1}{3} \cdot 5^n.$$

By the discussion above, this sequence still satisfies the recurrence relation because it is a real linear combination of sequences which satisfy the same linear recurrence relation.