MAT 108: PRACTICE PROBLEMS MIDTERM II

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ABSTRACT. This document contains practice problems for the second part of the course, with a view towards the 2nd Midterm Exam.

Purpose: The goal of this document is to provide practice problems on the different topics seen covered in the lectures after the 1st midterm. I have posted this document in order to help you practice problems on these topics, with a view towards the 2nd Midterm Exam on Monday Nov 25.

Suggestion: In the problems with several items, I would recommend that you practice the first few cases, and if you feel you need more practice, then do the rest. It is more important that you know how to do the first few cases in the first four problems than all the cases in one of these problems.

Problem 1. Prove the following four statements.

(a) Let $X_1 = \{x \in \mathbb{R} : -2 \leq x < 3\} = [-2, 3)$.
Then $\inf(X_1) = -2$ and $\sup(X_1) = 3$,

(b) Let $X_2 = \{x \in \mathbb{Q} : -\sqrt{3} \leq x < \sqrt{2}\} = [-\sqrt{3}, \sqrt{2}) \cap \mathbb{Q}$.
Then $\inf(X_2) = -\sqrt{3}$ and $\sup(X_2) = \sqrt{2}$,

(c) Let $X_3 = \{2^{-n} : n \in \mathbb{N}\}$, then $\inf(X_3) = 0$ and $\sup(X_3) = 1/2$,

(d) Let $X_4 = \left\{\frac{n}{5n+3} : n \in \mathbb{N}\right\}$, then $\inf(X_4) = 1/8$ and $\sup(X_4) = 1/5$.

Problem 2. Compute the following limits using the ε-definition of the limit of a sequence.

(a) $\lim_{n\to\infty} \frac{1}{n^3} = 0$,

(b) $\lim_{n\to\infty} \frac{2^n}{7^n} = 0$,

(c) $\lim_{n\to\infty} \frac{(-1)^n \cdot 4^n}{n!} = 0$,

(d) $\lim_{n\to\infty} \sqrt[n]{n} = 1$,

(e) $\lim_{n\to\infty} \frac{3n+7}{8n+1} = \frac{3}{8}$,

(f) $\lim_{n\to\infty} \frac{5n^2+3n+7}{9n^2+17n} = \frac{5}{9}$,

Problem 3. For each of the following sequences, first whether the following sequences are eventually **increasing** or **decreasing** (or neither), and second whether they are **bounded above** or **below** (or neither). Use this to decide whether each of the sequences (x_n) converges.

(a) $x_n = \frac{1}{4n}$,

(b) $x_n = \frac{n^2(-1)^{n+3}}{3n+2}$,

(c) $x_n = \frac{n}{n^2+1}$,

(d) $x_n = \frac{5n-7}{8^n}$,

(e) $x_n = \frac{n^n}{(2n)!}$,

(f) $x_n = \frac{n+1}{n-1}$,

(g) $x_n = \frac{10^n}{n!}$,

(h) $x_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n-1} + \frac{1}{n}$,

(i) $x_n = \frac{1}{2} + \frac{1}{6} + \ldots + \frac{1}{(n-1)n} + \frac{1}{n(n+1)}$,

Problem 4. Consider each of the following recursive sequences. Show that they converge by using the **Monotone Convergence Theorem**. Then, find their limit $\lim_{n\to\infty} x_n$.

- $x_{n+1} = \frac{x_n+1}{4}$, with $x_1 = 7$.
- $x_{n+1} = \frac{1}{2}x_n + 2$, with $x_1 = 1/2$.
- $x_{n+1} = \frac{1}{3-x_n}$, with $x_1 = 2$.
- $x_{n+1} = \sqrt{3+x_n}$, with $x_1 = 1$.
- $x_{n+1} = \sqrt{17+x_n}$, with $x_1 = \sqrt{17}$.
- $x_{n+1} = \frac{x_n^2-63}{2}$, with $x_1 = 10$.
- $x_{n+1} = 7 - \frac{10}{x_n}$, with $x_1 = 4$.

Problem 5. Prove that the following numbers are **not** rational:

(a) $\sqrt{5}$,

(b) $\sqrt[7]{11}$,

(c) $\sqrt{2}+\sqrt{3}$.

Problem 6. Show that for any real number $r \in \mathbb{R}$, there exists a sequence of rational numbers $(x_n) \subseteq \mathbb{Q}$ such that $\lim_{n\to\infty} x_n = r$. In particular, notice that a sequence of rational numbers can converge to an irrational number.

Problem 7. For each of the following sets X, Y and function $f : X \longrightarrow Y$, determine whether f is an **injection**, a **surjection** or a **bijection**.

(a) $f : \mathbb{N} \longrightarrow \mathbb{N}$, $f(x) = x^2$,

(b) $f : \mathbb{Z} \longrightarrow \mathbb{Z}$, $f(x) = x^2$,

(c) $f : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$, $f(x) = x^2$,

(d) $f : \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = x^3$,

(e) $f : \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = x^4 + x^2 + 3$,

(f) $f : \mathbb{Z}^{\neq 0} \longrightarrow \mathbb{Q}$, $f(x) = x^{-5}$,

(g) $f : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{Q}$, $f(x, y) = x \cdot y$,

(h) $f : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$, $f(x, y) = x \cdot y$,

(i) $f : \mathbb{Z} \longrightarrow \mathbb{Z} \times \mathbb{Z}$, $f(x) = (x, x)$.

Problem 8. Determine the cardinality of each of the following sets.

(a) $X_1 = \{(x, y) : x \in \mathbb{N}, y \in \mathbb{Z}\} = \mathbb{N} \times \mathbb{Z}$,

(b) $X_2 = \{(x, y, z) : x, y, z \in \mathbb{Q}\} = \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}$,

(c) $X_3 = \{(x, y) : x, y \in \mathbb{R}\} = \mathbb{R} \times \mathbb{R}$,

(d) $X_4 = \{x \in \mathbb{Z} : x \equiv 3 \mod 5\}$,

(e) $X_5 = \{p \in \mathbb{N} : p \text{ is a prime}\}$,

(f) $X_6 = \{x : x \in \mathbb{R} \setminus \mathbb{Q}, -5 < x < 7\} = (-5, 7) \cap \mathbb{I}$,

(g) $X_7 = \{x : x \in \mathbb{Q} \text{ or } x^2 \in \mathbb{Q} \text{ or } x^3 \in \mathbb{Q}\}$.

Problem 9. Define a *word* to be a finite sequence of letters, where the letters are taken from a finite set, oftentimes called the abecedary.

(a) Show that the set of all words is countable.

(b) Define a *text* as a subset of the set of all words. Is the set of texts countable ?

Problem 10. Prove or disprove the following statements:

(a) Let X be a countable set, and $S \subseteq X$ a subset. Then S is itself countable,

(b) Let X, Y be countable sets, then $X \times Y$ is countable,

(c) Let X be a set and $P(X)$ its power set. If $P(X)$ is uncountable then X is countable,

(d) Let X, Y be such that $X \times Y$ is uncountable, then X **or** Y are uncountable,

(e) Let X, Y be such that $X \times Y$ is uncountable, then X **and** Y are uncountable,

(f) There exist sets X, Y and maps $f, g : X \longrightarrow Y$, such that f is an injection but *not* a surjection, **and** g is a bijection.

(g) There exist finite sets X, Y and maps $f, g : X \longrightarrow Y$, such that f is an injection but *not* a surjection, **and** g is a bijection.